APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

ON A GENERALIZATION OF THE MARTINGALE MAXIMAL THEOREM

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1. Introduction

In this paper we give a generalization of the martingale maximal theorem (see e.g. [11]). A generalization of the notion of stopping time is also introduced. Among other things, we show that the inequality with respect to the $P_k(\cdot, \omega)$ polynomials, playing a basic role in Carleson's method (see [1]–[10]), can be closely related to the above-mentioned maximal theorem. These results are also used in an essential way in papers [8] and [9].

2. Stopping times

Let $(X, \mathscr{A}, \mu)$ be a probability space and $(\Gamma, \leqslant)$ a countable directed index-set. $\alpha < \beta$ will mean that $\alpha \leqslant \beta$ and $\alpha \neq \beta$. We assume that every non-empty subset Γ_0 of Γ has a minimal element. The set of minimal elements of Γ_0 is denoted by $\min \Gamma_0$. If $\Gamma_0 = \emptyset$ is the empty set, then let $\min \Gamma_0 = \infty$. Further, let $\overline{\Gamma} = \Gamma \cup \{\infty\}$, and $\gamma < \infty$ for every $\gamma \in \Gamma$.

Denote by $A = (\mathscr{A}_\gamma, \gamma \in \Gamma)$ a stochastic basis, i.e. a nondecreasing sequence of sub-σ-fields of $\mathscr{A}$. $\mathscr{A}_\infty = \bigvee_{\gamma \in \Gamma} \mathscr{A}_\gamma$ will denote the σ-algebra generated by $(\mathscr{A}_\gamma, \gamma \in \Gamma)$.

DEFINITION. A relation $\tau \subset X \times \overline{\Gamma}$ with the domain X is called a *stopping time of the stochastic basis* A if

(i) $\{\tau = \alpha\} := \{x \in X\colon (x, \alpha) \in \tau\} \in \mathscr{A}_\alpha$ for each $\alpha \in \Gamma$, and

(ii) for every $x \in X$ $\alpha, \beta \in \{\tau = x\} := \{\gamma \in \Gamma\colon (x, \gamma) \in \tau\}$ implies $\alpha \nless \beta$ and $\beta \nless \alpha$.

If Γ is a linearly ordered set, then τ is a function, and in this case τ is a stopping time in the usual sense. We note that (ii) gives

$$\{\tau = \alpha\} \cap \{\tau = \beta\} = \emptyset \quad \text{if} \quad \alpha < \beta. \tag{1}$$

For every $\alpha \in \Gamma$ let $f_\alpha\colon X \to \boldsymbol{K}$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$) be an $\mathscr{A}_\alpha$-measurable function and B a Borel subset of K. Then

$$\{\tau_B = x\} := \min\{\alpha \in \Gamma: f_\alpha(x) \in B\} \qquad (x \in X) \tag{2}$$

defines a stopping time τ_B of $(\mathscr{A}_\gamma, \gamma \in \Gamma)$.

Denote by $\mathscr{T} = \mathscr{T}(A)$ the set of stopping times of A. In the set $\mathscr{T}$ we introduce an ordering denoted by $\leqslant$.

Definition. For $\tau, \sigma \in \mathscr{T}$ let $\tau \leqslant \sigma$ if for every $x \in X$ the following holds: for each $\beta \in \{\sigma = x\}$ there exists an $\alpha \in \{\tau = x\}$ such that $\beta \leqslant \alpha$.

An elementary calculation shows, that in this way an ordering is defined in the set $\mathscr{T}$. If Γ is linearly ordered, then $\tau, \sigma \in \mathscr{T}$ are functions, and in this case $\leqslant$ gives the usual ordering of functions. It is easy to see that for the stopping times introduced in (2) the following holds:

$$B_1 \subseteq B_2 \subseteq K \quad \text{implies} \quad \tau_{B_2} \leqslant \tau_{B_1}. \tag{3}$$

3. A generalization of the concept of orthogonality

Let $G = \{g_\alpha: \alpha \in \Gamma\} \subset L^2 := L^2(X, \mathscr{A}, \mu)$ be a system of functions. For a sub-σ-algebra $\mathscr{B}$ of $\mathscr{A}$ let $E(f|\mathscr{B})$ denote the conditional expectation of f relative to $\mathscr{B}$. Further, let $\mathscr{A}_\alpha \vee \mathscr{A}_\beta$ be the σ-algebra generated by $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$.

Definition (see also [7]).

(i) The system of functions G is called an *A-orthogonal system* (briefly A-OS) if, for every $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$,

$$E(g_\alpha \bar{g}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = 0. \tag{4}$$

(ii) If there exists a system of sets $I_\alpha \in \mathscr{A}_\alpha (\alpha \in \Gamma)$ such that

$$E(|g_\alpha|^2 | \mathscr{A}_\alpha) = I(I_\alpha)\,(^1) \qquad (\alpha \in \Gamma), \tag{5}$$

then the system G is said to be an *A-normed system*. Systems which are A-orthogonal and A-normed are referred to as *A-orthonormal systems* (for the brevity A-ONS).

We note that any system $\{g_\alpha: \alpha \in \Gamma\} \subset L^2$ can be made A-normed by multiplication g_α by an appropriate $\mathscr{A}_\alpha$-measurable function. If, for every $\alpha \in \Gamma$, $\mathscr{A}_\alpha = \{X, \emptyset\}$, then $E(g_\alpha \bar{g}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = \int_X g_\alpha \bar{g}_\beta \, d\mu$, and so in this case the above definition reduces to that of usual ONS.

We can prove the following generalization of Bessel's identity for A-ONS (see [7], Theorem 1).

Theorem 1. *Let $G = \{g_\alpha: \alpha \in \Gamma\} \subset L^2$ be an A-ONS, $\Gamma_0 \subset \Gamma$ a finite subset. Then for any function $f \in L^2$ we have*

$$\inf\Big\{\int_X \Big|f - \sum_{\alpha \in \Gamma_0} \lambda_\alpha g_\alpha\Big|^2 d\mu: \lambda_\alpha \in L^2(X, \mathscr{A}_\alpha, \mu)\Big\} = \int_X |f|^2 d\mu - \sum_{\alpha \in \Gamma_0} \int_X |E(f\bar{g}_\alpha | \mathscr{A}_\alpha)|^2 d\mu \tag{6}$$

and the infimum is attained in the case $\lambda_\alpha = E(f\bar{g}_\alpha | \mathscr{A}_\alpha)$ $(\alpha \in \Gamma_0)$.

(1) $I(A)$ denotes the characteristic function of the set $A \subset X$.

In the case $\mathscr{A}_\alpha = \{X, \emptyset\}$ $(\alpha \in \Gamma)$ this identity reduces to the usual Bessel identity. (6) immediately implies the following generalization of Bessel's inequality:

Corollary. *If $G = \{g_\alpha: \alpha \in \Gamma\} \subset L^2$ is an A-ONS, then for any function $f \in L^2$*

$$\sum_{\alpha \in \Gamma} \int_X |E(f\bar{g}_\alpha | \mathscr{A}_\alpha)|^2 d\mu \leqslant \int_X |f|^2 d\mu. \tag{7}$$

Besides let us introduce the following generalization of the concept of Fourier coefficients and that of Fourier expansion.

Definition. Let $G = \{g_\alpha: \alpha \in \Gamma\}$ be an A-ONS. The function $E(f\bar{g}_\alpha | \mathscr{A}_\alpha)$ $(\alpha \in \Gamma)$ is called the *α-th A-Fourier coefficient of the function $f \in L^2$ with respect to the system* G, and the series $\sum_{\alpha \in \Gamma} E(f\bar{g}_\alpha | \mathscr{A}_\alpha) g_\alpha$ the *A-Fourier series of the function f with respect to the system G.*

4. A maximal inequality

Let $\Gamma = N := \{0, 1, 2, \ldots\}$ and denote by $\leqslant$ the usual ordering of N. Then for every $\lambda > 0$ the function

$$\nu(x) = \min\{n \in N: |E(f|\mathscr{A}_n)(x)| > \lambda\}$$

is a stopping time relative to $(\mathscr{A}_n, n \in N)$ and

$$\mu\{\sup_n |E(f|\mathscr{A}_n)| > \lambda\} = \mu\{\nu < \infty\} \leqslant \frac{1}{\lambda^2} \int_{\{\nu < \infty\}} |f|^2 d\mu. \tag{8}$$

Indeed, using

$$|E(f|\mathscr{A}_n)|^2 \leqslant E(|f|^2 | \mathscr{A}_n), \qquad \int_{\{\nu = n\}} |f|^2 d\mu = \int_{\{\nu = n\}} E(|f|^2 | \mathscr{A}_n) \, d\mu$$

(see e.g. [11]), we obtain

$$\mu\{\nu < \infty\} = \sum_{n \in N} \mu\{\nu = n\} \leqslant \sum_{n \in N} \frac{1}{\lambda^2} \int_{\{\nu = n\}} |E(f|\mathscr{A}_n)|^2 d\mu$$
$$\leqslant \sum_{n \in N} \frac{1}{\lambda^2} \int_{\{\nu = n\}} |f|^2 d\mu = \int_{\{\nu < \infty\}} |f|^2 d\mu,$$

which is the desired inequality.

(8) is the well-known martingale maximal inequality. We give a generalization of this inequality.

Theorem 2. *Let $\{h_\alpha: \alpha \in \Gamma\} \subset L^2$ be a system of functions for which*

$$E(h_\alpha \bar{h}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = \delta_{\alpha\beta}\,(^2), \tag{9}$$

if $\alpha \nless \beta$ and $\beta \nless \alpha$. Further, let

$$\{\tau = x\} := \min\{\alpha \in \Gamma: |E(f\bar{h}_\alpha | \mathscr{A}_\alpha)(x)| > \lambda\} \qquad (x \in X),$$

(2) $\delta_{\alpha\beta}$ is the Kronecker symbol.

where $\lambda > 0$. Then τ is a stopping time of the stochastic basis $(\mathscr{A}_\alpha, \alpha \in \Gamma)$ and

$$\sum_{\alpha \in \Gamma} \mu\{\tau = \alpha\} \leqslant \frac{1}{\lambda^2} \int_{\{\tau < \infty\}} |f|^2 d\mu. \tag{10}$$

Proof. We first show that $g_\alpha := I\{\tau = \alpha\} h_\alpha$ $(\alpha \in \Gamma)$ is a $(\mathscr{A}_\alpha, \alpha \in \Gamma)$-ONS. Since $I\{\tau = \alpha\}$ is $\mathscr{A}_\alpha$-measurable, by (9) we have

$$E(g_\alpha \bar{g}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = I\{\tau = \alpha\} I\{\tau = \beta\} E(h_\alpha \bar{h}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = 0$$

if $\alpha \not< \beta$ and $\beta \not< \alpha$. If $\alpha < \beta$ or $\beta < \alpha$, then by (1) $I\{\tau = \alpha\}. I\{\tau = \beta\} = 0$; therefore in this case $g_\alpha \bar{g}_\beta = 0$. Finally, in the case $\alpha = \beta$ by (9) we get

$$E(|g_\alpha|^2 | \mathscr{A}_\alpha) = I\{\tau = \alpha\} E(|h_\alpha|^2 | \mathscr{A}_\alpha) = I\{\tau = \alpha\},$$

i.e. $\{g_\alpha \colon \alpha \in \Gamma\}$ is $\boldsymbol{A}$-normed.

Applying the generalization of Bessel's inequality to the function $I\{\tau < \infty\} f$ and using the equality

$$I\{\tau = \alpha\}\, E(I\{\tau < \infty\} f \bar{h}_\alpha | \mathscr{A}_\alpha) = I\{\tau = \alpha\} E(f \bar{h}_\alpha | \mathscr{A}_\alpha),$$

we obtain

$$\sum_{\alpha \in \Gamma} \int_X I\{\tau = \alpha\} |E(f \bar{h}_\alpha | \mathscr{A}_\alpha)|^2 d\mu \leqslant \int_X I\{\tau < \infty\} |f|^2 d\mu.$$

Since by the definition of τ we have $|E(f\bar{h}_\alpha | \mathscr{A}_\alpha)| > \lambda$ on the set $\{\tau = \alpha\}$, the last inequality indeed implies (10), and Theorem 2 is proved.

If $\Gamma = \boldsymbol{N}$ and $h_n = 1$ $(n \in \boldsymbol{N})$, then $\tau = \nu$ and in this case from (10) we obtain (8).

5. Examples

In this section we give some examples of the concepts introduced before.

Let $\mathscr{A}_0 = \{X, \emptyset\} \subset \mathscr{A}_1 \subset \ldots \subset \mathscr{A}_n \subset \ldots \subset \mathscr{A}$ be a sequence of σ-algebras. Further, we investigate function systems

$$\Phi_n = \{\varphi_n^k \colon k \in J_n\} \subset L^2(X, \mathscr{A}_n, \mu) \quad (n \in \boldsymbol{N}^* := \boldsymbol{N} \setminus \{0\})$$

having the property $\varphi_n^0 = 1$ and

$$E(\varphi_n^i \bar{\varphi}_n^j | \mathscr{A}_{n-1}) = \delta_{ij} \quad (i, j \in J_n, n \in \boldsymbol{N}^*), \tag{11}$$

(i.e. Φ_n is an $B_n = (\mathscr{B}_n^k, k \in J_n)$-ONS, where $\mathscr{B}_n^k = \mathscr{A}_{n-1}$ $(k \in J_n)$). Here $J_n \subset \{n, -n \colon n \in \boldsymbol{N}\}$ and $0 \in J_n$. For

$$p = (p_1, p_2, \ldots) \in J = \bigcup_{n=1}^{\infty} (J_1 \times J_2 \times \ldots \times J_n \times \{0\} \times \{0\} \times \ldots)$$

we set

$$\psi_p = \prod_{j=1}^{\infty} \varphi_j^{p_j}.$$

The system $\Psi = \{\psi_p \colon p \in J\}$ is called the *product system* of systems $\{\Phi_n, n \in \boldsymbol{N}^*\}$. In the case of $\Phi_n = \{1, \varphi_n\}$ this definition gives the usual concept of the product system.

Further, let $p^n := (0, 0, \ldots, 0, p_{n+1}, p_{n+2}, \ldots)$ if $p = (p_1, p_2, \ldots) \in J$ and we set $\Gamma := \{(p^n, n) \colon p \in J, n \in \boldsymbol{N}\}$.

In the set Γ we introduce an ordering (denoted by $\leqslant$). For $\alpha = (p^n, n)$, $\beta = (q^m, m) \in \Gamma$ let $\alpha < \beta$ if $n < m$ and $p^m = q^m$. Clearly, in this way a linear ordering is defined in Γ, Γ is countable and any non-empty subset of Γ has a minimal element.

For $\alpha = (p^n, n) \in \Gamma$ let $\mathscr{A}_\alpha = \mathscr{A}_n$ and $\psi_\alpha = \psi_{p^n}$. Then $(\mathscr{A}_\alpha, \alpha \in \Gamma)$ is a non-decreasing sequence of σ-algebras and

$$E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = \delta_{\alpha\beta} \quad \text{if} \quad \alpha \not< \beta \quad \text{and} \quad \beta \not< \alpha. \tag{12}$$

Indeed, if $\alpha = (p^n, n)$, $\beta = (q^m, m)$ and $n < m$, then $\alpha \not< \beta$ implies that there exists an $s > m$ such that $p^s = q^s$ and $p_s \neq q_s$. Then, by a well-known property of the conditional expectation (see e.g. [11]), we have

$$E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_{s-1}) = \prod_{j=n+1}^{s-1} \varphi_j^{p_j} \prod_{j=m+1}^{s-1} \bar{\varphi}_j^{q_j} E(\varphi_s^{p_s} \bar{\varphi}_s^{q_s} |\psi_{p^s}|^2 | \mathscr{A}_{s-1}).$$

Since

$$E(\varphi_s^{p_s} \bar{\varphi}_s^{q_s} |\psi_{p^s}|^2 | \mathscr{A}_{s-1}) = E(\varphi_s^{p_s} \varphi_s^{q_s} E(|\psi_{p^s}|^2 | \mathscr{A}_s) | \mathscr{A}_{s-1})$$

and, by (11),

$$E(|\psi_{p^s}|^2 | \mathscr{A}_s) = E(|\varphi_{s+1}^{p_{s+1}}|^2 E(|\varphi_{s+2}^{p_{s+2}}|^2 \ldots | \mathscr{A}_{s+1}) | \mathscr{A}_s) = 1,$$

we get $E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_{s-1}) = 0$. This implies

$$E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_\alpha \vee \mathscr{A}_\beta) = E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_m) = E(E(\psi_\alpha \bar{\psi}_\beta | \mathscr{A}_{s-1}) | \mathscr{A}_m) = 0,$$

which proves (12) in the case of $\alpha \not< \beta$ and $\beta \not< \alpha$. If $\alpha = \beta$, then the proof is similar.

Let us now consider the following special cases:

1. Put $X = [0, 1)$, μ the Lebesgue measure and $\mathscr{A}$ the class of Lebesgue measurable sets of X. Denote by $(\varphi_n, n \in \boldsymbol{N}^*)$ the Rademacher system, i.e. define

$$\varphi_1(x) = \begin{cases} 1 & (0 \leqslant x < 1/2), \\ -1 & (1/2 \leqslant x < 1), \end{cases} \quad \varphi_1(x+1) = \varphi_1(x) \quad (x \in \boldsymbol{R}),$$

$\varphi_n(x) = \varphi_1(2^{n-1}x)$ $(n \in \boldsymbol{N}^*)$ and let $\Phi_n = \{1, \varphi_n\}$ $(n \in \boldsymbol{N}^*)$. For every n define $\mathscr{A}_n$ to be the σ-algebra generated by the functions $\varphi_1, \ldots, \varphi_n$. It is easy to verify condition (11) for Φ_n.

In this special case (10) gives an inequality of P. Billard [1] (see also R. A. Hunt [5], inequality (8)), playing a basic role in papers [1], [5] and [10].

In this example the Rademacher system can be replaced by any system $\{\varphi_n, n \in \boldsymbol{N}^*\} \subset L^2(X, \mathscr{A}, \mu)$ consisting of independent functions having the property $\int_X \varphi_n d\mu = 0$ $(n \in \boldsymbol{N}^*$; see [8], [9]).

2. It is easy to see [8] that every Vilenkin system is the product system of systems Φ_n $(n \in \boldsymbol{N}^*)$ satisfying condition (11). Applying Theorem 2 to such systems, we obtain some inequality of R. A. Hunt and M. Taibleson [6] and J. Gosselin [3].

3. Let $X = [0, 1]$, let $\mathscr{B}_{-n}$ $(n \in N^*)$ be the class of 2^{-n}-periodic Borel measurable sets of X and μ the Lebesgue measure. Then $\mathscr{B}_0 \supset \mathscr{B}_{-1} \supset \ldots$ and it can easily be proved (see e.g. [12]) that for every function $f \in L^1(X, \mathscr{A}, \mu)$ we have

$$E(f|\mathscr{B}_{-n}) = 2^{-n} \sum_{=0k}^{2^n-1} f(x \dotplus k2^{-n}),$$

where $\dotplus$ denotes the addition mod 1.

Let $N \in N^*$ be a fixed number and let us introduce the notation $\mathscr{A}_n = \mathscr{B}_{n-N}$ $(n = 0, 1, \ldots, N)$, $\varphi_n(x) = \exp(2\pi i 2^{N-n} x)$ $(x \in X;\ n = 1, 2, \ldots, N)$. Then we have $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \ldots \subset \mathscr{A}_N = \mathscr{B}_0$, and the systems $\Phi_n = \{1, \varphi_n\}$ $(n = 1, 2, \ldots, N)$ obviously satisfy condition (11). The product system of the systems Φ_n $(n = 1, 2, \ldots, N)$ is the same as the system $\{\exp(2in)\colon n = 0, 1, \ldots, 2^N - 1\}$. In this case (10) gives an inequality of Carleson (see e.g. [2] and [4]).

Further examples and applications can be found in [7], [8] and [9].

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Presented to the Semester
Approximation Theory
September 17–December 17, 1975

APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

LOWER BOUNDS FOR SPLINE APPROXIMATION

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1. Introduction

The purpose of this paper is to discuss how well certain classes of smooth functions can be approximated by some linear spaces of piecewise functions which include broad classes of splines. In particular, we shall give lower bounds which agree in order with known upper bounds for generalized spline approximation, thus showing that these upper bounds are asymptotically sharp.

To be more specific, let $\Delta = \{a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b\}$ be a partition of the interval $[a, b]$. Let $U_m = \{u_i\}_1^m$ be a set of continuous functions defined on the interval $[a, b]$. Consider

$$\mathscr{P}U_m(\Delta) = \{f\colon f_i = f|_{(x_i, x_{i+1})} \in \operatorname{span}(U_m),\ i = 0, 1, \ldots, k\}. \tag{1.1}$$

This is a linear space of functions which are piecewise elements in $\operatorname{span}(U_m)$. Given a function F defined on $[a, b]$, we are interested in the distance

$$d_q(F, \mathscr{P}U_m(\Delta)) = \inf_{s \in \mathscr{P}U_m(\Delta)} \|F - s\|_q, \tag{1.2}$$

where $\|\cdot\|_q$ is one of the usual q-norms, $1 \leqslant q \leqslant \infty$. We are especially interested in this distance for smooth functions; for example, members of $C^{r-1}[a, b]$ with $1 \leqslant r$, or members of the Sobolev space $L_p^r[a, b] = \{f \in AC^{r-1}[a, b]\colon f^{(r)} \in L_p[a, b]\}$ for some $1 \leqslant p \leqslant \infty$.

To state our main results, we need some additional notation. As usual, we denote by $\bar{\Delta}$ the maximum subinterval length associated with the partition Δ. Given a function $f \in L_p^r[a, b]$, we define the usual Sobolev norm by

$$\|f\|_{L_p^r[a, b]} = \sum_{j=0}^{r} \|f^{(j)}\|_{L_p[a, b]}. \tag{1.3}$$

Our main results are the following two theorems.

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