3. Let $X = [0, 1]$, let $\mathcal{B}_{-n}$ $(n \in N^*)$ be the class of 2^{-n}-periodic Borel measurable sets of X and μ the Lebesgue measure. Then $\mathcal{B}_0 \supset \mathcal{B}_{-1} \supset \ldots$ and it can easily be proved (see e.g. [12]) that for every function $f \in L^1(X, \mathcal{A}, \mu)$ we have

$$E(f|\mathcal{B}_{-n}) = 2^{-n} \sum_{k=0}^{2^n-1} f(x \dotplus k2^{-n}),$$

where $\dotplus$ denotes the addition mod 1.

Let $N \in N^*$ be a fixed number and let us introduce the notation $\mathcal{A}_n = \mathcal{B}_{n-N}$ $(n = 0, 1, \ldots, N)$, $\varphi_n(x) = \exp(2\pi i 2^{N-n} x)$ $(x \in X;\ n = 1, 2, \ldots, N)$. Then we have $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_N = \mathcal{B}_0$, and the systems $\Phi_n = \{1, \varphi_n\}$ $(n = 1, 2, \ldots, N)$ obviously satisfy condition (11). The product system of the systems Φ_n $(n = 1, 2, \ldots, N)$ is the same as the system $\{\exp(2in)\colon n = 0, 1, \ldots, 2^N - 1\}$. In this case (10) gives an inequality of Carleson (see e.g. [2] and [4]).

Further examples and applications can be found in [7], [8] and [9].

References

[1] P. Billard, *Sur la convergence presque partout des séries de Fourier Walsh des fonctions de l'espace* $L^2(0, 1)$, Studia Math. 28 (1967), pp. 336–388.

[2] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), pp. 135–157.

[3] J. Gosselin, *Almost everywhere convergence of Vilenkin–Fourier series*, Trans. Amer. Math. Soc. 185 (1973), pp. 345–370.

[4] R. A. Hunt, *On the convergence of Fourier series. Orthogonal Expansions and their Continuous Analogues*, SIU Press, Carbondale, Illinois 1968.

[5] —, *Almost everywhere convergence of Walsh–Fourier series of* L^2 *functions*, Proc. Internat. Congress Math., Nice 1970, vol. 2., Gauthier-Villars, Paris 1971, pp. 655–661.

[6] R. A. Hunt and M. Taibleson, *On the almost everywhere convergence of Fourier series on the ring of integers of a local field*, SIAM J. Math. Anal. 2 (1971), pp. 607–625.

[7] F. Schipp, *On a generalization of the concept of orthogonality*, Acta Sci. Math. 37 (1975), pp. 279–285.

[8] —, *On* L^p*-norm convergence of series with respect to product systems*, Analysis Math. 2 (1976), pp. 49–64.

[9] —, *On a.e. convergence of expansion with respect to certain product systems*, ibid. 2 (1976), pp. 65–76.

[10] P. Sjölin, *An inequality of Paley and convergence of Walsh–Fourier series*, Ark. Math. 7 (1969), pp. 551–570.

[11] E. M. Stein, *Topics in harmonic analysis related to the Littlewood–Paley theory*, Ann. of Math. Studies 63, Princeton Univ. Press, Princeton, N.J. 1970.

[12] W. F. Stout, *Almost sure convergence*, Academic Press, New York, San Francisco, London 1974.

[13] N. Ja. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk SSSR, Ser. Mat. 11 (1947), pp. 363–400.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975

APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

LOWER BOUNDS FOR SPLINE APPROXIMATION

LARRY L. SCHUMAKER*

The University of Texas at Austin, Austin, Texas 78712, U.S.A.

1. Introduction

The purpose of this paper is to discuss how well certain classes of smooth functions can be approximated by some linear spaces of piecewise functions which include broad classes of splines. In particular, we shall give lower bounds which agree in order with known upper bounds for generalized spline approximation, thus showing that these upper bounds are asymptotically sharp.

To be more specific, let $\Delta = \{a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b\}$ be a partition of the interval $[a, b]$. Let $U_m = \{u_i\}_1^m$ be a set of continuous functions defined on the interval $[a, b]$. Consider

$$\mathcal{P}U_m(\Delta) = \{f\colon f_i = f|_{(x_i, x_{i+1})} \in \operatorname{span}(U_m),\ i = 0, 1, \ldots, k\}. \tag{1.1}$$

This is a linear space of functions which are piecewise elements in $\operatorname{span}(U_m)$. Given a function F defined on $[a, b]$, we are interested in the distance

$$d_q(F, \mathcal{P}U_m(\Delta)) = \inf_{s \in \mathcal{P}U_m(\Delta)} \|F - s\|_q, \tag{1.2}$$

where $\|\cdot\|_q$ is one of the usual q-norms, $1 \leqslant q \leqslant \infty$. We are especially interested in this distance for smooth functions; for example, members of $C^{r-1}[a, b]$ with $1 \leqslant r$, or members of the Sobolev space $L_p^r[a, b] = \{f \in AC^{r-1}[a, b]\colon f^{(r)} \in L_p[a, b]\}$ for some $1 \leqslant p \leqslant \infty$.

To state our main results, we need some additional notation. As usual, we denote by $\overline{\Delta}$ the maximum subinterval length associated with the partition Δ. Given a function $f \in L_p^r[a, b]$, we define the usual Sobolev norm by

$$\|f\|_{L_p^r[a, b]} = \sum_{j=0}^{r} \|f^{(j)}\|_{L_p[a, b]}. \tag{1.3}$$

Our main results are the following two theorems.

* Department of Mathematics and Center for Numerical Analysis, University of Texas, Austin, Texas 78712. Supported in part by the United States Air Force Office of Scientific Research under grant AFOSR 74–2598B.

THEOREM 1.1. *Let $0 \leqslant r \leqslant m$ and suppose that U_m is a Tchebycheff system. Then, for every partition Δ of $[a, b]$, there exists a function $F \in L_\infty^r[a, b]$ with*

$$d_\infty(F, \mathscr{P}U_m(\Delta)) \geqslant K\,(\bar{\Delta})^r \|F\|_{L_\infty^r[a,b]}, \tag{1.4}$$

where

$$K = \left(\frac{1}{4(m+1)}\right)^r \frac{1}{2^{r(r-1)/2}[1+(b-a)+\ldots+(b-a)^r]}. \tag{1.5}$$

THEOREM 1.2. *Let $0 \leqslant r \leqslant m$, $1 \leqslant q < \infty$, and $1 \leqslant p \leqslant \infty$. Then for every partition Δ of $[a, b]$, there exists a function $F \in L_p^r[a, b]$ with*

$$d_q(F, \mathscr{P}U_m(\Delta)) \geqslant \hat{K}\,(\bar{\Delta})^{r+1/q-1/p} \|F\|_{L_p^r[a,b]}, \tag{1.6}$$

where

$$\hat{K} = \left(\frac{1}{8(m+1)}\right)^r \frac{2}{[2(m+1)]^{1/q} 2^{r(r-1)/2}[1+(b-a)+\ldots+(b-a)^r]}. \tag{1.7}$$

Theorem 1.1 is proved in Section 2 below, along with several related results and corollaries which show the direct relationship with upper bounds for spline approximation. Theorem 1.2 and related results are discussed in Section 3. We close the paper with a section containing a variety of remarks with references.

2. Lower bounds for d_∞

Our approach to proving Theorem 1.1 is to explicitly construct a function F belonging to $L_\infty^r[a, b]$ for which zero is the best approximation from $\mathscr{P}U_m(\Delta)$. A key tool in our construction is the following lemma.

LEMMA 2.1. *Let $0 \leqslant r$. Then there exists a function $g_r(t)$ defined on $[0, 1]$ with the following properties:*

(2.1) $g_r(t) \geqslant 0$ *for* $t \in [0, 1]$;

(2.2) $g_r(t)$ *is symmetric about the point* $t = \frac{1}{2}$;

(2.3) $g_r^{(j)}(0) = g_r^{(j)}(1) = 0$, $j = 0, 1, \ldots, r-1$;

(2.4) $|g_r^{(r)}(t)| = 2^{r(r-1)/2}$ *for* $t \in [0, 1]$;

(2.5) $g_r(t) \geqslant \frac{1}{4}(1/8)^{r-1}$ *for* $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$, $r \geqslant 1$;

(2.6) $g_r(\frac{1}{2}) \geqslant (\frac{1}{4})^r$.

Proof. Let $g_0 = 1$, and define $g_r(t)$ inductively by

$$g_r(t) = \begin{cases} \displaystyle\int_0^t g_{r-1}(2u)\,du, & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ g_r\left(\dfrac{1}{2}\right) - \displaystyle\int_{1/2}^t g_{r-1}(2u-1)\,du, & \dfrac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{2.7}$$

Clearly, g_0 satisfies the stated properties. Suppose now that g_{r-1} satisfies (2.1)–(2.6). We wish to show that g_r also does. First, it is clear from Definition (2.7) that g_r is symmetric about the point $\frac{1}{2}$. Since $g_r(0) = 0$ and

$$D^j g_r(t) = \begin{cases} D^{j-1} g_{r-1}(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ -D^{j-1} g_{r-1}(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

property (2.3) follows. If we carry out these differentiations, we obtain

$$D^r g_r(t) = \begin{cases} 2^{r-1} g_r^{(r-1)}(2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ -2^{r-1} g_r^{(r-1)}(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Now (2.4) follows from the induction hypothesis since $2^{r-1} \cdot 2^{(r-1)(r-2)/2} = 2^{r(r-1)/2}$

Suppose that we write $Q_r = g_r(\frac{1}{4})$. Since g_r is clearly monotone increasing on $[0, \frac{1}{2}]$, it follows that $g_r(t) \geqslant Q_r$ for $\frac{1}{4} \leqslant t \leqslant \frac{1}{2}$. But $Q_1 = \frac{1}{4}$, while

$$Q_r = \int_0^{1/4} g_{r-1}(2u)\,du \geqslant \int_{1/8}^{1/4} Q_{r-1}\,du \geqslant Q_{r-1}/8,$$

and (2.5) follows. Finally, suppose we write $C_r = g_r(\frac{1}{2})$. Then

$$C_r = \int_0^{1/2} g_{r-1}(2u)\,du \geqslant \int_{1/4}^{1/2} Q_{r-1}\,du \geqslant Q_{r-1}/4,$$

and we have proved (2.6). ∎

The functions constructed in this lemma are modelled after similar functions constructed in [7], p. 134 for the purpose of establishing lower bounds on n-widths.

Proof of Theorem 1.1. Let Δ be a partition of $[a, b]$, and suppose that v is chosen so that $x_{v+1} - x_v = \bar{\Delta}$. Let I_v denote the interval $[x_v, x_{v+1}]$. We further subdivide I_v into $m+1$ equal subintervals given by

$$I_{v,i} = \left[x_v + \frac{i\bar{\Delta}}{(m+1)},\ x_v + \frac{(i+1)\bar{\Delta}}{(m+1)}\right], \quad i = 0, 1, \ldots, m.$$

Now, define

$$F(t) = \begin{cases} (-1)^i \left(\dfrac{\bar{\Delta}}{m+1}\right)^r g_r\left(\dfrac{x - x_v - \dfrac{i\bar{\Delta}}{(m+1)}}{\bar{\Delta}/(m+1)}\right), & t \in I_{vi},\ i = 0, 1, \ldots, m, \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

By the properties of g_r, we see that $F \in C^{r-1}[a, b]$ while

$$\|F^{(r)}\|_\infty = 2^{r(r-1)/2}, \tag{2.9}$$

so that $F \in L_\infty^r[a, b]$. The fact that F and each of its derivatives vanish at x_v allows us to conclude that

$$\|F\|_{L_\infty^r[a,b]} \leqslant 2^{r(r-1)/2}[1+\bar{\Delta}+\ldots+\bar{\Delta}^r]. \tag{2.10}$$

(For example,

$$|F^{r-1}(t)| \leqslant \int_{x_v}^t |F^{(r)}(u)|\,du \leqslant \bar{\Delta}\,\|F^{(r)}\|_\infty, \tag{2.11}$$

and so forth.)

The construction of F also implies that

$$F\left(x_v+\frac{(i+1/2)\bar{\Delta}}{(m+1)}\right) \geqslant (-1)^i\left(\frac{\bar{\Delta}}{4(m+1)}\right)^r, \quad i = 0, 1, \ldots, m. \tag{2.12}$$

Since U_m is a Tchebycheff system, we conclude from (2.12) and the well-known theorem of de la Valleé Poussin, that

$$d_\infty(F, \mathscr{P}U_m(\Delta)) \geqslant \left(\frac{\bar{\Delta}}{4(m+1)}\right)^r. \tag{2.13}$$

Combining (2.10) and (2.12) yields (1.4)–(1.5). ∎

In the case where the functions in U_m are sufficiently differentiable, a similar proof can be used to establish a more general result. Before stating it, we need to agree on how to define the norms of derivatives of functions which are defined only piecewise. If φ is a function whose jth derivative exists in each of the subintervals of $[a, b]$ defined by some partition Δ, then we define

$$\|\varphi^{(j)}\|_{L_\infty[a,b]} = \max_{0\leqslant i\leqslant k} \|\varphi^{(j)}\|_{L_\infty[x_i, x_{i+1}]}. \tag{2.14}$$

Given $0 \leqslant j \leqslant m-1$, and assuming U_m consists of sufficiently differentiable functions, we define

$$d_{j,q}(F, \mathscr{P}U_m(\Delta)) = \inf_{s\in\mathscr{P}U_m(\Delta)} \|D^j(F-s)\|_{L_q[a,b]}. \tag{2.15}$$

THEOREM 2.2. *Let $0 \leqslant r \leqslant m$ and suppose that U_m is a set of functions in $C^{r-1}[a, b]$. Suppose in addition that each of the sets $V_j = \{D^j u_i\}_{i=1}^m$ is a Tchebycheff system on $[a, b]$. Then, for every partition Δ of $[a, b]$ there exists a function $F \in L^r_\infty[a, b]$ with*

$$d_{j,\infty}(F, \mathscr{P}U_m(\Delta)) \geqslant K_j\, (\bar{\Delta})^{r-j}\|F\|_{L^r_\infty[a,b]}, \quad 0 \leqslant j \leqslant r-1, \tag{2.16}$$

where

$$K_j = \left(\frac{1}{4(m+1)}\right)^{r-j} \frac{2^{j(j-1)/2}}{2^{r(r-1)/2}[1+(b-a)+\ldots+(b-a)^r]}. \tag{2.17}$$

Proof. Given Δ, let F be the function defined in (2.8). We already know that $F \in L^r_\infty[a, b]$. Moreover, it also follows that $F^{(j)}$ alternates between $\pm\|F^{(j)}\|_\infty$ at least $m+1$ times on the interval I_v. Since V_j is a Tchebycheff system, de la Valleé Poussin's theorem again assures that

$$d_{j,\infty}(F, \mathscr{P}U_m(\Delta)) \geqslant \|F^{(j)}\|_\infty.$$

Arguing as before, we see that

$$\|F^{(j)}\|_\infty \geqslant \left(\frac{\bar{\Delta}}{4(m+1)}\right)^{r-j} 2^{(j-1)j/2}. \tag{2.18}$$

Combining (2.18) with (2.10), we obtain (2.16)–(2.17). ∎

We should emphasize that in Theorem 2.2 the function F is such that it and its derivatives are all *simultaneously* hard to approximate. We can now state some

immediate corollaries of Theorem 1.1 and 2.2 which will help show the connection with upper bounds for generalized spline approximation. Given a continuous function φ, we define its usual modulus of continuity by

$$\omega(\varphi, h) = \sup_{\substack{x, x+t\in[a,b]\\ |t|\leqslant h}} |\varphi(x+t)-\varphi(x)|. \tag{2.19}$$

COROLLARY 2.3. *Under the hypotheses of Theorem 1.1, given any partition Δ of $[a, b]$, there exists a function $F \in C^{r-1}[a, b]$ such that $\|F\|_{L^{r-1}_\infty} = 1$ with*

$$d_\infty(F, \mathscr{P}U_m(\Delta)) \geqslant \frac{K\,(\bar{\Delta})^{r-1}}{2}\omega(F^{(r-1)}, \bar{\Delta}), \tag{2.20}$$

where K is the constant in (1.5).

Proof. First, we observe that in view of (2.11), (1.4) implies

$$d_\infty(F, \mathscr{P}U_m(\Delta)) \geqslant K\,(\bar{\Delta})^{r-1}\|F\|_{L^{r-1}_\infty[a,b]},$$

for the function F in Theorem 1.1. Now, if we normalize this function so that $\|F\|_{L^{r-1}_\infty[a,b]} = 1$, and observe that

$$\omega(F^{r-1}, \bar{\Delta}) \leqslant 2\|F^{(r-1)}\|_\infty \leqslant 2\|F\|_{L^{r-1}_\infty[a,b]},$$

we obtain (2.20). ∎

Similarly, we may establish

COROLLARY 2.4. *Under the hypotheses of Theorem 2.2, given any partition Δ of $[a, b]$, there exists a function $F \in L^r_\infty[a, b]$ such that $\|F\|_{L^{r-1}_\infty[a,b]} = 1$ and*

$$d_{j,\infty}(F, \mathscr{P}U_m(\Delta)) \geqslant \frac{K_j}{2}\,(\bar{\Delta})^{r-1-j}\omega(F^{(r-1)}, \bar{\Delta}), \tag{2.21}$$

for all $j = 0, 1, \ldots, r-1$, where K_j is the constant in (2.17).

We now may compare Corollary 2.4 with results in [4] concerning estimates on approximation with a class of generalized splines. In particular, in [4] the class of splines

$$\mathscr{S}(U_m; \Lambda; \Delta) = \{s\colon s_i = s|_{[x_i, x_{i+1}]} \in U_m,\ i = 0, 1, \ldots, m \text{ and } \lambda_{ij}s_{i-1} = \lambda_{ij}s_i,\ j = 1, 2, \ldots, m_i,\ i = 1, 2, \ldots, k\} \tag{2.22}$$

was considered, where U_m was assumed to be the set of fundamental solutions associated with a mth-order linear differential operator $L = D^m + \sum_{j=0}^{m-1} a_j D^j$ with coefficients $a_j \in C^m[a, b]$, and where $\Lambda = \{\lambda_{ij} = \sum_{v=0}^{m-1} \alpha_{ijv} e^{(v)}_{x_i}\}_{j=1,i=1}^{m_i,k}$ is a set of Extended–Hermite–Birkhoff linear functionals.

This set of generalized splines is, of course, a subset of $\mathscr{P}U_m(\Delta)$. This implies

$$d_{j,\infty}(F, \mathscr{S}) \geqslant d_{j,\infty}(F, \mathscr{P}U_m(\Delta)),$$

so the lower bounds in this section are also valid for $\mathscr{S}$. Now, we may observe that the lower bound in (2.22) is an exact match (with a different constant, of course) for an upper bound established in Theorem 2.1 of [4] for the class of splines $\mathscr{S}$.

3. Lower bounds for d_q with $1 < q < \infty$

In this section we shall establish Theorem 1.2 and related results. To prove Theorem 1.2 we shall explicitly construct a function $F \in L_p^r[a,b]$ for which zero is the best approximation from $\mathscr{P}U_m(\Delta)$, and thus

$$d_q(F, \mathscr{P}U_m(\Delta)) \geqslant \|F\|_{L_q[a,b]}. \tag{3.1}$$

In this case, however, we do not have the usual alternation theorem at our disposal to characterize best approximations. In its place we have the fact that $u^* \in \operatorname{span}(U_m)$ is a best approximation of F in the q-norm on an interval $[c,d]$ if and only if

$$\int_c^d |F-u^*|^{q-1}\operatorname{sgn}(F-u^*)u\,dt = 0, \quad \text{all } u \in \operatorname{span}(U_m). \tag{3.2}$$

Thus, we are seeking F satisfying

$$\int_{x_v}^{x_{v+1}} |F|^{q-1}\operatorname{sgn}(F)u_i\,dt = 0, \quad i = 1, 2, \dots, m,\ v = 0, 1, \dots, k. \tag{3.3}$$

The construction in this case is by no means as easy as in the uniform norm case. The main tool will be the following generalization of a result of Hobby and Rice [3].

LEMMA 3.1. *Let $\{u_i\}_1^m$ be a set of m linearly independent functions in $L_1[c,d]$, and let $\theta, \varphi \in C[0,1]$. Then there exist points $c = t_0 \leqslant t_1 \leqslant \dots \leqslant t_{m+1} = d$ and signs $\varepsilon_1, \dots, \varepsilon_m$ such that the function*

$$G(t) = \varepsilon_i\theta(t_{i+1}-t_i)\varphi\left(\frac{t-t_i}{t_{i+1}-t_i}\right), \quad t_i \leqslant t \leqslant t_{i+1}, i = 0, 1, \dots, m \tag{3.4}$$

satisfies

$$\int_c^d G(t)u_i(t)\,dt = 0, \quad i = 1, 2, \dots, m. \tag{3.5}$$

Proof. Let $S = \left\{\xi = (\xi_0, \xi_1, \dots, \xi_m) \in R^{m+1}\colon \sum_{i=0}^m |\xi_i| = d-c\right\}$. For each $\xi \in S$, define $\tau_i = c + |\xi_0| + |\xi_1| + \dots + |\xi_{i-1}|$, $i = 1, 2, \dots, m$, and

$$G_\xi(t) = \begin{cases} \operatorname{sgn}(\xi_0)\cdot\theta(|\xi_0|)\varphi\left(\dfrac{t-c}{|\xi_0|}\right), & c \leqslant t \leqslant \tau_1, \\ \operatorname{sgn}(\xi_1)\cdot\theta(|\xi_1|)\varphi\left(\dfrac{t-\tau_1}{|\xi_1|}\right), & \tau_1 \leqslant t \leqslant \tau_2, \\ \vdots & \\ \operatorname{sgn}(\xi_m)\cdot\theta(|\xi_m|)\varphi\left(\dfrac{t-\tau_m}{|\xi_m|}\right), & \tau_m \leqslant t \leqslant d. \end{cases}$$

Now define a function $\psi\colon S \to R^m \subseteq R^{m+1}$ by

$$[\psi(\xi)]_i = \int_c^d G_\xi(t)u_i(t)\,dt, \quad i = 1, 2, \dots, m.$$

Clearly, ψ is continuous on S, and is odd; i.e. $\psi(\xi) = -\psi(-\xi)$. Since S is the boundary of an open, bounded, symmetric set in R^{m+1} and $\psi(S)$ is contained in the proper subspace R^m of R^{m+1}, it follows from Corollary 3.29 in [12], p. 81 that ψ must take on the value 0 for some $\xi^* \in S$. We may take $t_i = \tau_i^*$, $i = 1, 2, \dots, m$. ∎

The original Hobby–Rice result corresponds to the special case where $\theta \equiv \varphi \equiv 1$. The proof in [3] is quite difficult. Here we have followed the much simpler proof of the Hobby–Rice result recently discovered by Allan Pinkus [9]. Our extension may be of interest in its own right.

Proof of Theorem 1.2. Let Δ be a partition of $[a,b]$, and let v be such that $x_{v+1}-x_v = \bar{\Delta}$. Choose $c = x_v$, $d = x_{v+1}$, $\theta(t) = t^{r(q-1)}$, and $\varphi(t) = |g_r(t)|^{q-1}$ in Lemma 3.1. By the lemma we know there exist $x_v = t_0 \leqslant t_1 \leqslant \dots \leqslant t_{m+1} = x_{v+1}$ and signs $\varepsilon_1, \dots, \varepsilon_m$ such that the function

$$F(t) = \begin{cases} \varepsilon_j(t_{j+1}-t_j)^r g_r\left(\dfrac{t-t_j}{t_{j+1}-t_j}\right), & t_j \leqslant t \leqslant t_{j+1}, j = 0, 1, \dots, m, \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

satisfies (3.3). We conclude that 0 is the best approximation of F from $\mathscr{P}U_m(\Delta)$; i.e., (3.1) holds.

To estimate $\|F\|_q$ from below, we note that at least one of the intervals $I_{vj} = [t_j, t_{j+1}]$ must be of length at least $\bar{\Delta}/(m+1)$. Moreover, on the middle one-half of this interval, call it I_{vj}^*, $|F(t)| \geqslant 2(\bar{\Delta}/8(m+1))^r$. Hence,

$$\|F\|_q \geqslant \left(\int_{I_{vj}^*}\left|2\left(\frac{\bar{\Delta}}{8(m+1)}\right)^r\right|^q dt\right)^{1/q} \geqslant 2\left(\frac{\bar{\Delta}}{8(m+1)}\right)^r\left(\frac{\bar{\Delta}}{2(m+1)}\right)^{1/q}. \tag{3.7}$$

On the other hand, F is clearly in $L_p^r[a,b]$, and in fact,

$$\|F^{(r)}\|_p \leqslant \left(\int_{I_v} 2^{r(r-1)/2}dt\right)^{1/p} = \bar{\Delta}^{1/p}2^{r(r-1)/2}. \tag{3.8}$$

Now,

$$|F^{(r-1)}(t)| = \left|\int_{x_v}^t F^{(r)}(u)\,du\right| \leqslant \|F^{(r)}\|_p\bar{\Delta}^{1/q},$$

so

$$\left(\int_{I_v}|F^{(r-1)}|^p dt\right)^{1/p} \leqslant \|F^{(r)}\|_p\bar{\Delta}^{1/q}\bar{\Delta}^{1/p} = \bar{\Delta}\|F^{(r)}\|_p.$$

Since the lower derivatives can also be estimated in this way, we obtain

$$\|F\|_{L_p^r[a,b]} \leqslant \bar{\Delta}^{1/p}2^{r(r-1)/2}[1+\bar{\Delta}+\dots+\bar{\Delta}^r]. \tag{3.9}$$

Combining (3.9) with (3.1) and (3.7) yields (1.8)–(1.9). ∎

We can also give a result concerning derivatives. If φ is defined piecewise on a partition Δ and is sufficiently differentiable on each subinterval, we define

$$\|\varphi^{(j)}\|_{L_q[a,b]} = \left[\sum_{v=0}^k (\|\varphi^{(j)}\|_{L_q[x_v, x_{v+1}]})^q\right]^{1/q}.$$

Suppose $d_{j,q}(F, \mathscr{P}U_m(\Delta))$ is defined as in (2.15) when the elements of $\mathscr{P}U_m(\Delta)$ are sufficiently smooth piecewise.

THEOREM 3.2. *Let $0 \leqslant r \leqslant m$ and suppose that the set $U_m \subseteq C^{r-1}[a,b]$. Fix $0 \leqslant j \leqslant r-1$, and suppose that the set $V_j = \{D^j u_i\}_{i=1}^m = \{v_{ji}\}_{i=1}^{m-j}$ is a set of $m-j$ linearly independent functions on $[a,b]$. Then for every partition Δ of $[a,b]$ there exists a function $F \in L_p^r[a,b]$ with*

$$d_{j,q}(F, \mathscr{P}U_m(\Delta)) \geqslant \hat{K}_j\ (\bar{\Delta})^{r-j+1/q-1/p}\|F\|_{L_p^r[a,b]}, \tag{3.10}$$

where

$$\hat{K}_j = \left(\frac{1}{8(m+1)}\right)^{r-j} \frac{2}{[2(m+1)]^{1/q}2^{(r-j)(r-j-1)/2}[1+(b-a)+\ldots+(b-a)^r]}. \tag{3.11}$$

Proof. Let Δ be a partition of $[a,b]$, and let v be such that $x_{v+1}-x_v = \bar{\Delta}$. In Lemma 3.1 let $c = x_v$, $d = x_{v+1}$, $\theta(t) = t^{(r-j)(q-1)}$, and $\varphi(t) = |g_{r-j}(t)|^{q-1}$. By the lemma, there exist $x_v = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{m+1} = x_{v+1}$ and signs $\varepsilon_1, \ldots, \varepsilon_m$ such that the function

$$h(t) = \begin{cases} \varepsilon_i(t_{i+1}-t_i)^{r-j}g_{r-j}\left(\dfrac{t-t_i}{t_{i+1}-t_i}\right), & t_i \leqslant t \leqslant t_{i+1},\ i = 0,1,\ldots,m, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies

$$\int_{x_v}^{x_{v+1}} |h(t)|^{q-1}\operatorname{sgn}(h(t))\,v_{ji}(t)\,dt = 0, \quad i = 1,2,\ldots,m-j,\ v = 0,1,\ldots,m. \tag{3.12}$$

Now let

$$F(t) = \int_{x_v}^{t}\int_{x_v}^{u_1}\ldots\int_{x_v}^{u_{j-1}} h(u_j)\,du_j\ldots du_1. \tag{3.13}$$

Clearly, $F \in L_p^r[a,b]$ and

$$\|F^{(r)}\|_p \leqslant 2^{(r-j)(r-j-1)/2}\bar{\Delta}^{1/p},$$

so that

$$\|F\|_{L_p^r[a,b]} \leqslant 2^{(r-j)(r-j-1)/2}\bar{\Delta}^{1/p}[1+\bar{\Delta}+\ldots+\bar{\Delta}^r]. \tag{3.14}$$

On the other hand, in view of (3.12),

$$\begin{aligned} d_{j,q}(F, \mathscr{P}U_m(\Delta)) &= d_q(F^{(j)}, \mathscr{P}V_j(\Delta)) \\ &= d_q(h, \mathscr{P}V_j(\Delta)) \geqslant \|h\|_q \geqslant 2\left(\frac{\bar{\Delta}}{8(m+1)}\right)^{r-j}\left(\frac{\bar{\Delta}}{2(m+1)}\right)^{1/q}. \end{aligned} \tag{3.15}$$

Combining (3.13) and (3.14) yields (3.10)–(3.11). ∎

The result of Theorem 3.2 is not quite as strong as that in Theorem 2.2 inasmuch as there a single function worked for all $0 \leqslant j \leqslant r-1$. Here it is not clear how to construct a function F whose derivatives $F^{(j)}$ are simultaneously hard to approximate from the spaces V_j. We can now state corollaries of Theorems 1.2 and 3.2 which

give lower bounds in terms of p-moduli of continuity. If $\varphi \in L_p[a,b]$, we define

$$\omega(\varphi, h)_p = \sup_{|t|\leqslant h}\left(\int |\varphi(x+t)-\varphi(x)|^p\,dx\right)^{1/p},$$

where the integral is taken over only those x such that x and $x+t$ are both in $[a,b]$. (An alternate definition allows the integral to be defined over all of $[a,b]$, but the function is first extended to a slightly larger interval, see [2], [6].)

COROLLARY 3.3. *Under the hypotheses of Theorem 1.2, given any partition Δ of $[a,b]$, there exists a function $F \in L_p^r[a,b]$ such that $\|F\|_{L_p^r[a,b]} = 1$ and*

$$d_q(F, \mathscr{P}U_m(\Delta)) \geqslant \frac{\hat{K}}{2}\ (\bar{\Delta})^{r+1/q-1/p}\omega(F^{(r)}, \bar{\Delta})_p, \tag{3.16}$$

where $\hat{K}$ is the constant in (1.7).

Proof. If the function F in the proof of Theorem 1.2 is normalized so that $\|F\|_{L_p^r[a,b]} = 1$, then (1.6)–(1.7) coupled with the fact that $\omega(F^{(r)}, \bar{\Delta})_p \leqslant 2\|F^{(r)}\|_p$ yields (3.16). ∎

Similarly, we may establish

COROLLARY 3.4. *Under the hypotheses of Theorem 3.2, given any partition Δ of $[a,b]$, there exists a function $F \in L_p^r[a,b]$ such that $\|F\|_{L_p^r[a,b]} = 1$ and*

$$d_{j,q}(F, \mathscr{P}U_m(\Delta)) \geqslant \frac{\hat{K}_j}{2}\ (\bar{\Delta})^{r-j+1/q-1/p}\omega(F^{(r)}, \bar{\Delta})_p, \tag{3.17}$$

where $\hat{K}_j$ is the constant in (3.11).

The result of Corollary 3.4 may now be compared with upper bounds on approximation of functions in $L_p^r[a,b]$ by splines in the class $\mathscr{S}$ defined in (2.22). Since $\mathscr{S} \subseteq \mathscr{P}U_m(\Delta)$, (3.17) implies

$$d_{j,q}(F, \mathscr{S}) \geqslant \frac{\hat{K}_j}{2}(\bar{\Delta})^{r-j+1/q-1/p}\omega(F^{(r)}, \bar{\Delta})_p. \tag{3.18}$$

For $1 \leqslant p \leqslant q \leqslant \infty$ this compares exactly with the upper bounds established in Theorem 2.3 of [4]. For $1 \leqslant q < p \leqslant \infty$, the upper estimate in Theorem 2.3 of [4] is of order $(\bar{\Delta})^{r-j}$, while the lower bound in (3.18) is of the smaller order $(\bar{\Delta})^{-j+1/q-1/p}$. It would be desirable to show that there is a corresponding lower bound for this second case. The following theorem gives a lower bound of the correct order $r-j$, but in terms of the quantity $\underline{\Delta} = \min_{0\leqslant i\leqslant k}(x_{i+1}-x_i)$ instead of $\bar{\Delta}$.

THEOREM 3.5. *Let $0 \leqslant r \leqslant m$, $0 \leqslant j \leqslant r-1$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant q < \infty$. Then for every partition Δ of $[a,b]$, there exists a function $F \in L_p^r[a,b]$ with*

$$d_{j,q}(F, \mathscr{P}U_m(\Delta)) \geqslant K_j^*(\underline{\Delta})^{r-j}\|F\|_{L_p^r[a,b]} \geqslant \frac{K_j^*}{2}\ (\underline{\Delta})^{r-j}\omega(F^{(r)}; \underline{\Delta})_p, \tag{3.19}$$

where

$$K_j^* = \left(\frac{1}{8(m+1)}\right)^{r-j} \frac{2\left(\dfrac{b-a}{2(m+1)}\right)^{1/q}}{(b-a)^{1/p}2^{(r-j)(r-j-1)/2}[1+(b-a)+\ldots+(b-a)^r]}. \tag{3.20}$$

Proof. Let $\Delta = \{a = x_0 < x_1 < \ldots < x_{k+1} = b\}$ be a partition of $[a, b]$. For every $v = 0, 1, \ldots, k$, let $x_v = t_0^{(v)} \leqslant t_1^{(v)} \leqslant \ldots \leqslant t_{m+1}^{(v)} = x_{v+1}$ and signs $\varepsilon_1^{(v)}, \ldots, \varepsilon_m^{(v)}$ be such that

$$h(t) = \varepsilon_j^{(v)}(t_{j+1}^{(v)} - t_j^{(v)})^{r-j} g_{r-j}\left(\frac{t - t_j^{(v)}}{t_{j+1}^{(v)} - t_j^{(v)}}\right),$$

$$t_j^{(v)} \leqslant t \leqslant t_{j+1}^{(v)}, \quad j = 0, 1, \ldots, m \text{ and } v = 0, 1, \ldots, k,$$

satisfies (3.12). Let F be defined as in (3.13). Now for each v, let j_v be chosen so that $|I_{vj_v}| \geqslant |I_v|/(m+1)$. Let I_v^* denote the middle one-half of the interval I_{vj_v}. Then (cf. (3.15))

$$(3.21) \qquad d_{j,q}(F, \mathscr{P}U_m(\Delta)) \geqslant ||h||_q$$

$$\geqslant 2\left(\frac{\underline{\Delta}}{8(m+1)}\right)^{r-j} \left(\sum_{v=0}^{k} \int_{I_v^*}\right)^{1/q} = 2\left(\frac{\underline{\Delta}}{8(m+1)}\right)^{r-j} \left(\frac{b-a}{2(m+1)}\right)^{1/q}.$$

Moreover,

$$(3.22) \qquad ||F^{(r)}||_p \leqslant 2^{(r-j)(r-j-1)/2}\left(\sum \int_{I_v}\right)^{1/p} = 2^{(r-j)(r-j-1)/2}(b-a)^{1/p}.$$

Thus, $F \in L_p^r[a, b]$, and converting (3.22) to an estimate on the L_p^r norm in the same way as before, we obtain (3.19)–(3.20). ∎

4. Remarks

(1) If we take $U_m = \{x^{i-1}\}_1^m$, we recover known results on piecewise polynomial approximation (cf. [1], [2], [5], [10], [13], [14]). Our approach is, however, completely different from that taken in these papers inasmuch as for general U_m we do not have $p(x+t) \in U_m$ whenever $p(x) \in U_m$ (which, of course, does hold for polynomials). We also note that we have obtained explicit constants which are new even for the polynomial case.

(2) The constants in the theorems of this paper are not best possible. They can be easily improved, for example, by obtaining better estimates for $||g_r||$ in the norms of interest.

(3) Lower bounds for $d_{j,\infty}(F, \mathscr{P}U_m(\Delta))$ similar to those in (2.21), but with $\underline{\Delta}$ instead of $\overline{\Delta}$, were established in [4] using results from the theory of n-widths. Corollary 2.4 is thus a significant improvement as it shows that the upper bounds are asymptotically optimal for all partitions, not just quasiuniform ones.

(4) Corollary 3.4 extends and improves the lower bound in Theorem 1.3 of [4] in two ways. First, we have $\overline{\Delta}$ here instead of $\underline{\Delta}$, and moreover, the result is valid for all $1 \leqslant p \leqslant q \leqslant \infty$ (whereas in [4], the result was established only for $p = q$). Theorem 3.5 provides the associated lower bounds for $1 \leqslant q < p \leqslant \infty$, so that in this case the upper bounds in Theorem 1.3 of [4] are asymptotically sharp for all quasiuniform meshes; i.e. meshes with $\overline{\Delta}/\underline{\Delta} \leqslant$ const. $< \infty$. It remains an open question as to whether a bound of the form (3.19) can be established using $\overline{\Delta}$ instead of $\underline{\Delta}$.

(5) It is quite easy to obtain further corollaries of the results of this paper giving lower bounds in terms of higher-order moduli of continuity. For example, if we define

$$\omega_n(\varphi, h) = \sup_{\substack{x, x+nt \in [a,b] \\ |t| \leqslant h}} |\Delta_t^n \varphi(x)|,$$

then we may use the fact (cf. [6]) that

$$\omega_n(\varphi, h) \leqslant h^n \omega(\varphi^{(n)}, h)$$

to obtain lower bounds involving $\omega_n(F, \overline{\Delta})$. The same can be done for an nth-order modulus of continuity with respect to the p-norm.

(6) In this paper we have concentrated on actual lower bounds. There is also an inverse theory for generalized spline approximation; e.g. see [10], [15].

References

[1] G. Birkhoff, M. H. Schultz, and R. S. Varga, *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations*, Numer. Math. 11 (1968), pp. 232–256.

[2] S. Demko and R. S. Varga, *Extended L_p-error bounds for spline and L-spline interpolation*, J. Approximation Theory 12 (1974), pp. 242–264.

[3] C. R. Hobby and J. R. Rice, *A moment problem in L_1 approximation*, Proc. Amer. Math. Soc. 65 (1965), pp. 665–670.

[4] J. W. Jerome and L. L. Schumaker, *On the distance to a class of generalized splines*, in ISNM 25, Birkauser Verlag, Basel 1974, pp. 503–517.

[5] J. W. Jerome and R. S. Varga, *Generalizations of spline functions and applications to nonlinear boundary value and eigenvalue problems*, in: Theory and Applications of Spline Functions, T.N.E. Greville, ed., Academic Press, N.Y. 1969, pp. 103–155.

[6] H. Johnen, *Inequalities connected with the moduli of smoothness*, Mat. Vesnik 9 (1972), pp. 289–303.

[7] G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston, New York 1966.

[8] T. Lyche and L. L. Schumaker, *Local spline approximation methods*, J. Approximation Theory 15 (1975), pp. 294–325.

[9] A. Pinkus, *A simple proof of the Hobby–Rice theorem*, Proc. Amer. Math. Soc. 60 (1976), pp. 82–84.

[10] H. Johnen and K. Scherer, *Direct and inverse theorems for best approximation by* 1*-splines*, in: Spline functions, Karlsruhe 1975, Lecture Notes 501, Springer-Verlag, pp. 116–131.

[11] M. H. Schultz and R. S. Varga, *L-splines*, Numer. Math. 10 (1967), pp. 345–369.

[12] J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York 1969.

[13] Yu. N. Subbotin, *A certain linear method of approximation of differentiable functions*, Mat. Zametki 7 (1970), pp. 423–430; also translated in Math. Notes. 7 (1970), pp. 256–260.

[14] B. Swartz and R. S. Varga, *Error bounds for spline and L-spline interpolation*, J. Approximation Theory 6 (1972), pp. 6–49.

[15] R. de Vore and F. Richards, *The degree of approximation by Tchebysheffian* splines, Trans. Amer. Math. Soc. 181 (1973), pp. 401–418.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975