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COMPARATIVE APPROXIMATION IN TWO TOPOLOGIES

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1. Introduction

This paper is a preliminary exploration of an approximation problem which can be roughly described as follows. Suppose a linear manifold E of functions spans a function f in some "weak" topology. Must E span f in a "strong" topology? In general the answer is of course no, but we shall show that if E consists of *linear combinations of exponential functions*, then (putting the matter very vaguely, and with gross oversimplification) if E spans f in *some* topology, however weak, it does so in the *strongest* topology which is meaningful for f.

To be more specific, let X and Y denote topological linear spaces, and E a linear manifold in $X \cap Y$. We shall assume that *X is a subset of Y, the X topology being stronger than the Y topology*, although the problem to be studied is meaningful whenever the topologies which X and Y induce in E are non-equivalent. We shall say E is *an (X, Y) manifold* if, whenever $x \in X \cap Y$ lies in the Y-closure of E, x also lies in the X-closure of E.

It is well first to enumerate certain trivial aspects of our problem:

(i) If E is dense in Y but not in X (in their respective topologies), it is not an (X, Y) manifold.

(EXAMPLE: $X = C(I)$, $Y = L^1(I)$, $I = [0, 1]$, and E is the set of polynomials p such that $p(0) = 0$.)

(ii) If E is dense in X, it is (trivially) an (X, Y) manifold.

(iii) If X, Y are Banach spaces, and the X and Y norms are equivalent on E (in particular, if E is finite-dimensional), then E is an (X, Y) manifold.

(EXAMPLE: E consists of all finite linear combinations of Rademacher functions, $X = L^2(I)$, $Y = L^1(I)$.)

In connection with (iii), one might hastily surmise that our problem is essentially that of finding linear manifolds E on which the X and Y topologies are equivalent, but this is by no means the case, as we shall see from examples later.

Because of considerations (i) and (ii), we can hereafter restrict attention to the case where E is not dense in either of the spaces X, Y.

2. An example

To see what sort of concrete problems our general point of view may lead to, let Λ be a compact subset of $\boldsymbol{R}$, and E the set of finite linear combinations of exponential functions $e_\lambda\colon t \to e^{i\lambda t}$, $t \in \boldsymbol{R}$ where $\lambda \in \Lambda$. Let X denote the Banach space of bounded uniformly continuous (complex-valued) functions on $\boldsymbol{R}$, endowed with the usual supremum norm. Let Y denote $L^\infty(\boldsymbol{R})$ endowed with its weak* topology. Let $f \in X$ and suppose f belongs to the Y-closure of E; must it belong to the X-closure of E? We have a bounded uniformly continuous function f on $\boldsymbol{R}$, and the assumption that f is in the Y-closure of E implies that its spectrum (in the sense that is customary when dealing with L^∞, namely the support of its distributional Fourier transform) is contained in Λ. Must f be in the uniform closure of E? Our problem thus takes the form: *if a bounded uniformly continuous function on $\boldsymbol{R}$ has its spectrum in the compact set Λ, must it be Bohr almost-periodic*? The answer is known to be *yes* if Λ is at most countable; if, on the other hand, Λ has a perfect subset, it supports a nondiscrete bounded measure and the answer is *no*.

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Let us fix a subset Λ of integers, and let E be the set of all finite linear combinations of the functions $e_n\colon t \to e^{int}$, $t \in T$ where $n \in \Lambda$. (As usual, T denotes the "circle" $\boldsymbol{R}/2\pi\boldsymbol{Z}$.) To fix ideas, we consider two specific topologies; our "weak" topology is that of $Y = L^1(T)$, our "strong" topology is that of $X = C(T)$, but we shall see from the following argument that the final result is nearly insensitive to how strong the "strong" topology is, and how weak the "weak" topology is. Let us prove:

Let $f \in C(T)$, and suppose f is spanned by E in $L^1(T)$. Then f is spanned by E in $C(T)$; in other words E is a (C, L^1) manifold for every choice of Λ.

Proof. Let μ be a measure on T such that $\int e^{int}\,d\mu(t) = 0$, $n \in \Lambda$. We have to show $\int f\,d\mu = 0$. Now let σ_m denote the Fejér mean of μ of order m. Then clearly $\int e^{int}\sigma_m(t)\,dt = 0$ for $n \in \Lambda$ and so, since $\sigma_m \in L^\infty(T)$ and we have assumed that f is in the L^1 closure of E, we have $\int \sigma_m(t) f(t)\,dt = 0$. But $\sigma_m dt$ tends to μ in the weak* topology of $M(T)$; hence $\int f\,d\mu = 0$, as was to be shown.

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Now we wish to look at something a little more interesting, namely the case where our functions are defined on $\boldsymbol{R}^+$ and E is spanned by exponentials $e^{-\lambda t}$, $\lambda \in \Lambda$ where Λ is some subset of the right half plane. Actually, to make certain technical matters a little simpler, we prefer to have our exponential functions defined on *the set $\boldsymbol{N}$ of non-negative integers* rather than $\boldsymbol{R}^+$, that is, we fix a subset Λ of the open unit disc and let E be the set of finite linear combinations of the vectors

$$(1, \lambda, \lambda^2, \ldots), \quad \lambda \in \Lambda.$$

Moreover, it is technically more convenient to work not with these vectors directly, but with their Laplace transforms

$$k_\lambda(z) = \sum_{n=0}^{\infty} \lambda^n z^n = \frac{1}{1-\lambda z} \tag{4.1}$$

considered as analytic functions on the open unit disc U. (Of course, some norms which seem "natural" for functions on $\boldsymbol{N}$ (such as l^∞) will seem "sophisticated" as applied to their Laplace transforms, and vice versa, but from a logical point of view the problems are the same.)

Our first choice of topologies will be this: Y is the Hardy space $H^1(U)$, and X the Hardy space $H^p(U)$, where $1 < p < \infty$.

Proposition. *Let Λ be any subset of U, and $E = E_\Lambda$ the set of all functions k_λ defined by* (4.1), *where $\lambda \in \Lambda$. Then E is an (H^p, H^1) manifold for every p, $1 < p < \infty$.*

Proof. Assume that $f \in H^p$ is in the H^1 closure of E. We must show it is in the H^p closure as well. Without loss of generality we may assume that $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is countable and satisfies the Blaschke condition

$$\sum (1-|\lambda_n|) < \infty, \tag{4.2}$$

since otherwise E is dense in H^p, in H^p norm, and the problem trivializes. Now, let $\varphi \in L^q(T)$ where $q = p/(p-1)$ and suppose

$$\int_0^{2\pi} \frac{\varphi(e^{i\theta})}{1-\lambda e^{i\theta}}\,d\theta = 0, \quad \lambda \in \Lambda. \tag{4.3}$$

If we can show

$$\int_0^{2\pi} \varphi(e^{i\theta}) f(e^{i\theta})\,d\theta = 0, \tag{4.4}$$

we shall have proved the proposition by duality. Let

$$g(w) = \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{1-we^{i\theta}} \cdot e^{i\theta}\,d\theta.$$

Then by virtue of a theorem of M. Riesz, $g \in H^q$. Moreover, $g(\lambda) = 0$ for $\lambda \in \Lambda$. Thus, $g = \mathrm{B}h$ where B is the Blaschke product formed with zeroes $\{\lambda_n\}$ (convergent because of (4.2)) and $h \in H^q$. To establish (4.4) we must show

$$\int_0^{2\pi} \mathrm{B}(e^{-i\theta}) h(e^{-i\theta}) f(e^{i\theta})\,d\theta. \tag{4.5}$$

Now, we see that (4.4) holds for any $\varphi \in L^\infty(T)$ which satisfies (4.3), and in particular this is the case if φ has the form $\varphi(e^{-i\theta}) = \mathrm{B}(e^{-i\theta})\Psi(e^{-i\theta})$, where $\Psi \in H^\infty(T)$. Hence the proof will be complete if we can make the H^q norm of $\mathrm{B}h - \mathrm{B}\Psi$ as small as desired. But this is certainly possible since H^∞ is dense in H^q, and the proof is finished.

Remarks. If, in the capacity of X in the above problem we had taken $C(T)$ rather than H^p (or, what comes to the same, $X = H^\infty \cap C$, the "disc algebra"), we would have an apparently more difficult problem and one which thus far I cannot solve. A similar difficulty arises if we take X to be L^∞ (or equivalently, H^∞) endowed with the weak* topology of $L^\infty(T)$. On the other hand, for another X "near" to H^∞, namely the set of functions analytic in U with a finite Dirichlet integral, we shall give a positive solution to the problem in the following section.

Another example where a seemingly more difficult problem arises is if, in the above analysis, we try to replace H^1 by a weaker topology, e.g. the "Bergman space" $L_a^2(U)$ of square integrable analytic functions on U (more discussion about this space will be given in the next chapter). Then the proof we gave for the above proposition breaks down because, with the pairing we have used between spaces and their duals, an element of L^∞, even one of the special form $\mathrm{B}\Psi$ used above, does not define a bounded linear functional on $L_a^2(U)$. Indeed, the proposition is false as it stands, since even when condition (4.2) holds, it may happen that E_Λ is dense in $L_a^2(U)$, or, what is equivalent, that no non-null analytic function with a finite Dirichlet integral can vanish at all the points $\{\lambda_n\}$. Thus, we have first of all to modify (4.2) to read: *E_Λ is not dense in $L_a^2(U)$*. Whether the proposition thus modified is true, I do not know. Another similar problem is to replace H^1 by H^p with $p < 1$.

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Let us continue along the lines of the preceding section. Suppose we took for X a topology stronger than all the H^p topologies with $p < \infty$, for example the D norm (Dirichlet integral) where

$$\|g\|_{\mathrm{D}}^2 = \sum_{n=1}^{\infty} n|b_n|^2 \quad \Big(g = \sum_{0}^{\infty} b_n z^n\Big)$$

or an even much stronger topology (H^2 norm of the derivative):

$$\|g'\|_2^2 = \sum_{n=1}^{\infty} n^2|b_n|^2.$$

In both of these cases (keeping H^1 as our "weak", or Y topology) one can show that our problem has a positive answer, that is, *E_Λ is an (X, Y) manifold if Λ satisfies* (4.2).

Let us consider, for example, the D norm, or (what is essentially the same, but a bit more convenient for our purposes) the D′ norm, whose square is $\sum_{n=0}^{\infty} (n+1)|b_n|^2$. The dual space of D′ is identified, in a natural way, with the space $L_a^2(U)$ of square-summable analytic functions on U. (That is, the bilinear form which expresses the duality between $f = \sum_0^\infty a_n z^n \in L_a^2(U)$ and $g = \sum_0^\infty b_n z^n \in \mathrm{D}'$ is $(f, g) = \sum_0^\infty a_n b_n$.) Repeating the reasoning in Section 4, we see that to prove our assertion we need only to establish:

PROPOSITION. *If $g \in L_a^2(U)$ vanishes on the zeroes of a given Blaschke product $\mathrm{B}(z)$ (i.e. g/B is analytic in U), then, for every $\varepsilon > 0$, we can find $\Psi \in H^\infty(U)$ such that $\Psi\mathrm{B}-g$ has $L^2(U)$-norm less than ε.*

Remark. In a paper in Mat. Sbornik 73 (1967) I claimed (in the remark following Theorem 7) to be in possession of a counterexample to the above proposition. That claim, it now appears, was mistaken. The proof of the above proposition was pointed out to me by Lennart Carleson. This proof is, in fact, valid in considerable generality and thus solves a large class of instances of our basic approximation problem, which we do not bother to formulate explicitly.

Let us consider a Banach space S of analytic functions in U satisfying the following conditions:

(a) Polynomials are dense in S.

(b) For each $\varphi \in H^\infty$, multiplication by φ is a bounded operator on S.

(c) If $f \in S$, $\lambda \in U$, and $f(\lambda) = 0$, the function $\dfrac{1-\bar{\lambda}z}{z-\lambda} \cdot f(z)$ is in S.

(d) (Dominated convergence). If $f_n \in S$, $f_n(z) \to 0$ for each $z \in U$ and moreover, for some $g \in S$,

$$|f_n(z)| \leqslant g(z), \quad \text{all } z \in U \text{ and all } n,$$

then $\|f_n\|_S \to 0$.

Obviously, if μ is any positive bounded measure on U, the space of all functions analytic in U such that $\int |f|^p d\mu < \infty$ satisfies the preceding conditions. In particular, this is the case for $L_a^2(U)$.

PROPOSITION. *Let S be a Banach space of analytic functions in U such that* (a), (b), (c), (d) *above hold. Suppose B is a Blaschke product, $g \in S$ and g/B is holomorphic in U. Then there exist polynomials P_n such that*

$$\lim_{n\to\infty} \|P_n\mathrm{B}-g\|_S = 0.$$

Proof. Write $\mathrm{B} = \mathrm{B}_m \boldsymbol{R}_m$ where B_m is the partial product consisting of the first m factors of B. Because of (c), g/B_m is in S for every m. Now, given $\varepsilon > 0$, we can (by virtue of (d)) choose m so large that $\|\boldsymbol{R}_m g - g\| < \varepsilon/2$, since $\boldsymbol{R}_m(z)g(z)-g(z)$ tends to zero for each z and is majorized by $2|g(z)|$. Because of (a), we can now find a polynomial P such that

$$\|P-(g/\mathrm{B}_m)\| < \varepsilon/2M,$$

where M is the norm of the operator $f \to \mathrm{B}f$. Hence, because of (b)

$$\|P\mathrm{B}-\boldsymbol{R}_m g\| < \varepsilon/2.$$

Thus, $\|P\mathrm{B}-g\| < \varepsilon$, and the proposition is proved.

6. Concluding remarks

6.1. A variant of the problems in Sections 4 and 5 arises if for the "strong" topology we choose a space X for which the failure of condition (4.2) does not necessarily force E_Λ to be dense in X, in other words sets Λ with $\sum (1-|\lambda_n|) = \infty$ must be taken into account. Then the use of Blaschke products is no longer possible, and we are led to what appear to be new and difficult problems. If, for instance, we choose $Y = \mathrm{D}$ and for X some space with a stronger topology, we fall into this situation.

6.2. Thus far, I know no example where E_Λ *fails* to be an (X, Y) manifold, except in the trivial way that it spans Y but not X.

6.3. One should observe that a number of interesting topological spaces of analytic functions in U (for instance, H^p with $p < 1$) contain k_λ even for $|\lambda| = 1$, so for these spaces the natural condition on Λ is that it is a subset of the *closed* unit disc.

6.4. To summarize somewhat the situation after Sections 4 and 5, we may put matters as follows. Be means of a duality argument, we arrive at two topological vector spaces (say, for simplicity, Banach spaces) X', Y' of analytic functions in U.

We are given moreover a subset Λ of U such that both X' and Y' contain non-null functions which vanish on Λ. Let $g \in Y'$, $g(z) = 0$ for all $z \in \Lambda$. Is there then some function $h \in X'$ which vanishes on Λ and is within ε of g, with respect to the Y' norm? This problem is of interest in its own right, and could itself be taken as the starting point of the investigation. The essence of what was done in Sections 4 and 5 is that when Λ satisfies (4.2), and X' contains all Blaschke products, we obtain a positive answer, insofar as the closure of the polynomial multiples of a given Blaschke product in a vast array of spaces Y' contains *all* functions in Y' which vanish on Λ. Blaschke products are of course specially attuned to H^p spaces and our problem is, in a sense, to find what kind of functions may replace them in other situations. Observe that, for instance, in the proof of the Proposition of Section 4, it was not essential that B was a Blaschke product. It would have been enough to take for B any *outer* function vanishing on $\{\lambda_n\}$. However, if we had chosen some B with a *singular* inner factor, the proof would not work, insofar as polynomial (or H^∞) multiples of B would then *not* span all functions in H^q vanishing on $\{\lambda_n\}$. When dealing with a space Y' where the usual notion of "inner-outer factorization" is not applicable, we have to search for something to replace it, and this in my opinion is what gives theoretical interest to the present problem.

6.5. Of course, these considerations apply also to (say) spaces of entire functions. Looking only at the dual problem, we have (recapitulating) the following situation. Let X', Y' be two t.v. spaces of entire functions; say for simplicity $Y' \subset X'$, and let Λ be a set of multiplicity for Y', i.e. the set $Y'(\Lambda)$ of non-null functions vanishing on Λ is not empty. Our problem is, does the X' closure of $Y'(\Lambda)$ contain *all* functions in X' which vanish on Λ?

A closely related problem was studied by Newman and Shapiro [1], [2] in the concrete situation where X' was the "Fischer space" F of all entire functions g such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x+iy)|^2 e^{-(x^2+y^2)}\,dx\,dy < \infty .$$

Indeed, the principal problem in [1], [2], was, *do the polynomial multiples of an entire function φ span (in F) all functions which vanish wherever φ vanishes*? (Here we assume, of course, that φ is such that its polynomial multiples lie in F, which is certainly the case if φ has order less than 2.) Thus far, no example of a function φ is known for which this problem has a negative answer. On the other hand, we cannot assert (on the basis of what is known) that for *every* φ of exponential type (say), or in any other "typical" space of entire functions, the problem has an affirmative answer. In the present paper we are asking for somewhat less, i.e. a problem of the sort: *does the set of all entire functions of exponential type which vanish on Λ span (in F) every function vanishing on Λ*?

Comparing these two problems, another problem naturally suggests itself, which in its general formulation reads: *is the X'-closure of $Y'(\Lambda)$ identical with the X'-closure of the set of all polynomial multiples of some single function in $Y'(\Lambda)$*? Usually, this problem is non-trivial even when $X' = Y'$. Then it takes the form: *is there a "canonical" function associated with the set (of multiplicity) Λ, such that the closure of its polynomial multiples consists of all functions vanishing on Λ*? For H^p spaces we have a positive answer, thanks to the theory of Blaschke products.

6.6. We add two more examples which may serve to show the wide range of problems concerning "(X, Y) manifolds", and further demonstrate its fundamental character.

First, the "spectral synthesis" problem of harmonic analysis: here $X = A(T)$, $Y = C(T)$ and E is an (algebraic) ideal of functions in $A(T)$. Then, $f \in A(T)$ is in the Y-closure of E if and only if it vanishes on the compact set $K(E)$, the set of common zeroes of all functions in E. Thus, E is an (X, Y) manifold if and only if $K(E)$ is a set of synthesis.

This can also be put in a slightly different way: with X, Y as before, let K be some compact set, and E the set of functions in $A(T)$ which vanish on an open *neighbourhood* of K. Then the Y-closure of E is the set of all continuous functions vanishing on K, and so E is an (X, Y) manifold if and only if K is a set of synthesis.

Second, it is amusing to observe (Y. Katznelson called my attention to this point) that our problem has an affirmative answer when X is a Banach space (with its strong topology) and Y the same space, endowed with its weak topology (no matter what the linear manifold E is like). This is a corollary of the Hahn-Banach theorem.

6.7. As remarked in the Introduction, the present paper is to be understood in the spirit of a preliminary sketch, the first goal of which ought to be a correct formulation of the underlying problems in the right degree of generality. For this purpose, observe that the example in Section 3 can be formulated without mention of exponential functions. We can, instead, postulate E to be any translation-invariant linear manifold of functions in $C(T)$. The above argument establishes that the closure of E in $C(T)$ equals the intersection with $C(T)$ of the closure of E in $L^1(T)$.

Thus the essential feature of E is the invariance of its elements under rotations of T. In like manner, the example in Section 4 can be generalized, replacing E by any linear manifold in H^p invariant with respect to the backward shift operator, that is the operator

$$V\colon (a_0, a_1, a_2, \dots) \to (a_1, a_2, \dots)$$

acting on the Taylor coefficients.

Thus, we might hope for a theorem of some generality in which X and Y are topological spaces of functions defined on some semigroup S, and $E \subset X$ is assumed invariant with respect to some transformation(s) of S. However, I could not so far even extend the example in Section 5 in this spirit. The problem here is: *Suppose E is a linear manifold in (say) $H^2(U)$, $VE \subset E$, and E is not dense in $L_a^2(U)$. Is E an $(H^2(U), L_a^2(U))$ manifold?* (That is: does the L_a^2 closure of E, intersected with H^2, equal the H^2 closure of E?) I would guess the answer is affirmative. This is a crucial test problem for ascertaining whether or not a theorem of some generality ultimately can be hoped for.

References

[1] D. J. Newman and H. S. Shapiro, *A Hilbert space of entire functions related to the operational calculus*, (mimeographed notes) Ann Arbor 1964.

[2] —, —, *Fischer spaces of entire functions*, Proc. Symp. Pure Math., Amer. Math. Soc. Transl. 11 (1968), pp. 360–369.

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A COUNTEREXAMPLE IN HARMONIC ANALYSIS

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Let B be a commutative Banach algebra with identity, and suppose $x \in B$ has norm 1 and satisfies

$$|\hat{x}(m)| \geqslant \delta > 0, \quad \text{all } m \in M \tag{1}$$

where $\hat{x}$ denotes the Gelfand transform of x, and M the maximal ideal space of B. Then x is invertible; Gunnar Ehrling has raised the question, in connection with a problem arising in theoretical physics, whether there exists a constant $N(B; \delta)$ depending only on B and δ such that $\|x^{-1}\| \leqslant N(B; \delta)$ for all $x \in B$ satisfying (1).

In this note we show that the answer is *no* in the case where B is the algebra of absolutely convergent Taylor series. More precisely: let A denote the Banach algebra of functions f analytic on the open unit disc U whose series of Taylor coefficients is absolutely convergent:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \|f\| = \sum_{n=0}^{\infty} |a_n| < \infty$$

with multiplication in A defined as the usual pointwise multiplication of functions of U. The maximal ideal space of A is the closure U^- of U. We shall denote $\max_{z \in U^-} |f(z)|$ by $\|f\|_\infty$. Our result is, then:

THEOREM. *There exists a sequence $\{f_n\}_1^\infty \subset A$ and a positive absolute constant δ such that*

(i) $\|f_n\| = 1$, $n = 1, 2, \dots$,

(ii) $|f_n(z)| \geqslant \delta$, $n = 1, 2, \dots$; $z \in U^-$,

(iii) $\lim_{n\to\infty} \|f_n^{-1}\| = \infty$.

It will be convenient to precede the proof by some lemmas.

LEMMA 1. *For $f \in A$ we have*

$$\|f\| \leqslant |f(0)| + \frac{1}{2} \int_0^{2\pi} |f'(e^{i\theta})| d\theta. \tag{2}$$

Proof. This is a classical inequality of Hardy and Littlewood. (See [2], vol. I, p. 286, Theorem (8.7).)