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INDIRECT APPROXIMATION THEOREMS IN L^p-METRICS $(1 < p < \infty)$

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1. Preliminaries

Let L^p $(1 \leqslant p < \infty)$ be the space of all 2π-periodic real-valued functions Lebesgue-integrable with pth power over the interval $\langle -\pi, \pi\rangle$; the norm of $f \in L^p$ is defined by the formula

$$\|f(\cdot)\|_{L^p} = \Big\{\int_{-\pi}^{\pi} |f(x)|^p dx\Big\}^{1/p}.$$

Consider functions f belonging to L^p $(1 \leqslant p < \infty)$. Suppose that x, h are real, and set

$$\Delta_h^k f(x) = \sum_{\nu=0}^{\infty} (-1)\binom{k}{\nu} f(x+(k-\nu)h),$$

$$\omega_k(\delta; f)_{L^p} = \sup_{|h|\leqslant\delta} \|\Delta_h^k f(\cdot)\|_{L^p},$$

for arbitrary positive k. These quantities are called the *k-th difference* of f at the point x, and the *k-th modulus of smoothness* of this function in L^p-metrics, respectively (see [5]). Denote by $E_n(f)_{L^p}$ the constants of the best approximation of f by trigonometric polynomials

$$T_n(x) = \sum_{\nu=-n}^{n} d_\nu e^{i\nu x},$$

that is

$$E_n(f)_{L^p} = \inf_{T_n} \|f(\cdot)-T_n(\cdot)\|_{L^p}.$$

Write

$$E_n^\varrho(f)_{L^p} = (E_n(f)_{L^p})^\varrho \quad \text{for positive } \varrho.$$

Suppose now that

$$S[f] = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x} = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1}$$

is the Fourier series of $f\in L^p$ $(1\leqslant p<\infty)$. Put

$$f_\alpha(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-u)\Psi_\alpha(u)\,du\quad\text{for}\quad\alpha>0,$$

where

$$\Psi_\alpha(u)=\sum_{\nu=-\infty}^{\infty}{}'(i\nu)^{-\alpha}e^{i\nu u},\qquad (i\nu)^{-\alpha}=|\nu|^{-\alpha}\exp\Big(-\frac{1}{2}\pi i\alpha\operatorname{sign}\nu\Big);$$

the dash $'$ indicates that the term with $\nu=0$ is omitted in summation. As is well known ([9], I, pp. 36, 42, 69, 90–94; II, p. 134), the convolution $f_\alpha(x)$ is Lebesgue-integrable over $\langle-\pi,\pi\rangle$ and

$$S[f_\alpha]=\sum_{\nu=-\infty}^{\infty}{}'(i\nu)^{-\alpha}c_\nu e^{i\nu x}.$$

The last series converges, eventually, for almost every x; its sum

$$I_\alpha(x)=I_\alpha(x;f)$$

coincides with $f_\alpha(x)$ almost everywhere.

Given any $\alpha\in(0,1)$, we define the derivative $f^{(\alpha)}(x)$ of $f\in L^p$ $(1\leqslant p<\infty)$ by the formula

$$f^{(\alpha)}(x)=\frac{d}{dx}I_{1-\alpha}(x;f),$$

provided the right-hand side exists. We set

$$f^{(\alpha+r)}(x)=\big(f^{(\alpha)}(x)\big)^{(r)}=\frac{d^{r+1}}{dx^{r+1}}I_{1-\alpha}(x;f)\quad\text{if}\quad r=1,2,\ldots$$

For nth partial sums of the series (1) and its conjugate

$$\tilde S[f]=\sum_{\nu=0}^{\infty}(a_\nu\sin\nu x-b_\nu\cos\nu x),$$

the symbols $S_n(x;f)$ and $\tilde S_n(x;f)$ will be used, respectively. The function conjugate to f is defined as usual:

$$\tilde f(x)=-\lim_{\varepsilon\to0+}\frac{1}{2\pi}\int_{\varepsilon}^{\pi}\{f(x+t)-f(x-t)\}\cot\frac{1}{2}t\,dt.$$

In this paper we extend to positive k and r two indirect approximation theorems of M. F. Timan, given in [8], p. 126. We denote by $A_{\alpha,\beta,\ldots}$, $B_{\alpha,\beta,\ldots}$ etc., the suitable positive constants depending only on the parameters α, β, ... shown explicitly. As in [5], $C(k)$ will mean the sum of the series

$$\sum_{\nu=0}^{\infty}\left|\binom{k}{\nu}\right|\quad(k>0).$$

2. Fundamental lemmas

In the sequel the following three auxiliary results will be needed.

LEMMA 1. *Let γ_k, $\bar\gamma_k$ be the conjugate complex numbers, and let*

$$\gamma_0=0,\qquad\gamma_{-k}=\bar\gamma_k\quad(k=1,2,\ldots).$$

Write

$$\nabla_\mu(x)=\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\gamma_\nu e^{i\nu x},\qquad\nabla_{-\mu}(x)=\sum_{\nu=-2^\mu+1}^{-2^{\mu-1}}\gamma_\nu e^{i\nu x}\quad(\mu=1,2,\ldots).$$

Then, if $1<p<\infty$ and $n=1,2,\ldots$, we have

$$\int_{-\pi}^{\pi}\Big|\sum_{\nu=-2^n+1}^{2^n-1}\gamma_\nu e^{i\nu x}\Big|^p dx\leqslant A_p\int_{-\pi}^{\pi}\Big\{\sum_{\mu=1}^{n}|\nabla_\mu(x)+\nabla_{-\mu}(x)|^2\Big\}^{p/2}dx.$$

Proof. By Lemma 6 of [2], p. 351, there is a system of numbers $\eta_m=\pm1$ $(m=1,2,\ldots)$ such that

$$\Big|\sum_{m=1}^{n}\eta_m(\nabla_m(x)+\nabla_{-m}(x))\Big|^p\leqslant B_p\Big\{\sum_{m=1}^{n}|\nabla_m(x)+\nabla_{-m}(x)|^2\Big\}^{p/2}\quad(n=1,2,\ldots).$$

In view of the second inequality (1) of [1], p. 86,

$$\int_{-\pi}^{\pi}\Big|\sum_{\nu=-2^n+1}^{2^n-1}\gamma_\nu e^{i\nu x}\Big|^p dx=\int_{-\pi}^{\pi}\Big|\sum_{\mu=-n}^{n}\nabla_\mu(x)\Big|^p dx$$
$$\leqslant G_p\int_{-\pi}^{\pi}\Big|\sum_{m=1}^{n}\eta_m(\nabla_m(x)+\nabla_{-m}(x))\Big|^p dx.$$

Collecting these results, we get the desired estimate.

LEMMA 2. *Suppose that $q\geqslant0$, $k>0$ and $0<h\leqslant\pi/2^\mu$, where μ is a positive integer. Then,*

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1}|\nu^q\sin^k\nu h-(\nu+1)^q\sin^k(\nu+1)h|\leqslant B_{q,k}\cdot2^{\mu(q+k)}h^k.$$

Proof. The left-hand side of the last inequality does not exceed

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1}(\nu+1)^q|\sin^k\nu h-\sin^k(\nu+1)h|+\sum_{\nu=2^{\mu-1}}^{2^\mu-1}|\{\nu^q-(\nu+1)^q\}\sin^k\nu h|$$
$$\leqslant2^{\mu q}\sum_{\nu=2^{\mu-1}}^{2^\mu-1}|\sin^k\nu h-\sin^k(\nu+1)h|+\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\{(\nu+1)^q-\nu^q\}\nu^k h^k.$$

Further,

$$|\sin^k\nu h-\sin^k(\nu+1)h|\leqslant\begin{cases}k(\nu+1)^{k-1}h^{k-1}|\sin\nu h-\sin(\nu+1)h| & \text{if}\quad k\geqslant1,\\ |\sin\nu h-\sin(\nu+1)h|^k & \text{if}\quad 0\leqslant k<1\end{cases}$$

and

$$(\nu+1)^q-\nu^q \leqslant \begin{cases} q(\nu+1)^{q-1} & \text{if} \quad q \geqslant 1, \\ q\nu^{q-1} & \text{if} \quad 0 \leqslant q < 1. \end{cases}$$

Considering the cases

(a) $r \geqslant 1,\ k \geqslant 1$ (b) $0 \leqslant r < 1,\ k \geqslant 1$,

(c) $r \geqslant 1,\ 0 < k < 1$, (d) $0 \leqslant r < 1,\ 0 < k < 1$,

we conclude our argumentation.

LEMMA 3. *Let $f, f_n \in L^p$ $(1 \leqslant p < \infty,\ n = 0, 1, 2, \ldots)$. Suppose that, for a certain $\alpha \in (0, 1)$, the functions*

$$g_n(x) = I_{1-\alpha}(x; f_n) \qquad (n = 0, 1, 2, \ldots)$$

are absolutely continuous in $\langle -\pi, \pi \rangle$ and that their derivatives

$$g_n'(x) = f_n^{(\alpha)}(x) \qquad (n = 0, 1, 2, \ldots)$$

belong to L^p. Suppose further that

$$\|f_n(\cdot)-f(\cdot)\|_{L^p} \to 0, \quad \|g_n'(\cdot)-\varphi(\cdot)\|_{L^p} \to 0 \quad \text{as} \quad n \to \infty,$$

where φ is of class L^p. Then the function

$$g(x) = I_{1-\alpha}(x; f) \tag{2}$$

is equivalent to a 2π-periodic function h absolutely continuous in $\langle -\pi, \pi \rangle$, such that

$$h'(x) = \varphi(x) \text{ a.e.}$$

The proof of this lemma can be found in [6], § 2 (see also [3], pp. 532–534).

3. Main results

Passing to indirect approximation theorems, let us suppose that $k > 0$ and put

$$\varrho = p \quad \text{if} \quad 1 < p \leqslant 2, \qquad \varrho = 2 \quad \text{if} \quad 2 < p < \infty.$$

THEOREM 1. *Consider an arbitrary $f \in L^p$ $(1 < p < \infty)$. Then, for $n = 0, 1, 2, \ldots$,*

$$\omega_k\left(\frac{\pi}{n+1}; f\right)_{L^p} \leqslant \frac{M_{p,k}}{(n+1)^k}\left\{\sum_{\nu=0}^{n} (\nu+1)^{\varrho k-1} E_\nu^\varrho(f)_{L^p}\right\}^{1/\varrho}.$$

Proof. Evidently, we may suppose that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Then, the Fourier series of f is of the form

$$S[f] = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x}, \quad \text{with} \quad c_0 = 0.$$

Let σ be a positive integer such that

$$2^{\sigma-1} \leqslant 2n+1 < 2^\sigma. \tag{3}$$

Write

$$S_m(x) = S_m(x; f) = \sum_{\nu=-m}^{m} c_\nu e^{i\nu x}.$$

By Minkowski's inequality,

$$\|\Delta_h^k f(\cdot)\|_{L^p} \leqslant \|\Delta_h^k(f(\cdot)-S_{2^\sigma-1}(\cdot))\|_{L^p} + \|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p}.$$

Applying the well-known estimate

$$\left\{\int_{-\pi}^{\pi} |f(x)-S_m(x)|^p\,dx\right\}^{1/p} \leqslant M_p E_m(f)_{L^p}, \tag{4}$$

we obtain

$$\begin{aligned} \|\Delta_h^k(f(\cdot)-S_{2^\sigma-1}(\cdot))\|_{L^p} &\leqslant C(k)\|f(\cdot)-S_{2^\sigma-1}(\cdot)\|_{L^p} \\ &\leqslant C(k) M_p E_{2^\sigma-1}(f)_{L^p} \leqslant C(k) M_p E_{2n+1}(f)_{L^p}. \end{aligned}$$

Hence

$$\|\Delta_h^k f(\cdot)\|_{L^p} \leqslant C(k) M_p E_{2n+1}(f)_{L^p} + \|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p}. \tag{5}$$

It is easily seen that

$$\Delta_h^k S_{2^\sigma-1}\left(x-\frac{k}{2}h\right) = \sum_{\nu=-2^\sigma+1}^{2^\sigma-1} \left(2i\sin\nu\frac{h}{2}\right)^k c_\nu e^{i\nu x},$$

whenever $0 < h < 2\pi/(2^\sigma-1)$ (see [5], § 2). Putting

$$\nabla_\mu(x) = \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \left(2i\sin\nu\frac{h}{2}\right)^k c_\nu e^{i\nu x} \qquad (\mu = 1, 2, \ldots),$$

$$\nabla_{-\mu}(x) = \sum_{\nu=-2^\mu+1}^{-2^{\mu-1}} \left(2i\sin\nu\frac{h}{2}\right)^k c_\nu e^{i\nu x} \qquad (\mu = 1, 2, \ldots)$$

and applying Lemma 1, we obtain

$$\int_{-\pi}^{\pi} \left|\Delta_h^k S_{2^\sigma-1}\left(x-\frac{k}{2}h\right)\right|^p dx \leqslant A_p \int_{-\pi}^{\pi} \left(\sum_{\mu=1}^{\sigma} |\nabla_\mu(x)+\nabla_{-\mu}(x)|^2\right)^{p/2} dx.$$

Consequently,

$$\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant A_p^{1/p}\left\{\int_{-\pi}^{\pi} \left(\sum_{\mu=1}^{\sigma} |\nabla_\mu(x)+\nabla_{-\mu}(x)|^2\right)^{p/2} dx\right\}^{1/p}.$$

If $1 < p \leqslant 2$, the last inequality leads to

$$\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant A_p^{1/p}\Big\{\int_{-\pi}^{\pi}\sum_{\mu=1}^{\sigma}|\nabla_\mu(x)+\nabla_{-\mu}(x)|^p\,dx\Big\}^{1/p}$$

$$= A_p^{1/p}\Big\{\sum_{\mu=1}^{\sigma}\|\nabla_\mu(x)+\nabla_{-\mu}(x)\|_{L^p}^p\Big\}^{1/p}.$$

If $2 < p < \infty$, the Minkowski inequality gives

$$\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant A_p^{1/p}\Big\{\sum_{\mu=1}^{\sigma}\Big(\int_{-\pi}^{\pi}|\nabla_\mu(x)+\nabla_{-\mu}(x)|^p\,dx\Big)^{2/p}\Big\}^{1/2}$$

$$= A_p^{1/p}\Big\{\sum_{\mu=1}^{\sigma}\|\nabla_\mu(\cdot)+\nabla_{-\mu}(\cdot)\|_{L^p}^2\Big\}^{1/2}.$$

Since

$$\nabla_\mu(x)+\nabla_{-\mu}(x) = \sum_{\nu=2^{\mu-1}}^{2^\mu-1}\Big(2\sin\nu\frac{h}{2}\Big)^k\big(c_\nu e^{i(\nu x+k\pi/2)}+\bar{c}_\nu e^{-i(\nu x+k\pi/2)}\big)$$

$$= \sum_{\nu=2^{\mu-1}}^{2^\mu-1}\Big(2\sin\nu\frac{h}{2}\Big)^k\cdot 2\,\mathrm{Re}\big(c_\nu e^{i(\nu x+k\pi/2)}\big),$$

we have

$$2^{-k}\|\nabla_\mu(\cdot)+\nabla_{-\mu}(\cdot)\|_{L^p} = \Big\{\int_{-\pi}^{\pi}\Big|\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\Big(\sin\nu\frac{h}{2}\Big)^k U_\nu(x)\Big|^p dx\Big\}^{1/p},$$

where

$$U_\nu(x) = 2\,\mathrm{Re}\big(c_\nu e^{i\nu x}\cdot e^{ik\pi/2}\big).$$

By the Abel transformation,

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\Big(\sin\nu\frac{h}{2}\Big)^k U_\nu(x)$$

$$= \sum_{\nu=2^{\mu-1}}^{2^\mu-2}\Big\{\sin^k\nu\frac{h}{2}-\sin^k(\nu+1)\frac{h}{2}\Big\}\sum_{l=2^{\mu-1}}^{\nu}U_l(x)+\sin^k(2^\mu-1)\frac{h}{2}\cdot\sum_{l=2^{\mu-1}}^{2^\mu-1}U_l(x).$$

Therefore

$$2^{-k}\|\nabla_\mu(\cdot)+\nabla_{-\mu}(\cdot)\|_{L^p} \leqslant \sum_{\nu=2^{\mu-1}}^{2^\mu-2}\Big|\sin^k\nu\frac{h}{2}-\sin^k(\nu+1)\frac{h}{2}\Big|\Big\{\int_{-\pi}^{\pi}\Big|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\Big|^p dx\Big\}^{1/p}+$$

$$+\Big|\sin^k(2^\mu-1)\frac{h}{2}\Big|\cdot\Big\{\int_{-\pi}^{\pi}\Big|\sum_{l=2^{\mu-1}}^{2^\mu-1}U_l(x)\Big|^p dx\Big\}^{1/p}.$$

In the case $2^{\mu-1} \leqslant \nu \leqslant 2^\mu-1$ $(\mu = 1, 2, \ldots)$,

$$\Big\{\int_{-\pi}^{\pi}\Big|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\Big|^p dx\Big\}^{1/p}$$

$$= 2\Big\{\int_{-\pi}^{\pi}\Big|\mathrm{Re}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\cos k\frac{\pi}{2}-\mathrm{Im}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\sin k\frac{\pi}{2}\Big|^p dx\Big\}^{1/p}$$

$$\leqslant 2\Big\{\int_{-\pi}^{\pi}\Big|\mathrm{Re}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\Big|^p dx\Big\}^{1/p}+2\Big\{\int_{-\pi}^{\pi}\Big|\mathrm{Im}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\Big|^p dx\Big\}^{1/p}.$$

Further,

$$2\,\mathrm{Re}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx} = \sum_{l=2^{\mu-1}}^{\nu}(a_l\cos lx+b_l\sin lx) = S_\nu(x;f)-S_{2^{\mu-1}-1}(x;f),$$

$$2\,\mathrm{Im}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx} = \sum_{l=2^{\mu-1}}^{\nu}(a_l\sin lx-b_l\cos lx) = \tilde{S}_\nu(x;f)-\tilde{S}_{2^{\mu-1}-1}(x;f).$$

Consequently,

$$\Big\{\int_{-\pi}^{\pi}\Big|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\Big|^p dx\Big\}^{1/p} \leqslant \Big\{\int_{-\pi}^{\pi}\big|S_\nu(x;f)-S_{2^{\mu-1}-1}(x;f)\big|^p dx\Big\}^{1/p}+$$

$$+\Big\{\int_{-\pi}^{\pi}\big|\tilde{S}_\nu(x;f)-\tilde{S}_{2^{\mu-1}-1}(x;f)\big|^p dx\Big\}^{1/p} = G+H.$$

By inequality (4) and the well-known theorem of M. Riesz ([9], p. 253),

$$G \leqslant \Big\{\int_{-\pi}^{\pi}\big|S_\nu(x;f)-f(x)\big|^p dx\Big\}^{1/p}+\Big\{\int_{-\pi}^{\pi}|S_{2^{\mu-1}-1}(x;f)-f(x)|^p dx\Big\}^{1/p}$$

$$\leqslant M_p E_\nu(f)_{L^p}+M_p E_{2^{\mu-1}-1}(f)_{L^p} \leqslant 2M_p E_{2^{\mu-1}-1}(f)_{L^p},$$

$$H \leqslant \Big\{\int_{-\pi}^{\pi}|\tilde{S}_\nu(x;f)-\tilde{f}(x)|^p dx\Big\}^{1/p}+\Big\{\int_{-\pi}^{\pi}|\tilde{S}_{2^{\mu-1}-1}(x;f)-\tilde{f}(x)|^p dx\Big\}^{1/p}$$

$$\leqslant Q_p\Big\{\int_{-\pi}^{\pi}|S_\nu(x;f)-f(x)|^p dx\Big\}^{1/p}+Q_p\Big\{\int_{-\pi}^{\pi}|S_{2^{\mu-1}-1}(x;f)-f(x)|^p dx\Big\}^{1/p}$$

$$\leqslant 2Q_p M_p E_{2^{\mu-1}-1}(f)_{L^p}.$$

Thus,

$$\text{(6)}\qquad \Big\{\int_{-\pi}^{\pi}\Big|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\Big|^p dx\Big\}^{1/p} \leqslant N_p E_{2^{\mu-1}-1}(f)_{L^p} \qquad (2^{\mu-1} \leqslant \nu < 2^\mu-1).$$

Applying Lemma 2, we obtain

$$2^{-k}\|\nabla_\mu(\cdot)+\nabla_{-\mu}(\cdot)\|_{L^p} \leqslant N_p E_{2^{\mu-1}-1}(f)_{L^p}\left\{B_{0,k}\cdot 2^{\mu k}\left(\frac{h}{2}\right)^k+(2^\mu-1)^k\left(\frac{h}{2}\right)^k\right\},$$

i.e.

$$\|\nabla_\mu(\cdot)+\nabla_{-\mu}(\cdot)\|_{L^p} \leqslant N_p(B_{0,k}+1)h^k\cdot 2^{\mu k}E_{2^{\mu-1}-1}(f)_{L^p} \qquad (\mu = 1, 2, \ldots).$$

Hence

$$\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant A_p^{1/p}\cdot N_p(B_{0,k}+1)h^k\left\{\sum_{\mu=1}^{\sigma} 2^{\varrho\mu k}E^\varrho_{2^{\mu-1}-1}(f)_{L^p}\right\}^{1/\varrho}.$$

It can easily be observed that

$$\sum_{\mu=1}^{\sigma} 2^{\varrho\mu k}E^\varrho_{2^{\mu-1}-1}(f)_{L^p} \leqslant 2^{\varrho k}E_0^\varrho(f)_{L^p}+C_{\varrho,k}\sum_{\mu=2}^{\sigma}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{\varrho k-1}E_\nu^\varrho(f)_{L^p}.$$

This implies

(7) $$\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant D_{p,\varrho,k}\cdot h^k\left\{\sum_{\nu=0}^{2^\sigma-1}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}\right\}^{1/\varrho},$$

provided $0 < h \leqslant 2\pi/2^\sigma$.

Further, by Theorem 6 of [5] and the estimates (5), (7),

$$\omega_k\left(\frac{\pi}{n+1};f\right)_{L^p} = \omega_k\left(\frac{2\pi}{2n+2};f\right)_{L^p} \leqslant \omega_k\left(\frac{2\pi}{2n+1};f\right)_{L^p}$$

$$\leqslant \omega_k\left(\frac{2\pi}{2^{\sigma-1}};f\right)_{L^p} \leqslant C(k)\omega_k\left(\frac{2\pi}{2^\sigma};f\right)_{L^p} = C(k)\cdot\sup_{0<h\leqslant\frac{2\pi}{2^\sigma}}\|\Delta_h^k f(\cdot)\|_{L^p}$$

$$\leqslant C^2(k)M_pE_{2n+1}(f)_{L^p}+C(k)\cdot\sup_{0<h\leqslant\frac{2\pi}{2^\sigma}}\|\Delta_h^k S_{2^\sigma-1}(\cdot)\|_{L^p}$$

$$\leqslant C^2(k)M_pE_{2n+1}(f)_{L^p}+C(k)D_{p,\varrho,k}\left(\frac{2\pi}{2^\sigma}\right)^k\left\{\sum_{\nu=0}^{2^\sigma-1}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}\right\}^{1/\varrho}$$

$$\leqslant C^2(k)M_pE_{2n+1}(f)_{L^p}+C(k)D_{p,\varrho,k}\left(\frac{\pi}{n+1}\right)^k\left\{\sum_{\nu=0}^{2n}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}\right\}^{1/\varrho}.$$

But

$$\sum_{\nu=0}^{2n}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p} = \left(\sum_{\nu=0}^{n}+\sum_{\nu=n+1}^{2n}\right)(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}$$

$$\leqslant \sum_{\nu=0}^{n}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}+E_{n+1}^\varrho(f)_{L^p}\sum_{\nu=n+1}^{2n}(\nu+1)^{\varrho k-1}$$

$$\leqslant G_{\varrho,k}\sum_{\nu=0}^{n}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}$$

and

$$E_{2n+1}(f)_{L^p} \leqslant \frac{H_{\varrho,k}}{(n+1)^k}\left\{\sum_{\nu=0}^{n}(\nu+1)^{\varrho k-1}E_\nu^\varrho(f)_{L^p}\right\}^{1/\varrho}.$$

Thus the assertion follows.

THEOREM 2. *Let, for some positive p and α $(1 < p < \infty, 0 < \alpha < 1)$, the function f be of class L^p and*

$$\sum_{\nu=1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{L^p} < \infty.$$

Suppose that the function g defined by (2) is continuous in the interval $\langle -\pi, \pi\rangle$. Then g is absolutely continuous in $\langle -\pi, \pi\rangle$ and its derivative $g' = f^{(\alpha)}$ (existing almost everywhere) is of class L^p. Moreover, for $n = 0, 1, 2, \ldots$,

$$\omega_k\left(\frac{\pi}{n+1};f^{(\alpha)}\right)_{L^p} \leqslant M_{p,k,\alpha}\left\{\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{L^p}+\right.$$

$$\left.+\frac{1}{(n+1)^k}\left(\sum_{\nu=0}^{n}(\nu+1)^{\varrho(k+\alpha)-1}E_\nu^\varrho(f)_{L^p}\right)^{1/\varrho}\right\}.$$

Proof. Consider the positive integer σ satisfying condition (3). Retain the symbol $S_m(x)$ used above.

By the inequality of Bernstein–Civin type ([5], § 2), for $\nu = 1, 2, \ldots$,

$$\|S^{(\alpha)}_{2^\nu-1}(\cdot)-S^{(\alpha)}_{2^{\nu-1}-1}(\cdot)\|_{L^p}$$

$$\leqslant 2\cdot(2^\nu-1)^\alpha\left\{\int_{-\pi}^{\pi}|S_{2^\nu-1}(x)-S_{2^{\nu-1}-1}(x)|^p\,dx\right\}^{1/p}$$

$$\leqslant 2\cdot 2^{\nu\alpha}\left\{\left(\int_{-\pi}^{\pi}|S_{2^\nu-1}(x)-f(x)|^p\,dx\right)^{1/p}+\left(\int_{-\pi}^{\pi}|f(x)-S_{2^{\nu-1}-1}(x)|^p\,dx\right)^{1/p}\right\}$$

$$\leqslant 2\cdot 2^{\nu\alpha}\{M_pE_{2^\nu-1}(f)_{L^p}+M_pE_{2^{\nu-1}-1}(f)_{L^p}\} \leqslant 4M_p\cdot 2^{\nu\alpha}E_{2^{\nu-1}-1}(f)_{L^p};$$

whence, if $l > j \geqslant 0$,

$$\|S^{(\alpha)}_{2^l-1}(\cdot)-S^{(\alpha)}_{2^j-1}(\cdot)\|_{L^p} \leqslant 4M_p\cdot\sum_{\nu=j+1}^{l}2^{\nu\alpha}E_{2^{\nu-1}-1}(f)_{L^p} \leqslant 4M_pC_\alpha\sum_{\nu=2^{j-1}}^{2^{l-1}-1}\nu^{\alpha-1}E_\nu(f)_{L^p}.$$

Thus the sequence $\{S^{(\alpha)}_{2^l-1}(x)\}$ satisfies the Cauchy condition in L^p-metric. Consequently, there is a function $\varphi \in L^p$, such that

$$\|S^{(\alpha)}_{2^l-1}(\cdot)-\varphi(\cdot)\|_{L^p} \to 0 \quad \text{as} \quad l\to\infty.$$

Also

$$\|S_{2^l-1}(\cdot)-f(\cdot)\|_{L^p} \leqslant M_pE_{2^l-1}(f)_{L^p} \to 0 \quad \text{as} \quad l\to\infty.$$

Therefore, by Lemma 3, the function g is absolutely continuous in the interval $\langle -\pi, \pi\rangle$ and $g'(x) = \varphi(x)$ a.e.

Clearly,

$$\omega_k\left(\frac{\pi}{n+1}; f^{(\alpha)}\right)_{L^p} = \omega_k\left(\frac{\pi}{n+1}; g'\right)_{L^p} = \omega_k\left(\frac{\pi}{n+1}; \varphi\right)_{L^p}$$
$$\leqslant \omega_k\left(\frac{\pi}{n+1}; \varphi - S^{(\alpha)}_{2^\sigma-1}\right)_{L^p} + \omega_k\left(\frac{\pi}{n+1}; S^{(\alpha)}_{2^\sigma-1}\right)_{L^p} \quad \text{if} \quad n = 0, 1, 2, \ldots$$

It can easily be observed that

$$\omega_k\left(\frac{\pi}{n+1}; \varphi - S^{(\alpha)}_{2^\sigma-1}\right)_{L^p} \leqslant C(k)\Big\{\int_{-\pi}^{\pi} |\varphi(x) - S^{(\alpha)}_{2^\sigma-1}(x)|^p dx\Big\}^{1/p}$$
$$\leqslant C(k)\Big\{\int_{-\pi}^{\pi} \Big|\varphi(x) - \sum_{\nu=1}^{\sigma} \big(S^{(\alpha)}_{2^\nu-1}(x) - S^{(\alpha)}_{2^{\nu-1}-1}(x)\big)\Big|^p dx\Big\}^{1/p}$$
$$\leqslant C(k)\Big\{\int_{-\pi}^{\pi} \Big|\varphi(x) - \sum_{\nu=1}^{\sigma+\tau} \big(S^{(\alpha)}_{2^\nu-1}(x) - S^{(\alpha)}_{2^{\nu-1}-1}(x)\big)\Big|^p dx\Big\}^{1/p} +$$
$$+ C(k)\Big\{\int_{-\pi}^{\pi} \Big|\sum_{\nu=\sigma+1}^{\sigma+\tau} \big(S^{(\alpha)}_{2^\nu-1}(x) - S^{(\alpha)}_{2^{\nu-1}-1}(x)\big)\Big|^p dx\Big\}^{1/p} \quad \text{for} \quad \tau = 1, 2, \ldots$$

Letting $\tau \to \infty$, we obtain

$$\omega_k\left(\frac{\pi}{n+1}; \varphi - S^{(\alpha)}_{2^\sigma-1}\right)_{L^p} \leqslant C(k) \sum_{\nu=\sigma+1}^{\infty} \Big\{\int_{-\pi}^{\pi} |S^{(\alpha)}_{2^\nu-1}(x) - S^{(\alpha)}_{2^{\nu-1}-1}(x)|^p dx\Big\}^{1/p}$$
$$\leqslant C(k) \sum_{\nu=\sigma+1}^{\infty} 4M_p \cdot 2^{\nu\alpha} E_{2^{\nu-1}-1}(f)_{L^p}$$
$$\leqslant 4C(k)\, M_p\, C_\alpha \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{L^p}.$$

By Lemma 1,

$$\Big\{\int_{-\pi}^{\pi} |\Delta_h^k S^{(\alpha)}_{2^\sigma-1}(x)|^p dx\Big\}^{1/p} \leqslant A_p^{1/p}\Big\{\int_{-\pi}^{\pi} \Big(\sum_{\mu=1}^{\sigma} |\nabla_\mu(x) + \nabla_{-\mu}(x)|^2\Big)^{p/2} dx\Big\}^{1/p},$$

with

$$\nabla_\mu(x) = \sum_{\nu=2^{\mu-1}}^{2^\mu-1} (i\nu)^\alpha \left(2i \sin\nu\frac{h}{2}\right)^k c_\nu e^{i\nu x};$$
$$\nabla_{-\mu}(x) = \sum_{\nu=-2^\mu+1}^{-2^{\mu-1}} (i\nu)^\alpha \left(2i \sin\nu\frac{h}{2}\right)^k c_\nu e^{i\nu x}.$$

Since

$$\nabla_\mu(x) + \nabla_{-\mu}(x) = \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu^\alpha \left(2\sin\nu\frac{h}{2}\right)^k \big(c_\nu e^{i(\nu x + (\alpha+k)\pi/2)} + \bar{c}_\nu e^{-i(\nu x + (\alpha+k)\pi/2)}\big)$$
$$= \sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu^\alpha \left(2\sin\nu\frac{h}{2}\right)^k \cdot 2\mathrm{Re}\big(c_\nu e^{i(\nu x + (\alpha+k)\pi/2)}\big),$$

we have

$$2^{-k}\|\nabla_\mu(\cdot) + \nabla_{-\mu}(\cdot)\|_{L^p} = \Big\{\int_{-\pi}^{\pi} \Big|\sum_{\nu=2^{\mu-1}}^{2^\mu-1} \nu^\alpha \left(\sin\nu\frac{h}{2}\right)^k V_\nu(x)\Big|^p dx\Big\}^{1/p},$$

where

$$V_\nu(x) = 2\mathrm{Re}\big(c_\nu e^{i\nu x} \cdot e^{i(\alpha+k)\pi/2}\big).$$

The Abel transformation and Minkowski's inequality give

$$2^{-k}\|\nabla_\mu(\cdot) + \nabla_{-\mu}(\cdot)\|_{L^p}$$
$$\leqslant \sum_{\nu=2^{\mu-1}}^{2^\mu-2} \Big|\nu^\alpha \sin^k\nu\frac{h}{2} - (\nu+1)^\alpha \sin^k(\nu+1)\frac{h}{2}\Big| \Big\{\int_{-\pi}^{\pi} \Big|\sum_{l=2^{\mu-1}}^{\nu} V_l(x)\Big|^p dx\Big\}^{1/p} +$$
$$+ \Big|(2^\mu-1)^\alpha \sin^k(2^\mu-1)\frac{h}{2}\Big| \Big\{\int_{-\pi}^{\pi} \Big|\sum_{l=2^{\mu-1}}^{2^\mu-1} V_l(x)\Big|^p dx\Big\}^{1/p}.$$

Estimate (6) remains valid for $V_l(x)$, too. Hence

$$2^{-k}\|\nabla_\mu(\cdot) + \nabla_{-\mu}(\cdot)\|_{L^p}$$
$$\leqslant N_p\Big\{\sum_{\nu=2^{\mu-1}}^{2^\mu-2} \Big|\nu^\alpha \sin^k\nu\frac{h}{2} - (\nu+1)^\alpha \sin^k(\nu+1)\frac{h}{2}\Big| + \Big|(2^\mu-1)^\alpha \sin^k(2^\mu-1)\frac{h}{2}\Big|\Big\} E_{2^{\mu-1}-1}(f)_{L^p}.$$

Applying Lemma 2, we get

$$2^{-k}\|\nabla_\mu(\cdot) + \nabla_{-\mu}(\cdot)\|_{L^p} \leqslant N_p\Big\{B_{\alpha,k} \cdot 2^{\mu(\alpha+k)}\left(\frac{h}{2}\right)^k + 2^{\mu(\alpha+k)}\left(\frac{h}{2}\right)^k\Big\} E_{2^{\mu-1}-1}(f)_{L^p},$$

if $0 < h \leqslant 2\pi/2^\sigma$ $(\mu = 1, 2, \ldots, \sigma)$. Consequently,

$$\|\Delta_h^k S^{(\alpha)}_{2^\sigma-1}(\cdot)\|_{L^p} \leqslant A_p^{1/p} N_p (B_{\alpha,k}+1) h^k \Big\{\sum_{\mu=1}^{\sigma} 2^{\varrho\mu(\alpha+k)} E^{\varrho}_{2^{\mu-1}-1}(f)_{L^p}\Big\}^{1/\varrho}.$$

Proceeding further as in the proof of Theorem 1, we obtain

$$\omega_k\left(\frac{\pi}{n+1}; S^{(\alpha)}_{2^\sigma-1}\right)_{L^p} \leqslant \frac{L_{p,k,\alpha}}{(n+1)^k}\Big\{\sum_{\nu=0}^{n} (\nu+1)^{\varrho(\alpha+k)-1} E^{\varrho}_\nu(f)_{L^p}\Big\}^{1/\varrho},$$

and the desired result is now evident.

Analogously, the following related theorem can easily be deduced (cf. [7], pp. 346–348, 352–353; [8], pp. 126–130).

THEOREM 3. *Suppose that* $f \in L^p$ $(1 < p < \infty)$ *and that*

$$\sum_{\nu=1}^{\infty} \nu^{r-1} E_\nu(f)_{L^p} < \infty,$$

for some positive integer r. Then

$$f(x) = \lambda(x) \quad \text{for almost every } x \in (-\infty, \infty),$$

where λ *is a function having the derivatives* $\lambda^{(j)}$ $(j = 0, 1, \ldots, r-1)$ *absolutely continuous in* $\langle -\pi, \pi\rangle$. *Moreover, the derivative* $\lambda^{(r)}$ *(existing almost everywhere) belongs to* L^p *and, for* $n = 0, 1, 2, \ldots$,

$$\omega_k\left(\frac{\pi}{n+1}; \lambda^{(r)}\right)_{L^p}$$

$$\leqslant C_{p,k,r}\left\{\sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)_{L^p} + \frac{1}{(n+1)^k}\left(\sum_{\nu=0}^{n} (\nu+1)^{\varrho(k+r)-1} E_\nu^\varrho(f)_{L^p}\right)^{1/\varrho}\right\}.$$

Finally, we shall present the following

THEOREM 4. *Consider positive* p, α, r $(1 < p < \infty,\ 0 < \alpha < 1,\ r$ *is integer) and a continuous function f of period* 2π, *such that*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha+r-1} E_\nu(f)_{L^p} < \infty.$$

Then f is absolutely continuous in $\langle -\pi, \pi\rangle$ *and the function g defined by* (2) *possesses the derivatives*

$$g' = f^{(\alpha)}, \ldots, \quad g^{(r)} = f^{(\alpha+r-1)},$$

absolutely continuous in $\langle -\pi, \pi\rangle$. *Moreover,*

$$g^{(r+1)} = f^{(\alpha+r)}$$

is of class L^p *and, for* $n = 0, 1, 2, \ldots$,

$$\omega_k\left(\frac{\pi}{n+1}; f^{(\alpha+r)}\right)_{L^p}$$

$$\leqslant Q_{p,k,r,\alpha}\left\{\sum_{\nu=n+1}^{\infty} \nu^{\alpha+r-1} E_\nu(f)_{L^p} + \frac{1}{(n+1)^k}\left(\sum_{\nu=0}^{n} (\nu+1)^{\varrho(k+\alpha+r)-1} E_\nu^\varrho(f)_{L^p}\right)^{1/\varrho}\right\}.$$

Proof. In view of Theorem 3, the function f is absolutely continuous in $\langle -\pi, \pi\rangle$, whence the function g is continuous therein for every positive $\alpha \in (0, 1)$.

Applying the inequalities

$$E_\nu(g)_{L^p} \leqslant \frac{H_\alpha}{(\nu+1)^{1-\alpha}} E_\nu(f)_{L^p} \quad \text{for} \quad \nu = 0, 1, 2, \ldots \tag{8}$$

(see [4], § 4), we obtain

$$\sum_{\nu=1}^{\infty} \nu^r E_\nu(g)_{L^p} \leqslant H_\alpha \sum_{\nu=1}^{\infty} \nu^{\alpha+r-1} E_\nu(f)_{L^p} < \infty.$$

Therefore, by Theorem 3, the function g possesses the derivatives

$$g', \ldots, g^{(r)}$$

absolutely continuous in $\langle -\pi, \pi\rangle$ and $g^{(r+1)} \in L^p$. Moreover,

$$\omega_k\left(\frac{\pi}{n+1}; g^{(r+1)}\right)_{L^p}$$

$$\leqslant M_{p,k,r}\left\{\sum_{\nu=n+1}^{\infty} \nu^r E_\nu(g)_{L^p} + \frac{1}{(n+1)^k}\left(\sum_{\nu=0}^{n} (\nu+1)^{\varrho(k+r+1)-1} E_\nu^\varrho(g)_{L^p}\right)^{1/\varrho}\right\}.$$

Further, the inequalities (8) lead to

$$\omega_k\left(\frac{\pi}{n+1}; f^{(\alpha+r)}\right)_{L^p}$$

$$\leqslant M_{p,k,r}\left\{\sum_{\nu=n+1}^{\infty} \nu^r \frac{H_\alpha}{\nu^{1-\alpha}} E_\nu(f)_{L^p} + \frac{H_\alpha}{(n+1)^k}\left(\sum_{\nu=0}^{n} (\nu+1)^{\varrho(k+r+1)-1} \frac{E_\nu^\varrho(f)_{L^p}}{(\nu+1)^{(1-\alpha)\varrho}}\right)^{1/\varrho}\right\},$$

and the proof is completed.

References

[1] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series II*, Proc. London Math. Soc. 42 (1937), pp. 52–89.
[2] R. E. A. C. Paley and A. Zygmund, *On some series of functions*, Proc. Cambridge Phil. Soc. 26 (1930), pp. 337–357.
[3] E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. Journal 3 (1937), pp. 529–543.
[4] R. Taberski, *Approximation of functions possessing derivatives of positive orders*, Ann. Polon. Math. 34 (1977), pp. 13–23.
[5] —, *Differences, moduli and derivatives of positive orders*, Comment. Math. 19 (1977), pp. 389–400.
[6] —, *Two indirect approximation theorems*, Demonstratio Math. 9 (1976), pp. 243–255.
[7] [A. F. Timan] А. Ф. Тиман, *Теория приближения функций действительного переменного*, Москва 1960.
[8] [M. F. Timan] М. Ф. Тиман, *Обратные теоремы конструктивной теории функций в пространствах* L_p $(1 \leqslant p \leqslant \infty)$, Мат. Сборник 46 (1958), 125–132.
[9] A. Zygmund, *Trigonometric series, I, II*, Cambridge 1959.

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