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ON UNCONDITIONAL CONVERGENCE OF HAAR SERIES

G. TKEBUCHAVA

University of Tbilisi, Institute of Applied Mathematics, Tbilisi, U.S.S.R.

The following result of A. Pełczyński [10] is well known: there are no unconditional bases in the space $L(0, 1)$. In particular, the Haar system is not an unconditional basis in the space $L(0, 1)$. Moreover, as was shown by V. F. Gaposhkin [4], [5], the Haar system is an unconditional basis in an Orlicz space if and only if it is reflexive (for the suffiency of the theorem see also [3]). According to the results of A. M. Olevski [8], reflexivity is a necessary and sufficient condition for the existence of an unconditional basis in an Orlicz space. P. L. Uljanov [13] and M. B. Petrovskaia [9] considered the question under what conditions the function f has its Fourier–Haar series unconditionally convergent in the metric of the space $L(0, 1)$. Analogical questions about the unconditional convergence in classes of spaces, containing in particular the spaces $L\ln^{\alpha}(0, 1)$ $(\alpha > 0)$, have been considered in [8], [11], and [12]. In this paper we consider necessary conditions for the unconditional convergence of multidimensional Fourier–Haar series in certain nonreflexive spaces of functions.

Let $\boldsymbol{R}^n$ be the n-dimensional real Euclidean space of points $\bar{x} = (x_1, \ldots, x_n)$, $I^n = [0, 1; \ldots; 0, 1]$ the n-dimensional unit cube, and $N^n \subset \boldsymbol{R}^n$ the subset of positive integer points $\overline{m} = (m_1, \ldots, m_n)$ and $|\overline{m}| = \sum_{i=1}^{n} m_i$.

Moreover, we use below the following notation; $r\overline{m} = (rm_1, \ldots, rm_n)$, $\overline{1} = (1, \ldots, 1)$, $\{a_{\overline{m}}\}_{\overline{m}}^{\infty} \equiv \{a_{m_1, \ldots, m_n}\}_{m_1, \ldots, m_n = 1}^{\infty}$, $\sum_{\overline{m}=\overline{1}}^{\overline{p}} (\cdot) = \sum_{m_1=1}^{p_1} \ldots \sum_{m_n=1}^{p_n} (\cdot)$.

The convergence $\overline{M} \to \infty$ is understood in the sense of Pringsheim, i.e. it is equivalent to $M_i \to \infty$ for all $1 \leqslant i \leqslant n$, and $\Phi(L)$ denotes the class of measurable functions $f(\bar{x})$ on the cube I^n for which $\int_{I^n} \Phi(f(\bar{x}))d\bar{x} < \infty$. In the sequel, $L_{\Phi}^* \equiv L_{\Phi}^*(I^n)$ denotes a Banach space of functions defined on the cube I^n which is generated by N-functions with Φ as their principal part (for details cf. [7]). We assume that the function Φ considered below satisfies condition Δ_2 (i.e. $\Phi(2u) < C\Phi(u)$ for $u \geqslant u_0$). This condition is necessary and sufficient for the Orlicz space L_{Φ}^* to be separable.

In the sequel the following notation will be used: C — absolute positive constants, $\lg u \equiv \lg_2 u$,

$$\Delta^k(p(m)) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}p(m+j), \quad \Delta(p(m)) = \Delta^1(p(m)),$$

$$\Delta_i(p(\overline{m})) = p(\overline{m})-p(\overline{m}_{(i)}), \quad \overline{m}_{(i)} = (m_1, \ldots, m_{i-1}, m_i+1, m_{i+1}, \ldots, m_n),$$

$$\Delta_{j_1,\ldots,j_k}(\varrho(\overline{m})) = \Delta_{j_k}\big(\Delta_{j_{k-1}} \ldots \Delta_{j_1}(\varrho(\overline{m}))\big),$$

$\varrho(\overline{m}) = 2^{-|\overline{m}|}\Phi(2^{|\overline{m}|})$ $\forall \overline{m} \in N^n$ (the function Φ is defined below),

$b_{\overline{m}} = |\overline{m}|^{-n}\varrho^{-1}(\overline{m})$.*

DEFINITION 1. Let B be a Banach space and let A be a subset of B. The sequence $\{x_{\overline{m}}\}_{\overline{m}=\overline{1}}^{\infty}$ in the space B is called an *(unconditional) basis* of A in the norm of the space B if for every $x \in A$ there exists a unique sequence of scalars $\{a_{\overline{m}}\}_{\overline{m}=\overline{1}}^{\infty}$ such that the series $\sum_{\overline{m}=\overline{1}}^{\infty} a_{\overline{m}}x_{\overline{m}}$ (unconditionally) converges to x.

DEFINITION 2. The function Φ *satisfies condition* $(*)$ iff Φ is an N-function and the following conditions hold:

(1) $$0 \leqslant \varrho(m)\Delta^k(m^{-n}\varrho^{-1}(m)) \leqslant Cm^{-n-k}, \quad k = 1, \ldots, n.$$

LEMMA 1. *Let the function Φ satisfy condition $(*)$. Set*

(2) $$\Psi(u) = \begin{cases} |u|\Big(\int_1^{|u|} \Phi(t)t^{-2}f dt\Big)\ln^{n-1}(|u|+2) & \textit{for} \quad |u| > 1,\\ 0 & \textit{for} \quad |u| \leqslant 1.\end{cases}$$

Then the following inequalities hold:

(3) $$\Phi(u^2) \leqslant C|u|\Phi(u),$$

(4) $$\Phi(u) \leqslant C|u|\big(\lg(|u|+2)\big)^C,$$

(5) $$C^{-1}\Phi(u)(\lg^+|u|)^n \leqslant \Psi(u) \leqslant C\Phi(u)(\lg^+|u|)^n,$$

(6) $$\varrho(u+v) \leqslant C\big(\varrho(u)+\varrho(u)\big).$$

Proof. It follows from Definition 2 for $k = 1$ that

$$\Phi(2^{m+1}) \leqslant 2(1+Cm^{-1})\Phi(2^m),$$

whence for large $|u|$,

$\Phi(u^2) \leqslant$

$$2^{2+[\lg|u|]}\Big(1+\frac{C}{2[\lg|u|]+1}\Big)\Big(1+\frac{C}{2[\lg|u|]}\Big)\ldots\Big(1+\frac{C}{[\lg|u|]}\Big)\Phi(2^{[\lg|u|]}) \leqslant C|u|\Phi(u),$$

and this implies (3).

* Here and later on $\varrho^{-1}(\overline{m}) = 1/\varrho(\overline{m})$.

According to the results of P. L. Uljanov ([14], p. 664) (3) implies (4). Since, for $u > 1$,

$$\Psi(u) \geqslant u\Big(\int_{\sqrt{u}}^{u} \Phi(t)t^{-2}dt\Big)\ln^{n-1}u \geqslant (2C)^{-1}\Phi(u)\ln^n u,$$

it follows that the left-hand inequality in (5) holds. Since the function $u^{-1}\Phi(u)$ is monotone for $u > 1$, the right-hand inequality in (5) holds, too.

Finally, let $m_1 \geqslant m_2$; then

$$\Phi(2^{m_1+m_2}) \leqslant 2^{m_2}\Big(1+\frac{C}{m_1+m_2-1}\Big)\ldots\Big(1+\frac{C}{m_1}\Big)\Phi(2^{m_1})$$

$$\leqslant 2^{m_2}\Big(1+\frac{C}{m_1}\Big)^{m_2}\Phi(2^{m_1}) \leqslant C2^{m_2}\Phi(2^{m_1}).$$

Applying this estimation, we obtain (6).

Let $\{\chi_m(x)\}_{m=1}^{\infty}$ denote the Haar system on the interval $I^1 = [0, 1]$ defined as follows (cf. [13], pp. 54–55):

$\chi_1(x) = 1$ for $0 \leqslant x \leqslant 1$, and

$$\chi_m(x) = \begin{cases} \sqrt{2^p} & \text{for} \quad \dfrac{2k-2}{2^{p+1}} < x < \dfrac{2k-1}{2^{p+1}},\\ -\sqrt{2^p} & \text{for} \quad \dfrac{2k-1}{2^{p+1}} < x < \dfrac{2k}{2^{p+1}},\\ 0 & \text{elsewhere in} \quad I^1\end{cases}$$

for $m = 2^p+k$, $p = 0, 1, \ldots$, $k = 1, \ldots, 2^p$. Let $\{\chi_{\overline{m}}(\overline{x})\}_{\overline{m}=\overline{1}}^{\infty}$ denote the Haar system defined on I^n, where $\chi_{\overline{m}}(\overline{x}) = \chi_{m_1}(x_1) \ldots \chi_{m_n}(x_n)$.

We shall use below the following facts:

(7) $$\sum_{\overline{m}=\overline{1}}^{\infty}|\overline{m}|^{-n} = \infty,$$

(8) $$\sum_{\overline{m}=\overline{1}}^{\infty}|\overline{m}|^{-n-\alpha} < \infty \quad \forall \alpha > 0.$$

It follows directly from (7) that

(9) $$\sum_{\overline{m}=\overline{1}}^{\infty} b_{\overline{m}}\varrho(|\overline{m}|) = \infty$$

and (1), (3), (6) imply that

(10) $$\sum_{\overline{m}=\overline{1}}^{\infty}\Delta(b_{\overline{m}})\varrho(|\overline{m}|) = \sum_{\overline{m}=\overline{1}}^{\infty}\Delta^n(|\overline{m}|^{-n}\varrho^{-1}(\overline{m}))\varrho(|\overline{m}|) \leqslant C\sum_{\overline{m}=\overline{1}}^{\infty}|m|^{-2n} < \infty.$$

Let $\varepsilon(t)$ be such a function that $\varepsilon(t)\downarrow 0$ while $t\to\infty$, $0<\varepsilon(t)<1$ and let $\{l_k\}_{k=0}^{\infty}$ ($l_0=0$) be an increasing sequence of integers satisfying some conditions specified below.

Let us define the following sets:

(11) $$V_k=\{\overline{m}:\ l_k+1\leqslant \max_{1\leqslant i\leqslant n} m_i\leqslant l_{k+1}\},\quad k=0,1,2,\ldots,$$

(12) $$V_k^*=\{\overline{m}:\ l_k+1<\max_{1\leqslant i\leqslant n} m_i\leqslant l_{k+1}\},\quad k=0,1,2,\ldots,$$

(13) $$S_k=V_k\setminus V_k^*,\quad k=1,2,\ldots,$$

(14) $$W_k=\{\overline{m}:\ l_k+1\leqslant m_i\leqslant l_{k+1},\ 1\leqslant i\leqslant n\},\quad k=0,1,2,\ldots$$

In addition, we set

(15) $$D_k=\sum_{\overline{m}\in W_k}|\overline{m}|^{-n},\quad k=1,2,\ldots,$$

(16) $$A_k=\sum_{\overline{m}\in V_k}|\overline{m}|^{-n},\quad k=1,2,\ldots,$$

(17) $$a_{\overline{m}}=\begin{cases}A_k^{-1}\varrho_k^{-1}(|\overline{m}|)|\overline{m}|^{-n} & \text{for}\quad \overline{m}\in V_k,\ k\geqslant 1,\\ |\overline{m}|^{-n} & \text{for}\quad \overline{m}\in V_0.\end{cases}$$

Let us note that

(18) $$a_{\overline{m}}=\begin{cases}b_{\overline{m}}A_k^{-1} & \text{for}\quad \overline{m}\in V_k,\ k\geqslant 1,\\ |\overline{m}|^{-n} & \text{for}\quad \overline{m}\in V_0.\end{cases}$$

Since

(19) $$\Delta_{j_1,\ldots,j_k}(b_{\overline{m}})\geqslant 0,\quad 1\leqslant j_1<\ldots<j_k\leqslant n,$$

we have

(20) $$\Delta_{j_1,\ldots,j_k}(a_{\overline{m}})\geqslant 0,\quad 1\leqslant j_1<\ldots<j_k\leqslant n.$$

In view of $\varepsilon(t)\downarrow 0$ for $t\to\infty$, using (7) we can choose a sequence $\{l_k\}_{k=1}^{\infty}$ in such a way that the conditions introduced below are fulfilled for $k=1,2,\ldots$

(21) $$\varepsilon(|\overline{m}|)<k^{-2},\quad \text{for}\quad |\overline{m}|>l_k,$$

(22) $$D_k>k^2,$$

(23) $$D_{k+1}>4D_k,$$

(24) $$D_k>\tfrac{1}{2}A_k,$$

(25) $$A_{k+1}>A_k.$$

Consequently, the following inequalities hold:

(26) $$\sum_{\overline{m}\in V_k}|\overline{m}|^{-n}\varepsilon(|\overline{m}|)<A_k\cdot k^{-2},\quad k=1,2,\ldots$$

LEMMA 2. *Let Φ be a function satisfying condition $(*)$ and let $\varepsilon(t)\downarrow 0$ for $t\to\infty$, $0<\varepsilon(t)<1$. Then the sequence $\{a_{\overline{m}}\}_{\overline{m}=\overline{1}}^{\infty}$, constructed according to (18), (21)–(25), is decreasing for every m_i $(1\leqslant i\leqslant n)$, tends to 0, and satisfies the inequalities*

(27) $$\sum_{k=1}^{\infty}\sum_{\overline{m}\in W_k}a_{\overline{m}}\varrho(|\overline{m}|)=\infty,$$

(28) $$\sum_{\overline{m}=\overline{1}}^{\infty}\Delta_{j_1,\ldots,j_k}(a_{\overline{m}})\varrho(|\overline{m}|)<\infty,\quad 1\leqslant j_1<\ldots<j_k\leqslant n,$$

(29) $$\sum_{\overline{m}=\overline{1}}^{\infty}2^{-|\overline{m}|}\big(\ln^+(2^{|\overline{m}|}\Delta(a_{\overline{m}}))\big)^n\varepsilon(2^{|\overline{m}|}\Delta(a_{\overline{m}}))\Phi(2^{|\overline{m}|}\Delta(a_{\overline{m}}))<\infty.$$

Proof. (27) follows immediately from (16) and (24)

$$\sum_{k=1}^{\infty}\sum_{\overline{m}\in W_k}a_{\overline{m}}\varrho(|\overline{m}|)=\sum_{k=1}^{\infty}D_kA_k^{-1}\geqslant\sum_{k=1}^{\infty}\frac{1}{2}=\infty.$$

The proof of (28) runs as follows:

(30) $$\sum_{\overline{m}=\overline{1}}^{\infty}\Delta_{j_1,\ldots,j_k}(a_{\overline{m}})\varrho(|\overline{m}|)=\sum_{\overline{m}\in V_0}+\sum_{p=1}^{\infty}\sum_{\overline{m}\in V_p^*}+\sum_{p=1}^{\infty}\sum_{\overline{m}\in S_p}\equiv\Delta_1+\Delta_2+\Delta_3.$$

It follows from (1), (8), (12) and (16) that for $1\leqslant k\leqslant n$

(31) $$\Delta_2\leqslant C\sum_{p=1}^{\infty}A_p^{-1}\sum_{\overline{m}\in V_p^*}|\overline{m}|^{-n-k}\leqslant C\sum_{\overline{m}=\overline{1}}|\overline{m}|^{-n-k}<\infty.$$

Since for $\overline{m}\in S_p$

$$\Delta_{j_1,\ldots,j_k}(a_{\overline{m}})=A_p^{-1}b_{\overline{m}},$$

and moreover,

$$\sum_{\overline{m}\in S_p}|\overline{m}|^{-n}<\infty\qquad(\forall p\ \text{uniformly}),$$

we obtain, by (22) and by the fact that $D_k<A_k$,

(32) $$\Delta_3\leqslant\sum_{p=1}^{\infty}A_p^{-1}\sum_{\overline{m}\in S_p}|\overline{m}|^{-n}<\infty.$$

Δ_1 is finite, and so by (30)–(32) we have (28).

To prove (29) we set

(33) $$\begin{aligned}\tau_k^{(1)}&\equiv\{\overline{m}:\ \overline{m}\in V_k,\ 1\leqslant 2^{|\overline{m}|}\Delta(a_{\overline{m}})\leqslant|\overline{m}|\},\quad k=1,2,\ldots,\\ \tau_k^{(2)}&\equiv\{\overline{m}:\ \overline{m}\in V_k^*,\ 2^{|\overline{m}|}\Delta(a_{\overline{m}})>|\overline{m}|\},\quad k=1,2,\ldots,\\ \tau_k^{(3)}&\equiv\{\overline{m}:\ \overline{m}\in S_k,\ 2^{|\overline{m}|}\Delta(a_{\overline{m}})>|\overline{m}|\},\quad k=1,2,\ldots\end{aligned}$$

Now we have the following decomposition:

$$\sum_{\bar m=\bar 1}^{\infty} 2^{-|\bar m|}\Phi\big(2^{|\bar m|}\Delta(a_{\bar m})\big)\varepsilon\big(2^{|\bar m|}\Delta(a_{\bar m})\big)\big(\ln^+(2^{|\bar m|}\Delta(a_{\bar m}))\big)^n$$

$$= \sum_{\bar m\in V_0} + \sum_{k=1}^{\infty}\sum_{\bar m\in\tau_k^{(1)}} + \sum_{k=1}^{\infty}\sum_{\bar m\in\tau_k^{(2)}} + \sum_{k=1}^{\infty}\sum_{\bar m\in\tau_k^{(3)}} = \sum_{i=1}^{4}\Pi_i.$$

According to (4) and the properties of the function $\varepsilon(t)$ we have

$$\Pi_2 \leqslant \sum_{\bar m=\bar 1}^{\infty} 2^{-|\bar m|}\ln^n|\bar m|\Phi(|\bar m|) < \infty.$$

Further, by (1) and (26) we get

$$\Pi_3 \leqslant \sum_{k=1}^{\infty} A_k^{-1}\sum_{\bar m\in V_k^*}\varepsilon(|\bar m|)\Delta(b_{\bar m})\varrho(|\bar m|)|\bar m|^n \leqslant \sum_{k=1}^{\infty} A_k^{-1}\sum_{\bar m\in V_k}|\bar m|^{-n}\varepsilon(|\bar m|) < \infty.$$

Finally,

$$\Pi_4 \leqslant 2^n\sum_{k=1}^{\infty}\sum_{p=1}^{n-1}\Big(\sum_{m_1=1}^{l_k-1}\dots\sum_{m_p=1}^{l_k-1}\Big)\Delta(a_{m_1,\dots,m_p,l_k,\dots,l_k})\times$$

$$\times\Big(\sum_{i=1}^{p} m_i+l_k(n-p)\Big)^n\cdot\varrho\Big(\sum_{i=1}^{p} m_i+l_k(n-p)\Big)$$

$$\leqslant C\sum_{k=1}^{\infty}A_k^{-1}\sum_{p=1}^{n-1}\Big(\sum_{m_1=1}^{l_k-1}\dots\sum_{m_p=1}^{l_k-1}\Big)\Big(\sum_{i=1}^{p} m_i+l_k(n-p)\Big) \leqslant C\sum_{k=1}^{\infty}A_k^{-1} < \infty,$$

and this completes the proof of Lemma 2.

Let $\{c_{\bar m}\}$ denote either $\{a_{\bar m}\}$ or $\{b_{\bar m}\}$, and let us set, for a positive integer r,

$$T(c_{r\bar p}) = \sum_{k=0}^{n}(-1)^{n-k}\sum_{1\leqslant j_1<\dots<j_k\leqslant n}\Big(\sum_{m_{j_1}=1}^{p_{j_1}-1}\dots\sum_{m_{j_k}=1}^{p_{j_k}-1}\Big)2^{r(m_{j_1}+\dots+m_{j_k}+p_{j_{k+1}}+\dots+p_{j_n})}\times$$

$$\times c_{rm_{j_1},\dots,rm_{j_k},rp_{j_{k+1}},\dots,rp_{j_n}}.$$

The following equality will be useful: for arbitrary positive integer r and such $\bar x$ that $x_i\in(2^{-rp_i-1}, 2^{-rp_i})$, $1\leqslant p_i\leqslant M_i$, $1\leqslant i\leqslant n$,

$$\sum_{\bar m=\bar 1}^{\bar M} 2^{\frac{r}{2}|\bar m|}c_{r\bar m}\chi_{2^{r\bar m}+1}(\bar x) = T(c_{r\bar p}). \tag{34}$$

It is easy to verify this fact by induction with respect to n.

LEMMA 3. *For a sufficiently large r the following inequalities hold:*

$$T(b_{r\bar p}) \geqslant 2^{r|\bar p|-1}b_{r\bar p}, \tag{35}$$

$$T(b_{r\bar p}) \geqslant 2^{\frac{r}{2}|\bar p|}. \tag{36}$$

Proof. First we shall prove that for an arbitrary $\varepsilon>0$ one can find such an $R(\varepsilon)$ that for $r>R(\varepsilon)$,

$$G(b_{r\bar p}) \equiv T(b_{r\bar p})b_{r\bar p}^{-1}-2^{r|\bar p|} \leqslant \varepsilon 2^{r|\bar p|}. \tag{37}$$

Indeed, $G(b_{r\bar p})$ can be represented as a finite combination of sums of the form

$$H \equiv \sum_{m_{j_1}=1}^{[p_{j_1}/2]}\dots\sum_{m_{j_s}=1}^{[p_{j_s}/2]}\ \sum_{m_{j_{s+1}}=[p_{j_{s+1}}/2]}^{p_{j_{s+1}}-1}\dots\sum_{m_{j_k}=[p_{j_k}/2]}^{p_{j_k}-1} b_{\bar j\bar p}^{-1}2^{t(m_{j_1}+\dots+m_{j_s}+p_{j_{s+1}}+\dots+p_{j_n})}\times$$

$$\times b_{rm_{j_1},\dots,rm_{j_k},rp_{j_{k+1}},\dots,rp_{j_n}}\qquad(0\leqslant s\leqslant k\leqslant n,\ k\geqslant 1),$$

where the first group of sums is degenerated for $s=0$ and the second one for $s=k$. However, from (3), (4), (6) and the estimate

$$|\bar p|(m_{j_1}+\dots+m_{j_s}+\tfrac{1}{2}(p_{j_{s+1}}+\dots+p_{j_k})+p_{j_{k+1}}+\dots+p_{j_n})^{-1}\leqslant p_{j_1}+\dots+p_{j_s}+2$$

we obtain

$$H \leqslant C\Big(2+\sum_{i=1}^{s}p_{j_i}\Big)^{n}\varrho^{-1}\Big(\frac{r}{2}\sum_{i=s+1}^{n}p_{j_i}\Big)\Big(\varrho\Big(r\sum_{i=1}^{s}p_{j_i}\Big)+\varrho\Big(r\sum_{i=s+1}^{n}p_{j_i}\Big)\Big)2^{-\frac{r}{2}\sum_{i=1}^{s}p_{j_i}}\times$$

$$\times(2^r-1)^{s-k}2^{r|\bar p|}$$

$$\leqslant C2^{r|\bar p|}\Big(\Big(r\sum_{i=1}^{s}p_{j_i}\Big)^{C}+C\Big)\Big(2+\sum_{i=1}^{s}p_j\Big)^n 2^{-\frac{r}{2}\sum_{i=1}^{s}p_{j_i}}(2^r-1)^{s-k}.$$

It is clear that the last term tends to 0 while $r\to\infty$ (because of the first factor for $s=0$ and of the second one for $s>0$). Now it is easy to find such an $R(\varepsilon)$ that for $r>R(\varepsilon)$ we get (37). Setting $\varepsilon=1/2$, we obtain (35). Since for a sufficiently large r

$$b_{r\bar p}|r\bar p|^{-n-c}\geqslant 2^{1-\frac{r}{2}|\bar p|}, \tag{38}$$

(36) immediately follows from (37).

LEMMA 4. *There are a positive integer R and a set $P\subset N^n$ such that for $r>R$ we have*

$$\sum_{r\bar p\in P}a_{r\bar p}\varrho(r|\bar p|) = \infty, \tag{39}$$

$$T(a_{r\bar p})\geqslant 2^{|r\bar p|-1}a_{r\bar p}\quad \forall r\bar p\in P, \tag{40}$$

$$T(a_{r\bar p})\geqslant 2^{\frac{r}{2}|\bar p|}\quad \forall r\bar p\in P. \tag{41}$$

Proof. Set

$$P=\bigcup_{k=2}^{\infty}Y_k, \tag{42}$$

where

$$Y_k=\{\bar m:\ \bar m\in W_k;\ 4l_k\leqslant m_i\leqslant l_{k+1};\ 1\leqslant i\leqslant n,\ A_k\leqslant\min_{1\leqslant i\leqslant n}2^{m_i/4}\}, \tag{43}$$

(44) $$Z_k = \{\overline{m}\colon \overline{m} \in W_k,\ 4l_k \leqslant m_i \leqslant l_{k+1},\ 1 \leqslant i \leqslant n,\ A_k > \min_{1 \leqslant i \leqslant n} 2^{m_i/4}\}.$$

Now by (44) and (27) we get

$$\sum_{\overline{p} \in P} a_{\overline{p}} \varrho(|\overline{p}|) \geqslant \sum_{k=2}^{\infty} \sum_{\overline{p} \in W_k} a_p \varrho(|\overline{p}|) - \sum_{k=2}^{\infty} \sum_{\overline{p} \in Z_k} |\overline{p}|^{-n} A_k^{-1} - \sum_{i=1}^{n} \sum_{k=2}^{\infty} \sum_{p_1 = l_k}^{\infty} \ldots$$

$$\ldots \sum_{p_{i-1}=l_k}^{\infty} \sum_{p_i = l_k}^{4l_k} \sum_{p_{i+1}=l_k}^{\infty} \ldots \sum_{p_n = l_k}^{\infty} |\overline{p}|^{-n} A_k^{-1} \geqslant \sum_{k=2}^{\infty} \sum_{\overline{p} \in W_k} a_{\overline{p}} \varrho(|\overline{p}|) - \sum_{i=1}^{n} \sum_{k=2}^{\infty} \sum_{\overline{p} \in W_k} |\overline{p}|^{-n} 2^{-p_i}$$

$$-C \sum_{k=2}^{\infty} k^{-2} \sum_{p_1 = l_k}^{4l_k} (p_1 + (n-1) l_k)^{-1} > \sum_{k=2}^{\infty} \sum_{\overline{p} \in W_k} a_{\overline{p}} \varrho(|\overline{p}|) - C - C = \infty;$$

hence taking into account the monoto nicity of the function $a_{\overline{p}}\ \varrho(|\overline{p}|)$ (for separated coordinates p_i, $1 \leqslant i \leqslant n$), we obtain (39).

Now we shall prove (40). Since for an arbitrary positive integer r there exist infinitely many $r\overline{p} \in P$ (as follows from (39)), there is a k_0 such that $r\overline{p} \in Y_{k_0}$. Since the set P is symmetric with respect to the diagonal of an n-dimensional matrix, then it is enough to show that for any $\varepsilon > 0$, $1 \leqslant l \leqslant n$, and for a sufficiently large r, the following inequality is satisfied:

(45) $$U \equiv \sum_{m_{j_1}=1}^{p_{j_1}-1} \ldots \sum_{m_{j_l}=1}^{p_{j_l}-1} 2^{r(m_{j_1} + \ldots + m_{j_l} + p_{j_{l+1}} + \ldots + p_{j_n})} a_{rm_{j_1}, \ldots, rm_{j_l}, rp_{j_{l+1}}, \ldots, rp_{j_n}} \leqslant \varepsilon 2^{r|\overline{p}|} a_{r\overline{p}}.$$

Let

$$X \equiv \{\overline{q}\colon q_{j_i} \leqslant r(p_{j_i} - 1),\ 1 \leqslant i \leqslant k;\ q_{j_i} = p_{j_i},\ k+1 \leqslant i \leqslant n\};$$

hence we have the following composition (see (42)):

$$U \equiv \sum_{k=0}^{k_0 - 1} \sum_{r\overline{m} \in W_k} + \sum_{r\overline{m} \in X}$$

$$\leqslant a_{r\overline{p}} \Big\{ A_{k_0} \sum_{m_{j_1}=1}^{[4^{-1} p_{j_1}]} \ldots \sum_{m_{j_l}=1}^{[4^{-1} p_{j_l}]} 2^{r(m_{j_1} + \ldots + m_{j_l} + p_{j_{l+1}} + \ldots + p_{j_n})} b_{r\overline{p}}^{-1} b_{rm_{j_1}, \ldots, rm_{j_l}, rp_{j_{l+1}}, \ldots, rp_{j_n}} +$$

$$+ \sum_{m_{j_1}=1}^{p_{j_1}-1} \ldots \sum_{m_{j_l}=1}^{p_{j_l}-1} 2^{r(m_{j_1} + \ldots + m_{j_l} + p_{j_{l+1}} + \ldots + p_{j_n})} b_{r\overline{p}}^{-1} b_{rm_{j_1}, \ldots, rm_{j_l}, rp_{j_{l+1}}, \ldots, rp_{j_n}} \Big\}.$$

By estimations (37), (43) we get (45), and by (38) we obtain (41).

LEMMA 5. *Let the function Φ satisfy condition* $(*)$. *Then*

(1) *For*

$$\theta_{\overline{p}} = c_{\overline{p}+\overline{1}} - c_{\overline{p}} + \sum_{k=0} (-1)^{n-k} \sum_{1 \leqslant j_1 < \ldots \leqslant j_k \leqslant n} \Delta_{k+1, \ldots, n}(c_{p_{j_1}+1, \ldots, p_{j_k}+1, p_{j_{k+1}}, \ldots, p_{j_n}}),$$

we have

(46) $$\int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\infty} |2^{|\overline{m}|/2} \theta_{\overline{m}} \chi_{2^{\overline{m}}+\overline{1}}(\overline{x})|\Big) d\overline{x} < \infty.$$

(2) *Also we have*

(47) $$\sum_{\overline{p}=\overline{1}}^{\infty} 2^{-|\overline{p}|} \Phi(2^{|\overline{p}|} c_{\overline{p}}) = \infty.$$

Proof. Appling (6) and (4) for all $\overline{m}$ (uniformly), we get

$$\sum_{\overline{p}=\overline{m}}^{\infty} \varrho(|\overline{p}|) \varrho^{-1}(|\overline{m}|) 2^{|\overline{m}| - |\overline{p}|} \leqslant \sum_{\overline{p}=\overline{m}}^{\infty} \{\varrho(|\overline{p}| - |\overline{m}|) \varrho^{-1}(|\overline{m}|) + 1\} 2^{|\overline{m}| - |\overline{p}|} < C.$$

Hence, by the properties of Φ, we obtain

(48) $$\int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\infty} |2^{|\overline{m}|/2} \theta_{\overline{m}} \chi_{2^{\overline{m}}+\overline{1}}(\overline{x})|\Big) d\overline{x}$$

$$\leqslant C \sum_{\overline{p}=\overline{1}}^{\infty} 2^{-|\overline{p}|} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{p}} 2^{|\overline{m}|} \theta_{\overline{m}}\Big) \leqslant C \sum_{\overline{p}=\overline{1}}^{\infty} \varrho(|\overline{p}|) \sum_{\overline{m}=\overline{1}}^{\overline{p}} 2^{|\overline{m}| - |\overline{p}|} \theta_{\overline{m}}$$

$$\leqslant C \sum_{\overline{m}=\overline{1}}^{\infty} \varrho(|\overline{m}|) \theta_{\overline{m}} \sum_{\overline{p}=\overline{m}}^{\infty} \varrho(|\overline{p}|) \varrho^{-1}(|\overline{m}|) 2^{|\overline{m}| - |\overline{p}|} \leqslant C \sum_{\overline{m}=\overline{1}}^{\infty} \varrho(|\overline{m}|) \theta_{\overline{m}}.$$

Since $\theta_{\overline{m}}$ is estimated by the finite combination of elements $\Delta_{j_1, \ldots, j_k}(\overline{m})$, according to (28) we have

$$\sum_{\overline{m}=\overline{1}} \varrho(|\overline{m}|) \theta_{\overline{m}} < \infty,$$

whence by (48) we get (46).

Applying properties (3) and (4), we can easily get (47). Indeed,

$$C \sum_{\overline{p}=\overline{1}}^{\infty} 2^{-|\overline{p}|} \Phi(2^{|\overline{p}|} c_{\overline{p}}) \geqslant C \sum_{c_{\overline{p}} \geqslant 2^{-|\overline{p}|/2}} 2^{-|\overline{p}|/2} \Phi(2^{|\overline{p}|/2}) c_{\overline{p}}$$

$$\geqslant \Big(\sum_{\overline{p}=\overline{1}}^{\infty} - \sum_{c_{\overline{p}} < 2^{-|\overline{p}|/2}}\Big) c_{\overline{p}} \varrho(|\overline{p}|) = \sum_{\overline{p}=\overline{1}}^{\infty} c_{\overline{p}} \varrho(|\overline{p}|) - C.$$

Because of (9) and (27), the last equality is equal to ∞; hence (47) holds true and this completes the proof of Lemma 5.

THEOREM 1. *Under condition* $(*)$ *the function*

$$f(\overline{x}) = f(\overline{x}, \{c_{\overline{m}}\}) = \begin{cases} 2^{n+|\overline{m}|} \Delta(c_{\overline{m}}) & \text{for } \overline{x} \in (2^{-\overline{m}-\overline{1}}, 2^{-\overline{m}}), \\ 0 & \text{elsewhere in } I^n, \end{cases}$$

has the following property: there exists a subseries of Haar–Fourier series which converges to ∞ in the norm of the Orlicz space $L_\Phi^*(I^n)$, though $f \in L_\Phi^*(I^n)$.

Proof. According to the definition of the function f and according to conditions (1), (8) and (28), by using properties of function Φ, we obtain

$$\int_{I^n} \Phi(f(\overline{x})) d\overline{x} < \infty.$$

Thus $f \in L_\Phi^*(I^n)$.

Now, choosing by Lemma 3 and 4 a sufficiently large r, let us consider the series

$$\sum_{\overline{m}=\overline{1}}^{\infty} 2^{\frac{r}{2}|\overline{m}|} c_{r\overline{m}} \chi_{2^{r\overline{m}}+1}(\overline{x})$$

and let

$$E_{\overline{M}} = \begin{cases} P \cap \{\overline{m}\colon \overline{1} \leqslant \overline{m} \leqslant \overline{M}\} & \text{for} \quad c_{\overline{m}} = a_{\overline{m}}, \\ N^n \cap \{\overline{m}\colon \overline{1} \leqslant \overline{m} \leqslant \overline{M}\} & \text{for} \quad c_{\overline{m}} = b_{\overline{m}}. \end{cases}$$

Taking into account (34), (35) and (40) and the fact that Φ is even and using the definition of the set $E_{\overline{M}}$, we get

$$(49) \qquad B \equiv \int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{M}} 2^{\frac{r}{2}|\overline{m}|} c_{r\overline{m}} \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x})\Big) d\overline{x} \geqslant \sum_{\overline{p}=\overline{2}}^{\overline{M}} \int_{2^{-r\overline{p}}-\overline{1}}^{2^{-r\overline{p}}}$$

$$\Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{M}} 2^{\frac{r}{2}|\overline{m}|} c_{r\overline{m}} \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x})\Big) d\overline{x} \geqslant 2^{-n} \sum_{\overline{p}=\overline{2}}^{\overline{M}} 2^{-r\overline{p}} \Phi\big(T(c_{r\overline{p}})\big) \geqslant C \sum_{r\overline{p} \in E_{\overline{M}}} 2^{-r\overline{p}} \Phi(2^{r|\overline{p}|} c_{r\overline{p}}),$$

whence by (47) it follows that $B \to \infty$ for $\overline{M} \to \infty$ (49). Now let us compute the $d_{2\overline{p}+\overline{1}}(f)$ — Fourier coefficients of the function f:

$$(50) \qquad d_{2\overline{p}+\overline{1}}(f) = 2^{|\overline{p}|/2} \sum_{k=0}^{n} (-1)^{n-k} \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant n} \int_0^{2^{-p_{j_1}}-1} \ldots \int_0^{2^{-p_{j_k}}-1} \int_{2^{-p_{j_{k+1}}}-1}^{2^{-p_{j_{k+1}}}} \ldots$$

$$\ldots \int_{2^{-p_{j_n}}-1}^{2^{-p_{j_n}}} f(\overline{x})\, d\overline{x} = 2^{|\overline{p}|/2}(c_{\overline{p}} - \theta_{\overline{p}}).$$

Thus by the properties of the function Φ and by (46), (50) and (49) we obtain

$$(51) \quad \lim_{\overline{M}\to\infty} \int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{M}} d_{2^{r\overline{m}}+\overline{1}}(f) \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x})\Big) d\overline{x} \geqslant C \lim_{\overline{M}\to\infty} \int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{M}} 2^{\frac{r}{2}|\overline{m}|} c_{r\overline{m}} \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x})\Big) d\overline{x} -$$

$$- C \lim_{m\to\infty} \int_{I^n} \Phi\Big(\sum_{\overline{m}=\overline{1}}^{\overline{M}} 2^{\frac{r}{2}|\overline{m}|} \theta_{r\overline{m}} \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x})\Big) d\overline{x} = \infty.$$

However, it follows from (51) (see [7], p. 92) that for the Haar–Fourier series of the function f there exists a subseries which diverges in the norm of the Orlicz space $L_\Phi^*(I^n)$:

$$\sum_{\overline{m}=\overline{1}}^{\infty} d_{2^{r\overline{m}}+\overline{1}}(f) \chi_{2^{r\overline{m}}+\overline{1}}(\overline{x}).$$

According to the well-known theorem of Orlicz concerning unconditional convergence (see [6], p. 20) we get our statement.

THEOREM 2. *Let the function Φ satisfy condition* $(*)$. *If $\theta(u) = o(\Phi(u)\ln^n u)$ for $u \to \infty$ is an even, nonnegative and nondecreasing function on* $[0, \infty)$ *and $\Phi(u) = O(\theta(u))$, then the Haar system does not form an unconditional basis in the norm of the Orlicz space $L_\Phi^*(I^n)$ for the class $\theta(L)$.*

Proof. It follows that $\theta(L) \subset \Phi(L)$. It will be seen that by the assumption of the theorem there exists a function from the class $\theta(L)$ with the Haar–Fourier series which does not converge unconditionally in the norm of $L_\Phi^*(I^n)$. One can find such an $N \geqslant 2$ that $\theta(u) > 0$ for $n \geqslant N$. Then $\theta(u) = \varepsilon(u)\Phi(u)\ln^n u$ for $n \geqslant N$, where $\varepsilon(u) \to 0$ for $u \to \infty$. Set

$$\tilde{\varepsilon}(u) = \begin{cases} \sup\limits_{t \geqslant u} \varepsilon(t) & \text{for} \quad u \geqslant N, \\ \varepsilon(N) & \text{for} \quad 0 \leqslant u < N, \end{cases}$$

and

$$\tilde{\theta}(u) = \begin{cases} \varepsilon(u)\Phi(u)\ln^n u & \text{for} \quad u \geqslant N, \\ \theta(N) & \text{for} \quad 0 \leqslant u < N. \end{cases}$$

Since $\tilde{\varepsilon}(u) \geqslant \varepsilon(u)$ for $u \geqslant N$, $\tilde{\varepsilon}(u)\downarrow 0$ for $u \to \infty$, $\tilde{\theta}(u) \geqslant \theta(u)$ for $u > 0$, we can construct according to (18), (21)–(25) the desired sequence $\{a_{\overline{m}}\}$. Thus by Theorem 1 the function $f(\overline{x}, \{a_{\overline{m}}\}) \in L_\Phi^*(I^n)$, and its Haar–Fourier series does not converge unconditionally in the norm of this space. We shall show that $f \in \theta(L)$. Indeed, letting

$$E = \{\overline{x}\colon \overline{x} \in I^n,\ f(\overline{x}) > N\},$$

we infer by (29) and the definition of $\theta(u)$, that

$$\int_{I^n} \theta(f(\overline{x}))\, d\overline{x} \leqslant \Big(\int_{I^n \setminus E} + \int_E\Big) \tilde{\theta}(f(\overline{x}))\, d\overline{x}$$

$$\leqslant \tilde{\theta}(N) + C \sum_{\overline{m}=\overline{1}}^{\infty} 2^{-|\overline{m}|} \varepsilon\big(2^{-|\overline{m}|} \Delta(a_{\overline{m}})\big) \Phi\big(2^{|\overline{m}|}\Delta(a_{\overline{m}})\big) \big(\ln^+ (2^{|\overline{m}|}\Delta(a_{\overline{m}})\big)^n < \infty,$$

and this completes the proof.

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ТЕОРИЯ ЭКСТРЕМАЛЬНЫХ ЗАДАЧ И ТЕОРИЯ ПРИБЛИЖЕНИЙ

В. М. ТИХОМИРОВ

МГУ, Мехмат, Москва 117234, СССР

1. Постановки некоторых экстремальных задач теории приближений

Постановки экстремальных задач сопровождают всю историю теории приближений. Еще в 18 веке Лежандр нашел, выражаясь современным языком, полиномы наименее уклоняющиеся от нуля в метрике пространства $\mathscr{L}_2([-1, 1])$. Точнее говоря, фактически он разрешил следующую проблему минимизации [1]:

$$(1)\qquad f_{r2}(x) = \int_{-1}^{1}\Big(t^r+\sum_{k=1}^{r} x_k t^{k-1}\Big)^2 dt = ||t^r+p_r(t)||_{\mathscr{L}_2([-1,1])} \to \inf.$$

Получившиеся в результате решения задачи (1) полиномы имеют вид:

$$(1')\qquad T_{r2}(t) = \frac{r!}{(2r)!}\frac{d^r}{dt^r}(t^2-1)^r;$$

они пропорциональны полиномам $P_r(t)$, названных *полиномами Лежандра*. Чебышев решил аналогичную задачу в двух других метриках: $C([-1, 1])$ и $\mathscr{L}_1([-1, 1])$. Решением задачи

$$(2)\qquad f_{r\infty}(x) = \max_{t\in[-1,1]}\Big|t^r+\sum_{k=1}^{r} x_k t^{k-1}\Big| = ||t^r+p_r(t)||_{C([-1,1])} \to \inf$$

являются *полиномы Чебышева*: $T_{r\infty}(t) = 2^{-(r-1)}\cos(r\arccos t)$, а решением задачи

$$(3)\qquad f_{r1}(x) = \int_{-1}^{1}\Big|t^r+\sum_{k=1}^{r} x_k t^{k-1}\Big| dt = ||t^r+p_r(t)||_{\mathscr{L}_1([-1,1])} \to \inf$$

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[1] Если X — некоторое множество, $f\colon X\to \boldsymbol{R}$ — функционал на нем, а $C\subset X$ — подмножество X, называемое ограничением, то задача отыскания минимума f на C обозначается далее $f\to\inf;\ x\in C$.