4. Можно думать, что при $q < s$ величины d_N и λ_N связаны с какой-то новой интересной экстремальной проблемой. Но какой? Достоверно, что не с основной изопериметрической задачей из § 2, ибо d_n и λ_n имеет другую асимптотику в сравнении с λ_{nrqs}^{-1}.

5. Можно поставить задачи для многих переменных, подобные решенным нами выше в одновременном случае. Возникают изопериметрические задачи похожие на задачу из § 2. Имеет ли она прямое отношение к теории приближений? Что заменит сплайны в этом случае?

6. Ограничения на градиенты типа включений требуют разработки многомерной теории оптимального управления. Как будет выглядеть здесь принцип максимума?

7. Выше были решены задачи аппроксимации гладких функций. Можно получить аналогичные результаты для гармонических функций заданных в круге. При этом возникают подпространства, подобные сплайнам, но не являющиеся сплайнами. Хотелось бы разработать единую теорию, где сплайны выступали бы как нечто единое для гладких, гармонических, аналитических и т. д. функций

8. Следовало бы дать аналогичную теорию аппроксимации гладких комплексных функций.

9. Нет достаточно содержательной точной теории приближения классов аналитических функций в достаточно общих областях.

10. Совершенно не ясны ответы на простейшие вопросы, касающиеся приближения функций многих комплексных переменных.

Этот обзор составлен на базе четырех лекций, прочитанных автором в Центре им. С. Банаха на семестре по теории аппроксимации в декабре 1975 года. Мне хотелось бы в заключение выразить мою глубокую благодарность организаторам этого семестра и в особенности проф. Ч. Олеху и проф. З. Чисельскому за предоставленную мне возможность принять участие в работе семестра и за дружескую поддержку.

Примечание при корректуре. За истекшее время появилось много работ по этой тематике, в частности Ю. Н. Субботина, Х. Мичелли, А. Пинкуса, Т. Ривлина, С. Винограда и др. Особенно отметим замечательные работы: Ch. Miccelli, A. Pinkus, *On n-widths in* L^∞, Trans. Amer. Math. Soc. 234 (1) (1977), *Total positivity and the exact n-width on certain sets in* L^1, Pacific J. Math. 71 (2) (1977).

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A MULTIPLIER IN BESOV SPACES WHICH IS NOT A MULTIPLIER IN LEBESGUE SPACES

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1. Introduction

If $1 < p < \infty$, then M_p denotes the set of all multipliers in the Lebesgue space L_p, i.e. the set of all essentially bounded Lebesgue-measurable functions $m(x)$ in the n-dimensional Euclidean space such that $F^{-1}mF$ is a bounded operator from L_p into L_p (here F and F^{-1} are the Fourier transform and its inverse, respectively). In the same way one defines $\mathscr{M}_p$, the set of all multipliers in the isotropic Besov spaces $B^s_{p,q}$ (= Lipschitz spaces $\Lambda^s_{p,q}$), where $-\infty < s < \infty$, and $1 \leqslant q \leqslant \infty$ ($\mathscr{M}_p$ does not depend on s or q; [5]). In [5] it was proved that $M_p \subset \mathscr{M}_p$. In [6] we introduced the subclass $\mathscr{M}^H_p$ of $\mathscr{M}_p$ consisting of all multipliers $m(x) \in \mathscr{M}_p$ such that the norms of the multiplier-operators belonging to the multipliers $m(ax)$, where $0 < a < \infty$, are uniformly bounded. We have $M_p = \mathscr{M}^H_p$; [6]. But in the two above-cited papers we did not clarify whether $\mathscr{M}^H_p$ is strictly contained in $\mathscr{M}_p$ or not (or, in other words, whether M_p and $\mathscr{M}_p$ coincide or not). The aim of this paper is to give an explicit example of a multiplier $m(x)$ belonging to $\mathscr{M}_p$ but not to M_p.

2. Definitions

$\boldsymbol{R}_n$ is the n-dimensional real Euclidean space; its general point is denoted by $x = (x_1, \ldots, x_n)$. As usual, S is the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions in $\boldsymbol{R}_n$; its dual space S' is the space of tempered distributions. F and F^{-1} are the Fourier transform and its inverse, respectively. If $f \in S$, then

$$(Ff)(\xi) = (2\pi)^{-n/2} \int_{\boldsymbol{R}_n} e^{-ix\xi} f(x)\, dx; \qquad x\xi = \sum_{j=1}^{n} x_j \xi_j.$$

A corresponding formula holds for F^{-1}; one must replace $-i$ by i in the last formula.

If $1 \leqslant p \leqslant \infty$, then L_p is the usual space of all complex-valued Lebesgue-measurable functions defined in R_n and such that

$$\|f\|_{L_p} = \begin{cases} \left(\int_{R_n} |f(x)|^p dx\right)^{1/p} & \text{for} \quad 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{x \in R_n} |f(x)| & \text{for} \quad p = \infty \end{cases}$$

is finite.

Let

$$Q_0 = \{x \mid -2 \leqslant x_j \leqslant 2\}$$

and

$$Q_k = \{x \mid -2^{k+1} \leqslant x_j \leqslant 2^{k+1}\} - \{x \mid -2^k \leqslant x_j \leqslant 2^k\},$$

$k = 1, 2, 3, \ldots$ If χ_k is the characteristic function of Q_k, then, by definition,

$$(1) \qquad B^s_{p,q} = \Big\{f \mid f \in S',\ \|f\|_{B^s_{p,q}} = \Big(\sum_{k=0}^{\infty} 2^{sqk} \|F^{-1}\chi_k Ff\|^q_{L_p}\Big)^{1/q} < \infty\Big\}.$$

Here $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$. (For $q = \infty$ the l_q-norm in (1) must be replaced by the l_∞-norm.) These are the usual Besov spaces; see [2], p. 374, or [4], Lemma 2.11.2.

If $1 < p < \infty$, then $\mathscr{M}_p$ denotes the set of all tempered distributions m for which there exists a positive number C (depending on m) such that

$$(2) \qquad \|F^{-1}mFf\|_{B^s_{p,q}} \leqslant C \|f\|_{B^s_{p,q}},$$

for all $f \in S$. Here s, $-\infty < s < \infty$, and q, $1 \leqslant q < \infty$, are fixed. It was shown in [5] that $\mathscr{M}_p$ is independent of the choice of s and q (for $q = \infty$ a small modification of this definition is necessary, but this is unimportant here: we refer to [5]). Similarly one defines M_p, the set of all multipliers in L_p: one must replace $B^s_{p,q}$ in (2) by L_p. [In the same way one can define multipliers in the Lebesgue space (= Bessel-potential space = Liouville space) H^s_p, where $-\infty < s < \infty$ and $1 < p < \infty$. But it is easy to see that with a fixed p the set of all these multipliers is independent of s: it coincides with M_p.]

3. $\mathscr{M}_p$ and M_p

In [5] it was proved that if $1 < p < \infty$ and if $1/p + 1/p' = 1$, then

$$(3) \qquad M_p \subset \mathscr{M}_p,$$

$$(4) \qquad \mathscr{M}_p = \mathscr{M}_{p'},$$

$$(5) \qquad \mathscr{M}_p \subset \mathscr{M}_q \subset \mathscr{M}_2 = M_2 = L_\infty \quad \text{for} \quad |1/q - 1/2| \leqslant |1/p - 1/2|.$$

In particular, all multipliers belonging to $\mathscr{M}_p$ are essentially bounded Lebesgue-measurable functions. (4) and (5) are counterparts of the well-known relations, [1].

$$(6) \qquad M_p = M_{p'},$$

$$(7) \qquad M_p \subset M_q \subset M_2 = L_\infty \quad \text{for} \quad |1/q - 1/2| \leqslant |1/p - 1/2|.$$

The aim of this paper is to prove that M_p is strictly contained in $\mathscr{M}_p$.

THEOREM. *Let $1 < p < \infty$ and $p \neq 2$. Let $\psi_k(x)$ be the characteristic function of the cube*

$$\{x \mid -1 \leqslant x_j \leqslant 1 \text{ for } j = 1, \ldots, n-1; \text{ and } 2^k \leqslant x_n \leqslant 2^k + 2\},$$

$k = 1, 2, 3, \ldots$ Then

$$(8) \qquad m(x) = \sum_{k=1}^{\infty} e^{i2^k x_n} \psi_k(x)$$

belongs to $\mathscr{M}_p$, but does not belong to M_p.

Proof. Step 1. Prove $m \in \mathscr{M}_p$. By (1) it is sufficient to prove

$$(9) \qquad \|F^{-1}\chi_k mFf\|_{L_p} \leqslant C \|F^{-1}\chi_k Ff\|_{L_p}, \quad k = 1, 2, 3, \ldots,$$

where C is independent of k. Using the well-known multiplier properties for characteristic functions of cubes, [3], IV, § 4, we infer (9) from

$$\|F^{-1}\chi_k mFf\|_{L_p} = \|F^{-1}e^{i2^k x_n}\psi_k Ff\|_{L_p} = \|F^{-1}\psi_k Ff\|_{L_p}$$
$$= \|F^{-1}\psi_k F(F^{-1}\chi_k Ff)\|_{L_p} \leqslant C \|F^{-1}\chi_k Ff\|_{L_p}.$$

Step 2. Prove $m \notin M_p$. Let f_N be given by its Fourier transform

$$(10) \qquad Ff_N = \sum_{k=1}^{N} e^{-i2^k x_n} \psi_k(x).$$

Here $N = 1, 2, 3, \ldots$ Using the explicit formulas for $F^{-1}\psi_k$, [2], (1.5.7/10), we obtain

$$(11) \qquad f_N(x) = \sum_{k=1}^{N} (F^{-1}\psi_k)(x_1, \ldots, x_{n-1}, x_n - 2^k)$$

$$= c \sum_{k=1}^{N} e^{-ix_n(2^k+1)} \frac{\sin(x_n - 2^k)}{x_n - 2^k} \prod_{j=1}^{n-1} \frac{\sin x_j}{x_j} \in L_p.$$

The last formula yields

$$(12) \qquad \|f_N\|_{L_\infty} \leqslant c \sup_{x_n \in R_1} \sum_{k=1}^{\infty} \left| \frac{\sin(x_n - 2^k)}{x_n - 2^k} \right| \leqslant C,$$

where C is independent of N. Using $\|f_N\|_{L_2} = \|Ff_N\|_{L_2}$, we infer from (10)

$$\|f_N\|_{L_2} = 2^{n/2} N^{1/2}.$$

If $2 < p < \infty$, then one obtains

$$(13) \qquad \|f_N\|_{L_p} \leqslant \|f_N\|^{1-2/p}_{L_\infty} \|f_N\|^{2/p}_{L_2} \leqslant C' N^{1/p},$$

where C' is independent of N. On the other hand, using (10) and a Paley–Littlewood theorem for L_p; [2], 1.5.6, we have

$$\|F^{-1}mFf_N\|_{L_p} = \Big\|F^{-1}\sum_{k=1}^{N}\psi_k\Big\|_{L_p} \tag{14}$$

$$\geqslant c\Big\|\Big(\sum_{k=1}^{N}|(F^{-1}\psi_k)(x)|^2\Big)^{1/2}\Big\|_{L_p}$$

$$= c'\Big\|N^{1/2}\prod_{j=1}^{n}\frac{\sin x_j}{x_j}\Big\|_{L_p} = c''N^{1/2},$$

where c, c', and c'' are appropriate positive numbers independent of N. Assume that m belongs to M_p for $2 < p < \infty$. Then it follows from the definition of multipliers in L_p and from (13) and (14) that there exists a positive number C, such that for all $N = 1, 2, 3, \ldots$ we have

$$N^{1/2} \leqslant c_1\|F^{-1}mFf_N\|_{L_p} \leqslant c_2\|f_N\|_{L_p} \leqslant CN^{1/p}.$$

This is impossible. Hence, $m \notin M_p$ for $2 < p < \infty$. By (6) we also have $m \notin M_p$ for $1 < p < 2$. This proves the theorem.

Remarks. 1. The proof is based essentially on (12) and (14). Formulas of type (14) are also useful for the proof that $B^0_{p,p}$ (more generally $B^0_{p,q}$ for $q \neq 2$) and L_p do not coincide. Such an argumentation is due to O. V. Besov, see [2], 9.7. We refer also to [4], 2.12, and the papers cited there. But, obviously, this is also a consequence of the above theorem: since $\mathcal{M}_p \neq M_p$ for $p \neq 2$ it follows immediately that $L_p \neq B^0_{p,q}$ for $1 < p < \infty$, $p \neq 2$, and $1 \leqslant q \leqslant \infty$ (including $q = 2$).

2. As remarked above in [6] a subclass $\mathcal{M}_p^H$ of $\mathcal{M}_p$ was introduced by the following definition: $m(x) \in \mathcal{M}_p$ belongs to $\mathcal{M}_p^H$ if and only if there exists a number C such that for all a, $0 < a < \infty$, and all $f \in S$ we have

$$\|F^{-1}m(ax)Ff\,\|B^s_{p,q} \leqslant C\|f\,\|B^s_{p,q}. \tag{15}$$

(Again s and q are fixed, $-\infty < s < \infty$, $1 \leqslant q \leqslant \infty$, but $\mathcal{M}_p^H$ is independent of the choice of s and q.) (15) is a homogeneity property. Multipliers belonging to M_p satisfy such a homogeneity property automatically. In [6] it was proved that $\mathcal{M}_p^H = M_p$. Hence, M_p coincides with the set of all "homogeneous" multipliers of $\mathcal{M}_p$. The theorem can be re-formulated by $\mathcal{M}_p \neq \mathcal{M}_p^H$. Consequently, $m(x)$ defined by (8), does not satisfy (15) uniformly with respect to a.

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