APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

EQUIVALENCE, UNCONDITIONALITY AND CONVERGENCE A.E. OF THE SPLINE BASES IN L_p SPACES

Z. CIESIELSKI

Polish Academy of Sciences, Branch in Gdańsk, Institute of Mathematics, 81-825 Sopot, Poland

1. Introduction

In the joint paper by P. Simon, P. Sjölin, and the author [5] it was shown that the Franklin and Haar bases in $L_p(0, 1)$, $1 < p < \infty$, are equivalent, and a simplified proof of the result of S. B. Bočkariev [1] on the unconditionality of the Franklin system in $L_p(0, 1)$ was presented. Moreover, the convergence a.e. of the Fourier series with respect to the bounded orthonormal set of polygonals for functions in $L_p(0, 1)$, $1 < p < \infty$, was established. We have suggested in [5] the possibility of extending all the results to spline systems of higher orders. The main goal of this paper is to carry out this programm. The ideas of the presented proofs are not in principle new compared with [5]. However, in order to have more or less complete theory of the spline systems it seems necessary to publish this paper in addition to the works by Z. Ciesielski and J. Domsta [4], Z. Ciesielski [3], J. Domsta [6], and S. Ropela [9], [10], [11].

The main results of this paper require some comments. Theorem 3.1 is the crucial one. It implies in particular the unconditionality of the Haar and Franklin orthonormal sets in $L_p(0, 1)$ spaces $1 < p < \infty$ (cf. Corollary 3.1). Moreover, as a consequence we obtain Theorem 3.2 as well. The second non-trivial result is Theorem 5.1, and it follows essentially from Theorem 3.2 and the highly non-trivial maximal inequality for Walsh system proved by P. Sjölin in [12]. One would expect that a result like Theorem 3.1 should give Theorem 4.1. We were able to prove the equivalence of the spline bases by means of the C. L. Fefferman and E. M. Stein inequalities [7] only.

2. Preliminaries

The simplest way of defining the spline systems, in which we are interested, is by the Haar orthonormal functions χ_n, $n = 1, 2, \ldots$, given on $I = \langle 0, 1\rangle$. To do this let us denote by D the differantiation operator and let us define the following integration operators:

$$(Gf)(t) = \int_0^t f(u)\,du, \qquad (Hf)(t) = \int_t^1 f(u)\,du,$$

and the usual "scalar product"

$$(f, g) = \int_I fg, \qquad f \in L_p(I),\ g \in L_q(I),$$

where $1 \leqslant p,\ q \leqslant \infty$ and $p^{-1}+q^{-1} = 1$.

Let m be an arbitrary but fixed integer, $m \geqslant -1$.

It is clear that the functions $1, t, \ldots, t^{m+1}$, $G^{m+1}\chi_n(t)$, $n \geqslant 2$, are linearly independent over I. The Schmidt orthonormalization procedure applied to this set of functions gives the orthonormal set of splines $\{f_n^{(m)},\ n \geqslant -m\}$ of order (of smoothness) m. Following now [3] we define the remaing spline systems as follows. For $0 \leqslant k \leqslant m+1$ we set

$$f_n^{(m,k)} = D^k f_n^{(m)}, \qquad n \geqslant k-m,$$

and

$$g_n^{(m,k)} = H^k f_n^{(m)}, \qquad n \geqslant k-m.$$

For our purpose it is more convenient to have more unified notation for the systems $\{f_n^{(m,k)},\ n \geqslant k-m\}$ and $\{g_n^{(m,k)},\ n \geqslant k-m\}$ where $0 \leqslant k \leqslant m+1$. To compare these spline systems as bases in $L_p(I)$ it is good to normalize them suitably in the L_2 norm. Thus, let us define for $|k| \leqslant m+1$ and $n \geqslant |k|-m$

$$h_n^{(m,k)} = \begin{cases} f_n^{(m,k)} \| f_n^{(m,k)}\|_2^{-1} & \text{for} \quad 0 \leqslant k \leqslant m+1, \\ g_n^{(m,-k)} \| f_n^{(m,-k)}\|_2 & \text{for} \quad 0 \leqslant -k \leqslant m+1, \end{cases}$$

where

$$\|f\|_p = \Big(\int_I |f|^p\Big)^{1/p}, \qquad 1 \leqslant p \leqslant \infty.$$

With the help of the results established in [3] it is not hard to derive for $|k| \leqslant m+1$, $1 \leqslant p \leqslant \infty$, the following properties:

(1) *There is a constant C_m, depending on m only, such that*

$$n^{1/2-1/p} C_m^{-1} \leqslant \|h_n^{(m,k)}\|_p \leqslant C_m n^{1/2-1/p}.$$

(2) *The set $\{h_i^{(m,k)},\ h_j^{(m,-k)},\ i, j \geqslant |k|-m\}$ is biorthogonal, i.e. $(h_i^{(m,k)}, h_j^{(m,-k)}) = \delta_{ij}$ for $i, j \geqslant |k|-m$.*

(3) *The system $\{h_n^{(m,k)},\ n \geqslant |k|-m\}$ is a basis in $L_p(I)$, $1 \leqslant p < \infty$.*

(4) *Let $n = 2^\mu+\nu$, $1 \leqslant \nu \leqslant 2^\mu$, $t_n = (2\nu-1)2^{-(\mu+1)}$ and let $\|h_n^{(m,k)}\|_\infty = |h_n^{(m,k)}(s_n^{(m,k)})|$. Then $|t_n - s_n^{(m,k)}| = O(1/n)$ for large n.*

(5) *There are constants C_m and q_m, $0 < q_m < 1$, such that*

$$|h_n^{(m,k)}(t)| \leqslant C_m n^{1/2} q_m^{n|t-t_n|}$$

holds for $n \geqslant 1$, $t \in I$.

(6) *There is a constant C_m such that*

$$\sum_{2^\mu < n \leqslant 2^{\mu+1}} |h_n^{(m,k)}(t)| \leqslant C_m 2^{\mu/2}$$

holds for $|k| \leqslant m+1$, $\mu \geqslant 0$ and $t \in I$.

LEMMA 2.1 (S. Ropela [10]). *Let the integers m and k be given such that $|k| \leqslant m+1$. Then $\{h_n^{(m,k)},\ n \geqslant |k|-m\}$ is an unconditional basis, i.e. a Riesz basis in $L_2(I)$.*

LEMMA 2.2. *Let m, m', k, and k' be given integers such that $|k| \leqslant m+1$ and $|k'| \leqslant m'+1$. Then there is a constant $C_{m,m'}$, such that*

$$\sum_{|t-s| > 2^{-\mu}} 2^\mu \sum_{2^\mu < n \leqslant 2^{\mu+1}} |h_n^{(m,k)}(t)\, h_n^{(m',k')}(s)| \leqslant \frac{C_{m,m'}}{|t-s|^2}$$

holds for $t, s \in I$.

Proof. To each $n \geqslant 2$, $n = 2^\mu+\nu$, $1 \leqslant \nu \leqslant 2^\mu$, there corresponds the dyadic partition

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for} \quad i \leqslant 2\nu, \\[2mm] \dfrac{i-\nu}{2^{\mu+1}} & \text{for} \quad i > 2\nu. \end{cases}$$

Now, for given $t \in I$, $t > 0$, let $\langle t\rangle$ denote the unique solution of the inequality $s_{n,\langle t\rangle-1} < t \leqslant s_{n,\langle t\rangle}$. Then property (5) of $h_n^{(m,k)}$ gives

$$|h_n^{(m,k)}(t)| \leqslant C_m 2^{\mu/2} q_m^{|\langle t\rangle-\nu|}.$$

Consequently, for some $C_{m,m'}$ and $q_{m,m'}$ with $\max(q_m, q_{m'}) < q_{m,m'} < 1$, we have

$$\sum_{2^\mu < n \leqslant 2^{\mu+1}} |h_n^{(m,k)}(t)\, h_n^{(m',k')}(s)| \leqslant C_m C_{m'} 2^\mu \sum_{\nu=1}^{2^\mu} q_m^{|\langle t\rangle-\nu|} q_m^{|\langle s\rangle-\nu|} \leqslant C_{m,m'} 2^\mu q_{m,m'}^{2^\mu|t-s|}.$$

On the other hand, there is a constant $C_{m,m'}$ such that

$$\sum_{|t-s| > 2^{-\mu}} 2^{2\mu} q_{m,m'}^{2^\mu|t-s|} \leqslant \frac{C_{m,m'}}{|t-s|^2}.$$

Combining these two inequalities we complete the proof.

The *Walsh system* $\{w_n,\ n \geqslant 1\}$ is a bounded orthonormal set related to the Haar system $\{\chi_n,\ n \geqslant 1\}$ as follows:

$$w_1 = \chi_1$$

and

$$w_{2^\mu+\nu} = \sum_{\lambda=1}^{2^\mu} A_{\nu,\lambda}^{(\mu)} \chi_{2^\mu+\lambda}, \qquad 1 \leqslant \nu \leqslant 2^\mu,$$

$$\chi_{2^\mu+\lambda} = \sum_{\nu=1}^{2^\mu} A_{\nu,\lambda}^{(\mu)} w_{2^\mu+\nu}, \qquad 1 \leqslant \lambda \leqslant 2^\mu,$$

where $A_{\nu\lambda}^{(\mu)} = (w_{2^\mu+\nu}, \chi_{2^\mu+\lambda})$, $A_{\nu,\lambda}^{(\mu)} = A_{\lambda,\nu}^{(\mu)}$.

Starting with the system $\{h_n^{(m,k)}, n \geqslant |k|-m\}$ instead of the Haar system, we define $w_n^{(m,k)}$, $n \geqslant |k|-m$, in the same way as w_n, i.e. $w_n^{(m,k)} = h_n^{(m,k)}$ for $n = |k|-m, \ldots, 1$, and (cf. [2] and [9])

$$w_{2^\mu+\nu}^{(m,k)} = \sum_{\lambda=1}^{2^\mu} A_{\nu,\lambda}^{(\mu)} h_{2^\mu+\lambda}^{(m,(k)}, \quad 1 \leqslant \nu \leqslant 2^\mu.$$

It now follows that

$$h_{2^\mu+\lambda}^{(m,k)} = \sum_{\nu=1}^{2^\mu} A_{\nu,\lambda}^{(\mu)} w_{2^\mu+\nu}^{(m,k)}, \quad 1 \leqslant \lambda \leqslant 2^\mu.$$

Since $A_{\lambda,\nu}^{(\mu)} = \pm 2^{-\mu/2}$, we obtain by property (6) of $\{h_n^{(m,k)}\}$ that for some constant C_m

$$|w_n^{(m,k)}(t)| \leqslant C_m, \quad n \geqslant |k|-m, \ t \in I.$$

Thus, $\{w_n^{(m,k)}, n \geqslant |k|-m\}$, $|k| \leqslant m+1$, is a set of splines of order $m-k$ uniformly bounded on I. According to property (2) we find that the set $\{w_i^{(m,k)}, w_j^{(m,-k)}, i,j \geqslant |k|-m\}$ is biorthogonal whenever $|k| \leqslant m+1$.

LEMMA 2.3 (S. Ropela [9]). *Let the integers m and k be given such that $|k| \leqslant m+1$ and let $1 < p < \infty$. Then $\{w_n^{(m,k)}, n \geqslant |k|-m\}$ is a basis in $L_p(I)$.*

It seems to be a good place to identify the spline systems for particular choise of the parameters m and k with some of the known systems:

$\{h_n^{(-1,0)}\} \equiv \{\chi_n\}$	the Haar system,
$\{w_n^{(-1,0)}\} \equiv \{w_n\}$	the Walsh system,
$\{h^{(0,0)}\} = \{f_n\}$	the Franklin system,
$\{w_n^{(0,0)}\} = \{c_n\}$	for this system see [2].

For the proof of the equivalence of the spline bases we need an inequality of C. L. Fefferman and E. M. Stein [7]. To state it let us recall the definition of the maximal function. If $f \in L_1(I)$, then the *maximal function* is defined as

$$(Mf)(t) = \sup_\omega \frac{1}{|\omega|} \int_\omega |f|,$$

where the supremum is taken over all intervals $\omega \subset I$ such that $t \in \omega$.

LEMMA 2.4 (C. L. Fefferman and E. M. Stein [7]). *Let $1 < p < \infty$ and let $g_1, g_2, \ldots$ be a sequence of functions in $L_p(I)$ with the property that*

$$\Big(\sum_n |g_n|^2\Big)^{1/2} \in L_p(I).$$

Then there is a constant C_p depending on p only such that

$$\Big\|\Big(\sum_n (Mg_n)^2\Big)^{1/2}\Big\|_p \leqslant C_p \Big\|\Big(\sum_n |g_n|^2\Big)^{1/2}\Big\|_p.$$

In the proof of the maximal inequality for the basis $\{w_n^{(m,k)}, n \geqslant |k|-m\}$ an important role is played by

LEMMA 2.5. *Let m and k, $|k| \leqslant m+1$, be given, and let*

$$H_n^{(m,k)} f = \sum_{j=|k|-m}^{n} (f, h_j^{(m,-k)}) h_j^{(m,k)}.$$

Then there is a constant C_m such that

$$H_*^{(m,k)} f \leqslant C_m Mf, \quad f \in L_1(I),$$

where

$$H_*^{(m,k)} f = \sup_n |H_n^{(m,k)} f|.$$

For $0 \leqslant k \leqslant m+1$ the lemma was established in [3]. In the case of k: $-(m+1) \leqslant k \leqslant 0$, the proof is quite similar and therefore it is omitted.

Now, the well-known Hardy–Littlewood maximal inequality, (cf. [13], p. 5), combined with Lemma 2.5, gives

COROLLARY 2.1. *Let $|k| \leqslant m+1$ and let $1 < p < \infty$. Then for some C_m we have*

$$\|H_*^{(m,k)} f\|_p \leqslant C_m \frac{p}{p-1} \|f\|_p$$

and

$$\|H_*^{(m,k)} f\|_1 \leqslant C_m \||f| \log^+ |f|\|_1 + C_m.$$

3. Unconditionality of the basis $\{h_n^{(m,k)}\}$ in $L_p(I)$, $1 < p < \infty$

For given m and k, $|k| \leqslant m+1$, a function f is called *(m, k)-polynomial* if it is of the form

$$f = \sum_{j=|k|-m}^{n} a_j h_j^{(m,k)}.$$

Now, let the integers m, k, m', and k' be given such that $|k| \leqslant m+1$, $|k'| \leqslant m'+1$, and let $\varepsilon = \{\varepsilon_j = \pm 1, \ j \geqslant (|k|-m) \vee (|k'|-m')\}$ where $a \vee b = \max(a, b)$. The operator T is defined on $(m', -k')$-polynomials as follows

$$Tf = \sum_{j=(|k|-m)\vee(|k'|-m')}^{\infty} \varepsilon_j (f, h_j^{(m',k')}) h_j^{(m,k)}.$$

Let us notice that, according to Lemma 2.1, Tf is well defined for f in $L_2(I)$, as well.

THEOREM 3.1. *There exists a constant $C_{m,m'}$ such that*

$$|\{t\colon |Tf(t)| > y\}| \leqslant C_{m,m'} \frac{\|f\|_1}{y}, \quad y > 0, \tag{3.1}$$

holds for all $(m', -k')$-polynomials f, i.e. T is of weak type (1.1).

This theorem is being proved in almost the same way as Bočkariev's theorem in [5], and the proof below is given simply for the sake of completeness.

Proof. For fixed $y > 0$ let us define $Q = \{t \in I\colon Mf(t) > y\}$ and $P = I \setminus Q$. Since the operator M is of weak type (1.1), we have (cf. [13], p. 5)

$$|Q| \leqslant \frac{5}{y}\,||f||_1. \tag{3.2}$$

Notice that (3.1) holds with $C_{m,m'} = 5$ if only $0 < y \leqslant 5||f||_1$. However, in the opposite case, P is non-empty, and in this case let $(Q_i)_1^\infty$ be a Whitney decomposition of Q (see [13], pp. 167–168), i.e. each $Q_i = \langle\alpha_i, \beta_i\rangle$ is a dyadic interval of the form $\langle 2^{-\mu}(\nu-1), 2^{-\mu}\nu\rangle$ and

$$Q = \bigcup_{i=1}^{\infty} Q_i, \quad \operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset, \quad i \neq j, \tag{3.3}$$

$$|Q_i| \leqslant \operatorname{dist}(Q_i, P) \leqslant 4|Q_i|. \tag{3.4}$$

Now, (3.4) and the definition of P give

$$\int_{Q_i} |f| \leqslant 5y|Q_i|. \tag{3.5}$$

The next step is to decompose f in suitable way into a sum of two functions f_1 and f_2. To do this let T_i denote the orthogonal projection of $L_2(Q_i)$ onto the $(d+1)$-dimensional subspace spanned by $1, t, \ldots, t^d$ restricted to Q_i, where $d = 2[(m+1) \vee (m'+1)]$. Let

$$f_1(t) = \begin{cases} f(t) & \text{for} \quad t \in P, \\ T_i f(t) & \text{for} \quad t \in \operatorname{int} Q_i,\ i = 1, 2, \ldots, \\ 0 & \text{for} \quad t \in Q \setminus \bigcup_i \operatorname{int} Q_i, \end{cases}$$

and let $f_2 = f - f_1$.

The projections T_i can be represented in terms of the Legendre orthonormal polynomials $l_0, \ldots, l_d$, given on I. Indeed, let

$$l_{j,i}(t) = |Q_i|^{-1/2} l_j\left(\frac{t-\alpha_i}{|Q_i|}\right), \quad t \in Q_i;$$

then

$$T_i f(t) = \int_{Q_i} f(s)\Big(\sum_{j=0}^{d} l_{j,i}(t) l_{j,i}(s)\Big)ds, \quad t \in Q_i,$$

whence we infer

$$|T_i f(t)| \leqslant \frac{C_d}{|Q_i|}\int_{Q_i} |f|, \quad t \in Q_i,$$

where

$$C_d = \max_{s,t \in I}\Big|\sum_{j=0}^{d} l_j(t) l_j(s)\Big|.$$

Thus, the definitions of P and f_1, and (3.5) give

$$|f_1(t)| \leqslant 5C_d y \text{ a.e. in } t \in I. \tag{3.6}$$

The following properties of f_2 are going to be needed. According to definition, $f_2(t) = 0$ for $t \in P$, and f_2 restricted to Q_i is orthogonal to all polynomials of degree at most d, i.e.

$$\int_{Q_i} f_2 w = 0, \quad i = 1, 2, \ldots, \tag{3.7}$$

where w is a polynomial of degree less than or equal to d. Moreover, since $f_2 = f - f_1$, we have from (3.5) and (3.6)

$$\int_{Q_i} |f_2| \leqslant 5(C_d+1)y|Q_i|, \quad i = 1, 2, \ldots \tag{3.8}$$

The estimate for Tf_1. According to Lemma 2.1 we have

$$||Tf_1||_2^2 \sim \sum_{j=(|k|-m)\vee(|k'|-m')}^{\infty} (f_1, h_j^{(m',k')})^2 \leqslant \sum_{j=|k'|-m'}^{\infty} (f_1, h_j^{(m',k')})^2 \sim ||f_1||_2^2,$$

and therefore for some constant $C_{m,m'}$ we have

$$||Tf_1||_2 \leqslant C_{m,m'}||f_1||_2. \tag{3.9}$$

Consequently, (3.9), (3.6), and (3.2) give

$$|\{t\colon |Tf_1(t)| > y\}| \leqslant \frac{||Tf_1||_2^2}{y^2} \leqslant \frac{C_{m,m'}}{y^2}||f_1||_2^2 = \frac{C_{m,m'}}{y^2}\Big(\int_P |f_1|^2 + \int_Q |f_1|^2\Big)$$

$$\leqslant \frac{C_{m,m'}}{y}\Big(\int_P |f_1| + \int_Q |f_1|\Big) \leqslant \frac{C_{m,m'}}{y}\Big(\int_P |f_1| + y|Q|\Big) \leqslant \frac{C_{m,m'}}{y}||f||_1.$$

We denote, here and later on in each step, by the same letter different constants.

The estimate for Tf_2. Let us expand f_2 with respect to $\{h_j^{(m',-k')}, j \geqslant |k'|-m'\}$; f_2 is bounded, so we have

$$f_2 = \sum_{j=|k'|-m'}^{\infty} b_j h_j^{(m',-k')}$$

where $b_n = (f_2, h_n^{(m',k')})$. Since f_2 vanishes on P, it follows by (3.7) that $b_{|k'|-m'} = \ldots = b_1 = 0$, and consequently for $n \geqslant 2$ we have

$$b_n = \sum_{i=1}^{\infty} \int_{Q_i} f_2 h_n^{(m',k')}.$$

Now, if $2^\mu < n \leqslant 2^{\mu+1}$, then $h_n^{(m',k')}$ is a polynomial of degree at most $2(m'+1) \leqslant d$ on each dyadic interval of length $2^{-(\mu+1)}$. Thus, according to (3.7), for an i such that $|Q_i| \leqslant 2^{-(\mu+1)}$ we have

$$\int_{Q_i} f_2 h_n^{(m',k')} = 0,$$

and therefore

$$b_n = \sum_{i:\,|Q_i|\geqslant 2^{-\mu}} \int_{Q_i} f_2 h_n^{(m',k')}.$$

Consequently,

$$|Tf_2(t)| = \Big|\sum_{n=(|k|-m)\vee(|k'|-m')}^{\infty} \varepsilon_n b_n h_n^{(m,k)}(t)\Big| \leqslant \sum_{\mu=0}^{\infty}\sum_{2^\mu<n\leqslant 2^{\mu+1}} |b_n h_n^{(m,k)}(t)|$$

$$\leqslant \sum_{\mu=0}^{\infty}\sum_{2^\mu<n\leqslant 2^{\mu+1}} |h_n^{(m,k)}(t)| \sum_{i:\,|Q_i|\geqslant 2^{-\mu}} \int_{Q_i} |f_2(s) h_n^{(m',k')}(s)|\,ds$$

$$= \sum_{i=1}^{\infty}\int_{Q_i} |f_2(s)| \Big(\sum_{\mu:\,|Q_i|\geqslant 2^{-\mu}} \sum_{2^\mu<n\leqslant 2^{\mu+1}} |h_n^{(m,k)}(t) h_n^{(m',k')}(s)|\Big)\,ds$$

$$\leqslant \sum_{i=1}^{\infty}|Q_i| \int_{Q_i} |f_2(s)| \Big(\sum_{\mu;\,|Q_i|\geqslant 2^{-\mu}} 2^\mu \sum_{2^\mu<n\leqslant 2^{\mu+1}} |h_n^{(m,k)}(t) h_n^{(m',k')}(s)|\Big)\,ds.$$

Let us now assume that $t\in P$ and $s\in Q_i$. Then (3.4) implies that $|t-s|\geqslant |Q_i|$, and therefore, for $t\in P$, by Lemma 2.2 and by (3.8)

$$|Tf_2(t)| \leqslant \sum_{i=1}^{\infty}|Q_i| \int_{Q_i} |f_2(s)| \Big(\sum_{|t-s|\geqslant 2^{-\mu}} 2^\mu \sum_{2^\mu<n\leqslant 2^{\mu+1}} |h_n^{(m,k)}(t) h_n^{(m'k')}(s)|\Big)$$

$$\leqslant C_{m,m'} \sum_{i=1}^{\infty}|Q_i| \int_{Q_i} |f_2(s)| \frac{1}{|t-s|^2}\,ds$$

$$\leqslant C_{m,m'} \sum_{i=1}^{\infty}|Q_i|\,[\mathrm{dist}(t,Q_i)]^{-2} \int_{Q_i} |f_2|$$

$$\leqslant C_{m,m'}y \sum_{i=1}^{\infty}|Q_i|^2[\mathrm{dist}(t,Q_i)]^{-2}.$$

Notice that (3.4) implies $|Q_i|\leqslant \mathrm{dist}(s,P)$ for $s\in Q_i$ and that $\mathrm{dist}(t,Q_i)\geqslant |t-s|/2$ for $s\in Q_i$, $t\in P$. Thus, for each i we have

$$|Q_i|^2[\mathrm{dist}(t,Q_i)]^{-2} \leqslant 4\int_{Q_i} \frac{\mathrm{dist}(s,P)}{|t-s|^2}\,ds,$$

and therefore

$$|Tf_2(t)| \leqslant C_{m,m'}y \int_{Q} \frac{\mathrm{dist}(s,P)}{|t-s|^2}\,ds, \qquad t\in P.$$

It now follows from a property of Marcinkiewicz integral (see [13], pp. 14–15) that for some constant $C_{m,m'}$

$$\int_P |Tf_2| \leqslant C_{m,m'}y|Q|.$$

This and (3.2) give finally

$$|\{t\in I:\ |Tf_2(t)|>y\}| \leqslant |Q|+|\{t\in P:\ |Tf_2(t)|>y\}|$$
$$\leqslant |Q|+\frac{1}{y}\int_P |Tf_2| \leqslant C_{m,m'}|Q|$$
$$\leqslant C_{m,m'}\frac{\|f\|_1}{y},$$

and therefore the proof is complete.

THEOREM 3.2. *Let p, m, k, m', k' and ε be given such that $1<p<\infty$, $|k|\leqslant m+1$ and $|k'|\leqslant m'+1$. Then there exists a constant $C_{m,m'}$ such that*

$$\|Tf\|_p \leqslant C_{m,m'}\frac{p^2}{p-1}\|f\|_p. \tag{3.10}$$

Proof. Inequality (3.10) for $1<p<2$ is a consequence of Lemma 2.1, Theorem 3.1, and the interpolation theorem of Marcinkiewicz (cf. [14], vol. II, (4.6)). Let now $T^*\colon L_q(I)\to L_q(I)$, $p^{-1}+q^{-1}=1$, denote the conjugate operation to T. It is easily seen that

$$T^*g = \sum_{j=(|k|-m)\vee(|k'|-m')}^{\infty} \varepsilon_j(g, h_j^{(m,k)})h_j^{(m',k')}, \qquad g\in L_q(I).$$

According to Theorem 3.1, T^* is of weak type (1, 1) and, by Lemma 2.1, it is of strong type (2, 2) and therefore we have

$$\|T^*g\|_p \leqslant C_{m,m'}\frac{1}{p-1}\|g\|_p, \qquad 1<p\leqslant 2.$$

Now, simple conjugacy argument gives (3.10) for all p, $1<p<\infty$.

COROLLARY 3.1. *Let m, k, and p be given such that $|k|\leqslant m+1$, $1<p<\infty$. Then $\{h_n^{(m,k)},\ n\geqslant |k|-m\}$ is an unconditional basis in $L_p(I)$, and for some constant C_m we have*

$$\Big\|\sum_{j=|k|-m}^{\infty} \pm a_j h_j^{(m,k)}\Big\|_p \leqslant C_m\frac{p^2}{p-1}\Big\|\sum_{j=|k|-m}^{\infty} a_j h_j^{(m,k)}\Big\|_p.$$

To obtain Corollary 3.1 we use Theorem 3.2 with $m=m'$ and $k=-k'$.

Remarks. Corollary 3.1 gives in particular the unconditionality of Haar (J. Marcinkiewicz [8]) and Franklin (S. V. Bočkariev [1]) bases in $L_p(I)$, $1<p<\infty$. The first case corresponds to $m=-1$, $k=0$, and the second case to $m=k=0$. If

m is arbitrary and $k = 0$, then Corollary 3.1 can be obtained by direct extension of Bočkariev's result (cf. S. Ropela [11]).

COROLLARY 3.2. *Let* $|k| \leqslant m+1$, $1 < p < \infty$, *and let for* $f \in L_p(I)$

$$f = \sum_{n=|k|-m}^{\infty} a_n h_n^{(m,k)}.$$

Then there is a constant $C_{m,p}$ *such that*

$$C_{m,p}^{-1}\|f\|_p \leqslant \Big\|\Big(\sum_{n=|k|-m}^{\infty} (a_n h_n^{(m,k)})^2\Big)^{1/2}\Big\|_p \leqslant C_{m,p}\|f\|_p.$$

This follows from Corollary 3.1 by the known argument with the Khinchine inequality.

4. Equivalence of the spline bases in the $L_p(I)$ spaces

The main object of this section is to prove for given p, $1 < p < \infty$, the equivalence of all bases in $L_p(I)$ belonging to the family $\{h_n^{(m,k)}, n \geqslant |k|-m\}$ indexed by the pair of integers (m,k): $|k| \leqslant m+1$. It is sufficient of course to prove that the Haar system $\{\chi_n, n \geqslant 1\}$ is equivalent to $\{h_n^{(m,k)}, n \geqslant |k|-m\}$ for each pair (m,k): $|k| \leqslant m+1$. To do this we need two lemmas.

Let $S_n^m(I)$ denote the linear span of $\{h_j^{(m,0)}, -m \leqslant j \leqslant n\}$, and let $I_{n,j} = \langle s_{n,j-1}, s_{n,j})$ for $j = 1, \ldots, n-1$ and $I_{n,n} = \langle s_{n,n-1}, s_{n,n}\rangle$. The partition $0 = s_{n,0} < \ldots < s_{n,n} = 1$ is defined as in the proof of Lemma 2.2.

LEMMA 4.1. *Let* $m \geqslant -1$ *be given. Then for some constant* C_m,

$$\left|\frac{\phi(t)-\phi(s)}{t-s}\right| \leqslant C_m n \frac{1}{|I_{n,j}|}\int_{I_{n,j}} |\phi|, \quad t, s \in I_{n,j}$$

holds for $j = 1, \ldots, n$; $\phi \in S_n^m(I)$.

Proof. The case of $m = -1$ is trivial. Let us assume, therefore, that $m \geqslant 0$. The argument is now carried out in two steps.

First step. For given m we introduce the family of partitions $\pi_{m,j} = \{s_{-m-1}^{(j)}, \ldots, s_{m+2}^{(j)}\}$, $j = -m, \ldots, m+2$, which is described by the following properties: $s_{-m-1}^{(j)} < \ldots < s_{m+2}^{(j)}$, $s_0^{(j)} = 0$, $s_1^{(j)} = 1$, and $\delta_{-m}^{(j)} = \delta_{-m+1}^{(j)} = \ldots = \delta_j^{(j)} = \frac{1}{2}\delta_{j+1}^{(j)} = \ldots = \frac{1}{2}\delta_{m+2}^{(j)}$ for $j = -m, \ldots, m+2$, where $\delta_i^{(j)} = s_i^{(j)} - s_{i-1}^{(j)}$. To each partition there correspond the B-splines

$$B_i^{(j)}(t) = (s_{i+m+1}^{(j)} - s_{i-1}^{(j)})\,[s_{i-1}^{(j)}, \ldots, s_{i+m+1}^{(j)}; (s-t)_+^{m+1}], \quad i = -m, \ldots, 1,$$

where the square bracket stands for the devided difference. For each j, $-m \leqslant j \leqslant m+2$, $\{B_i^{(j)}, i = -m, \ldots, 1\}$ is a basis in $S_1^m(I)$, i.e. in the space of polynomials on I of degree not exceeding $m+1$. Thus, each $\psi \in S_1^m(I)$ has unique representation

$$\psi = \sum_{i=-m}^{1} a_i^{(j)} B_i^{(j)}.$$

Writing $a^{(j)} = (a_{-m}^{(j)}, \ldots, a_1^{(j)})$ and $\|a^{(j)}\|_1 = |a_{-m}^{(j)}| + \ldots + |a_1^{(j)}|$, we find easily that there is a constant C_m such that

$$C_m^{-1}\|a^{(j)}\|_1 \leqslant \|\psi\|_1 \leqslant C_m\|a^{(j)}\|_1, \quad j = -m, \ldots, m+2. \tag{4.1}$$

Second step. Let us assume that ϕ is in $S_n^m(I)$. The function ϕ has therefore the unique representation

$$\phi = \sum_{j=-m}^{n} \xi_i N_{n,i}^{(m)},$$

where $N_{n,i}^{(m)}$, $i = -m, \ldots, n$, are the B-splines corresponding to the dyadic partition $\{s_{n,i}, i = 0, \pm 1, \pm 2, \ldots\}$.

Let now $t, s \in \operatorname{int} I_{n,j}$. Then

$$\phi(t)-\phi(s) = \sum_{i=-m}^{n} \xi_i (N_{n,i}^{(m)}(t) - N_{n,i}^{(m)}(s)) = \sum_{i=j-m-1}^{j} \xi_i \int_s^t DN_{n,i}^{(m)}(s)\,ds,$$

whence we infer

$$|\phi(t)-\phi(s)| \leqslant |t-s| \sum_{i=j-m-1}^{j} |\xi_i|\,\|DN_{n,i}^{(m)}\|_\infty \leqslant |t-s|\,2n \sum_{i=j-m-1}^{j} |\xi_i|. \tag{4.2}$$

On the other hand,

$$\frac{1}{|I_{n,j}|}\int_{I_{n,j}} |\phi| = \frac{1}{|I_{n,j}|}\int_{I_{n,j}} \Big|\sum_{i=j-m-1}^{j} \xi_i N_{n,i}^{(m)}\Big| \tag{4.3}$$

$$= \int_0^1 \Big|\sum_{i=j-m-1}^{j} \xi_i N_{n,i}^{(m)}(t|I_{n,j}| + s_{n,j-1})\Big|\,dt.$$

Now, to complete the proof it is sufficient to apply the left-hand side of (4.1) to (4.3) and to combine the obtained result with (4.2).

COROLLARY 4.1. *If* $m \geqslant 0$ *and* $\phi \in S_n^m(I)$, *then*

$$|D\phi(t)| \leqslant C_m \cdot n \cdot \frac{1}{|I_{n,j}|}\int_{I_{n,j}} |\phi|, \quad t \in I_{n,j},\ j = 1, \ldots, n.$$

For further use let for given m, k, $|k| \leqslant m+1$, and $n > 1$, the integers r, μ and λ be given as follows: $r = m+1-|k|$, $2^{\mu-1} \leqslant r+1 < 2^\mu$ and $2^\lambda < n \leqslant 2^{\lambda+1}$. Notice that $r \geqslant 0$. Moreover, let $\hat{f}(t) = f(1-t)$, $t \in I$.

LEMMA 4.2. *Let* m *and* k, $|k| \leqslant m+1$ *be given. Moreover, let* $n > 2^{\mu+1}$, *i.e.* $\lambda > \mu$. *Then for some constant* C_m *we have*

(i) $|\chi_n| \leqslant C_m M h_{n-r}^{(m,k)}$ *for* $2^\lambda + r < n$,

(ii) $|\hat{\chi}_n| \leqslant C_m M \hat{h}_{n-r}^{(m,k)}$ *for* $n \leqslant 2^\lambda + r$,

(iii) $|h_{n-r}^{(m,k)}| \leqslant C_m M \chi_n$ *for* $2^\lambda + r < n$,

(iv) $|h_{n-r}^{(m,k)}| \leqslant C_m M \hat{\chi}_n$ *for* $n \leqslant 2^\lambda + r$.

Proof. Notice that $h_n^{(m,k)} \in S_n^{(m-k)}(I)$.

Let now assume at first $2^\lambda + r < n$. Then $n-r = 2^\lambda + l$, $1 \leqslant l \leqslant 2^\lambda$ and $n = 2^\lambda + l'$, $1 \leqslant l' = l+r \leqslant 2^\lambda$.

The proof of (i). In the case of $k = m+1$ we have $h_n^{(m,k)} \in S_n^{-1}(I)$ and therefore (i) is a consequence of the properties (4) and (1) ($p = \infty$) of Section 2. For $k < m+1$, inequality (i) follows by Corollary 4.1 and properties (1) ($p = \infty$) and (4) of Section 2.

The proof of (iii). This inequality follows by property (5) of Section 2.

The second case corresponds to $2^\lambda + r \geqslant n$, and therefore $2^{\lambda-1} < n-r \leqslant 2^\lambda$. Thus, $n-r = 2^{\lambda-1}+l$ with $1 \leqslant 2^{\lambda-1}-r < l \leqslant 2^{\lambda-1}$, and $n = 2^\lambda + l'$ with $1 \leqslant l' \leqslant r \leqslant 2^\lambda$. In particular, we infer that $r > 0$.

The proof of (ii). Corollary 4.1, properties (1) and (4) give (ii).

The proof of (iv). It is sufficient to apply property (5).

THEOREM 4.1. *Let m, k and p be given such that $|k| \leqslant m+1$, $1 < p < \infty$. Then the Haar system $\{\chi_n, n = 1, 2, \ldots\}$ and $\{h_n^{(m,k)}, n \geqslant |k|-m\}$ are equivalent bases in the space $L_p(I)$, i.e. the following two series*

$$\sum_{n=1}^{\infty} a_n \chi_n \tag{4.4}$$

and

$$\sum_{n=|k|-m}^{\infty} a_{n+m+1-|k|} h_n^{(m,k)} \tag{4.5}$$

are equiconvergent in the space $L_p(I)$.

Proof. Let f denote the sum of (4.4), and let g be the sum of (4.5). It is enough to show the existence of such $C_{m,p}$ that $C_{m,p}^{-1}\|f\|_p \leqslant \|g\|_p \leqslant C_{m,p}\|f\|_p$. However, this follows by Corollary 3.2, Lemma 4.2, and Lemma 2.4.

5. Almost everywhere convergence for bounded spline systems

In order to state the maximal inequality we introduce the following notation:

$$W_n^{(m,k)} f = \sum_{j=|k|-m}^{n} (f, w_n^{(m,-k)}) w_n^{(m,k)},$$

$$W_*^{(m,k)} f = \sup_n |W_n^{(m,k)} f|.$$

THEOREM 5.1. *Let m, k, and p be given such that $|k| \leqslant m+1$, $1 < p < \infty$. Then there exists constant C_m such that*

$$\|W_*^{(m,k)}\|_p \leqslant C_m \frac{p^8}{(p-1)^5} \|f\|_p, \quad f \in L_p(I).$$

Proof. The idea of the proof is related to the argument used in [5] and therefore the present reasoning is being restricted to the main steps only.

Let us define

$$Tf = \sum_{n=1}^{\infty} (f, h_n^{(m,-k)}) h_n^{(-1,0)}.$$

Then the following inequalities can be derived

$$\begin{aligned} W_*^{(m,k)} f &\leqslant H_*^{(m,k)} f + \sup_{\mu \geqslant 0} \sup_{1 \leqslant \nu \leqslant 2^\mu} |(W_{2^\mu+\nu}^{(m,k)} - W_{2^\mu}^{(m,k)}) f| \\ &\leqslant H_*^{(m,k)} f + C_m \cdot M\big(\sup_{\mu \geqslant 0} \sup_{1 \leqslant \nu \leqslant 2^\mu} |(W_{2^\mu+\nu}^{(-1,0)} - W_{2^\mu}^{(-1,0)}) Tf|\big) \\ &\leqslant H_*^{(m,k)} f + C_m \cdot M W_*^{(-1,0)} Tf + C_m \cdot M H_*^{(-1,0)} Tf. \end{aligned} \tag{5.1}$$

It now follows by Theorem 3.2 that

$$\|Tf\|_p \leqslant C_m \frac{p^2}{p-1} \|f\|_p. \tag{5.2}$$

Corollary 2.1 gives

$$\|H_*^{(m,k)}\|_p \leqslant C_m \frac{p}{p-1} \|f\|_p, \quad |k| \leqslant m+1. \tag{5.3}$$

The Hardy–Littlewood theorem implies

$$\|Mf\|_p \leqslant C \frac{p}{p-1} \|f\|_p. \tag{5.4}$$

Finally, P. Sjölin [12] proved that

$$\|W_*^{(-1,0)} f\|_p \leqslant C \frac{p^5}{(p-1)^3} \|f\|_p. \tag{5.5}$$

Combining the five inequalities (5.1)–(5.5), we obtain the required inequality in the statement of Theorem 5.2.

COROLLARY 5.1. *Let k, m and p be given such that $|k| \leqslant m+1$, $1 < p < \infty$. Then, for every $f \in L_p(I)$, we have*

$$f(t) = \sum_{n=|k|-m}^{\infty} (f, w_n^{(m,-k)}) w_n^{(m,k)}(t) \ \textit{a.e. in } I. \tag{5.6}$$

Moreover, the series converges in the L_p norm.

6. Final remarks

The limitting cases of $p = 1$ and $p = \infty$ could be discussed in similar way as in [5]. However, the results suggested by work [5] need not to be final, and the delicate discussion how sharp the estimates are requires new ideas.

The question whether (5.6) does not hold in the case of $k \neq 0$ for some $f \in L_1(I)$ remains open. For $k = 0$, $\{w_n^{(m,0)}, n \geqslant -m\}$ is a uniformly bounded orthonormal system and therefore, by a recent result of S. V. Bočkariev, there is a function $f \in L_1(I)$ such that the series in (5.6) diverges on a set of positive Lebesgue measure. (Cf. S. V. Bočkariev, *Divergent on a set of positive measure Fourier series for arbitrary bounded orthonormal set*, Mat. Sb. 98 (1975), pp. 436–449.)

References

[1] S. V. Bočkariev, *Some inequalities for Franklin series*, Analysis Mathematica 1 (1975), pp. 249–257.
[2] Z. Ciesielski, *A bounded orthonormal system of polygonals*, Studia Math. 31 (1968), pp. 339–346.
[3] —, *Constructive function theory and spline systems*, ibid. 53 (1975), pp. 277–302.
[4] Z. Ciesielski and J. Domsta, *Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$*, ibid. 41 (1972), pp. 211–224.
[5] Z. Ciesielski, P. Simon, and P. Sjölin, *Equivalence of Haar and Franklin bases in L_p spaces*, ibid. 60 (1977), pp. 195–210).
[6] J. Domsta, *A theorem on B-splines*, ibid. 41 (1972), pp. 291–314.
[7] C. L. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), pp. 107–115.
[8] J. Marcinkiewicz, *Quelques théoréms sur les series orthogonales*, Ann. Soc. Polon. Math. 16 (1937), pp. 85–96.
[9] S. Ropela, *Properties of bounded orthonormal spline bases*, this volume, pp. 195–203.
[10] —, *Spline bases in Besov spaces*, Bull. Acad. Polon. Sci. 24 (1976), pp. 319–325.
[11] —, *Decomposition lemma and unconditional spline bases*, ibid. 24 (1976), pp. 467–470.
[12] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh–Fourier series*, Ark. Mat. 7 (1968), pp. 551–570.
[13] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.
[14] A. Zygmund, *Trigonometric series*, Cambrige 1959.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975

APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

APPROXIMATION ON DOUBLE COSET SPACES

BERND DRESELER and WALTER SCHEMPP

Siegen University, Department of Mathematics, D-5900 Siegen 21, F.R.G.

To Professor Dr. Werner Meyer-König, Stuttgart, on his 65th birthday, 26 May 1977

1. Introduction

Since the fundamental work by É. Cartan it is well known that a wide class of "special functions" of mathematical physics admits a geometric interpretation. More precisely, É. Cartan and H. Weyl succeeded in identifying these functions with certain distinguished functions that live on a corresponding homogeneous G-manifold X, in such a way that their main properties can be obtained by the analysis to be done on X. In our opinion [20], this philosophy that dominates the modern theory of special functions is useful for the investigation of approximation processes by expansions into special functions, too. This comes from the fact that, roughly speaking, the coefficients of the expansions can be considered as the Fourier coefficients of a transformation determined by the symmetry group G of X.

The main object of the present paper is the investigation of some aspects of approximation on double coset spaces $K \diagdown G/K$, where G is an unimodular locally compact topological group and K a closed subgroup such that (G, K) forms a Gelfand pair (Dieudonné [12]). This choice is justified by the facts that, on the one hand, the theory of Gelfand pairs forms "le cadre naturel de la transformation de Fourier" (Godement [29]) and, on the other hand, that it covers a wide range of applications.

The contents of the paper may be summarized in the following way: In Section 2 we study multipliers T of the complex commutative semisimple Banach algebra $L^1(K \diagdown G/K)$. In particular, Theorem 1 gives several natural characterizations of the multipliers T. Theorem 2 deals with a characterization in the spirit of Eymard [24], [25] of the space $A(K \diagdown G/K)$, which is defined by the inverse Plancherel transformation $\mathscr{F}'$ associated with the space $Z = Z(G/K)$ of all continuous zonal K-spherical functions of positive type on G. This result (that is of interest in its own right), combined with some Plancherel formula arguments, yields our version of the Bochner–Schoenberg–Eberlein theorem, valid for multipliers of $L^1(K \diagdown G/K)$ (Theorem 3). As a consequence, a sort of continuity principle (Co-