The question whether (5.6) does not hold in the case of $k \neq 0$ for some $f \in L_1(I)$ remains open. For $k = 0$, $\{w_n^{(m,0)},\ n \geqslant -m\}$ is a uniformly bounded orthonormal system and therefore, by a recent result of S. V. Bočkariev, there is a function $f \in L_1(I)$ such that the series in (5.6) diverges on a set of positive Lebesgue measure. (Cf. S. V. Bočkariev, *Divergent on a set of positive measure Fourier series for arbitrary bounded orthonormal set*, Mat. Sb. 98 (1975), pp. 436–449.)

References

[1] S. V. Bočkariev, *Some inequalities for Franklin series*, Analysis Mathematica 1 (1975), pp. 249–257.
[2] Z. Ciesielski, *A bounded orthonormal system of polygonals*, Studia Math. 31 (1968), pp. 339–346.
[3] —, *Constructive function theory and spline systems*, ibid. 53 (1975), pp. 277–302.
[4] Z. Ciesielski and J. Domsta, *Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$*, ibid. 41 (1972), pp. 211–224.
[5] Z. Ciesielski, P. Simon, and P. Sjölin, *Equivalence of Haar and Franklin bases in L_p spaces*, ibid. 60 (1977), pp. 195–210).
[6] J. Domsta, *A theorem on B-splines*, ibid. 41 (1972), pp. 291–314.
[7] C. L. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), pp. 107–115.
[8] J. Marcinkiewicz, *Quelques théoréms sur les series orthogonales*, Ann. Soc. Polon. Math. 16 (1937), pp. 85–96.
[9] S. Ropela, *Properties of bounded orthonormal spline bases*, this volume, pp. 195–203.
[10] —, *Spline bases in Besov spaces*, Bull. Acad. Polon. Sci. 24 (1976), pp. 319–325.
[11] —, *Decomposition lemma and unconditional spline bases*, ibid. 24 (1976), pp. 467–470.
[12] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh–Fourier series*, Ark. Mat. 7 (1968), pp. 551–570.
[13] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.
[14] A. Zygmund, *Trigonometric series*, Cambrige 1959.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975

APPROXIMATION THEORY
BANACH CENTER PUBLICATIONS, VOLUME 4
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

APPROXIMATION ON DOUBLE COSET SPACES

BERND DRESELER and WALTER SCHEMPP

Siegen University, Department of Mathematics, D-5900 Siegen 21, F.R.G.

To Professor Dr. Werner Meyer-König, Stuttgart, on his 65th birthday, 26 May 1977

1. Introduction

Since the fundamental work by É. Cartan it is well known that a wide class of "special functions" of mathematical physics admits a geometric interpretation. More precisely, É. Cartan and H. Weyl succeeded in identifying these functions with certain distinguished functions that live on a corresponding homogeneous G-manifold X, in such a way that their main properties can be obtained by the analysis to be done on X. In our opinion [20], this philosophy that dominates the modern theory of special functions is useful for the investigation of approximation processes by expansions into special functions, too. This comes from the fact that, roughly speaking, the coefficients of the expansions can be considered as the Fourier coefficients of a transformation determined by the symmetry group G of X.

The main object of the present paper is the investigation of some aspects of approximation on double coset spaces $K\diagdown G/K$, where G is an unimodular locally compact topological group and K a closed subgroup such that (G, K) forms a Gelfand pair (Dieudonné [12]). This choice is justified by the facts that, on the one hand, the theory of Gelfand pairs forms "le cadre naturel de la transformation de Fourier" (Godement [29]) and, on the other hand, that it covers a wide range of applications.

The contents of the paper may be summarized in the following way: In Section 2 we study multipliers T of the complex commutative semisimple Banach algebra $L^1(K\diagdown G/K)$. In particular, Theorem 1 gives several natural characterizations of the multipliers T. Theorem 2 deals with a characterization in the spirit of Eymard [24], [25] of the space $A(K\diagdown G/K)$, which is defined by the inverse Plancherel transformation $\mathscr{F}'$ associated with the space $Z = Z(G/K)$ of all continuous zonal K-spherical functions of positive type on G. This result (that is of interest in its own right), combined with some Plancherel formula arguments, yields our version of the Bochner–Schoenberg–Eberlein theorem, valid for multipliers of $L^1(K\diagdown G/K)$ (Theorem 3). As a consequence, a sort of continuity principle (Corollary of Theorem 3) follows. Section 3 introduces the notions of prosaturation family, of saturation family and the notion of saturation structure with respect to approximation processes of the multiplier type on $L^1(K\diagdown G/K)$ and presents our main results (Theorems 4, 5 and 6), concerning the saturation behavior of these processes. Finally, Section 4 collects a list of various applications, whereas Section 5 adds some few perspectives towards further investigations in this field.

Throughout this paper, G denotes a locally compact unimodular topological group (with a Haar measure m_G) and K a closed subgroup of G (with the normalized Haar measure m_K) such that (G, K) forms a Gelfand pair. Moreover we shall keep to the notations introduced by Dieudonné [12]; all unexplained notations and conventions are as there.

2. Multipliers of $L^1(K\diagdown G/K)$

In view of the fact that $L^1(K\diagdown G/K)$ is a complex commutative semisimple Banach algebra, we have at our disposal a suitable notion of multiplier; cf. Larsen [36], [37]. Our present situation allows to establish the following characterization theorem.

THEOREM 1. *Let $T\colon L^1(K\diagdown G/K)\to L^1(K\diagdown G/K)$ be a linear mapping. The following assertions are pairwise equivalent:*

(i) *T is a multiplier of $L^1(K\diagdown G/K)$, i.e., $T\in \mathrm{Mult}\big(L^1(K\diagdown G/K)\big)$.*

(ii) *There is a Radon measure $\mu\in\mathscr{M}^1(K\diagdown G/K)$ such that $T(f)=f*\mu$ for all $f\in L^1(K\diagdown G/K)$.*

(iii) *$T(f*\nu)=T(f)*\nu$ for all $f\in L^1(K\diagdown G/K)$ and all measures $\nu\in\mathscr{M}^1(K\diagdown G/K)$.*

(iv) *$T\big((\delta(y)f)^\natural\big)=(\delta(y)T(f))^\natural$ for all $y\in G$ and $f\in L^1(K\diagdown G/K)$ ("Wendel condition").*

Proof. (i) ⇒ (ii): Let $(u_\alpha)_{\alpha\in I}$ be an approximate unit for $L^1(K\diagdown G/K)$ such that $\|u_\alpha\|_{L^1(K\diagdown G/K)}=1$ for all $\alpha\in I$. For each index $\alpha\in I$, define the linear form

$$\mu_\alpha\colon\ \mathscr{C}^0(K\diagdown G/K)\ni g \rightsquigarrow \int_G g\cdot T(u_\alpha)\,dm_G(x).$$

Then we have $\mu_\alpha\in\mathscr{M}^1(K\diagdown G/K)$ and $\|\mu_\alpha\|\leqslant\|T\|$. By Alaoglu's theorem, there exist a measure $\mu\in\mathscr{M}^1(K\diagdown G/K)$ and a subnet $(\mu_\alpha)_{\alpha\in J}$ such that $\mu=\lim\limits_{\alpha\in J}\mu_\alpha$, with respect to the vague topology of $\mathscr{M}^1(K\diagdown G/K)$. By standard arguments used for instance in [23], we obtain $T(f)=f*\mu$, for all $f\in L^1(K\diagdown G/K)$ as contented.

(ii) ⇒ (iii): Obviously

$$\begin{aligned} T(f*\nu) &= (f*\nu)*\mu = f*(\nu*\mu)\\ &= f*(\mu*\nu) = T(f)*\nu. \end{aligned}$$

(iii) ⇒ (i): Trivial.

(ii) ⇒ (iv): See the analogous arguments outlined in [23].

(iv) ⇒ (iii): It suffices to prove that

$$\int_G T(f*\nu)h\,dm_G(x)=\int_G (T(f)*\nu)h\,dm_G(x),$$

for all $f\in L^1(K\diagdown G/K)$, $h\in L^\infty(K\diagdown G/K)$ and $\nu\in\mathscr{M}^1(K\diagdown G/K)$. We have, by the Lebesgue–Fubini theorem,

$$\begin{aligned}\int_G T(f*\nu)h\,dm_G(x) &= \int_G (f*\nu)\,{}^tTh\,dm_G(x)\\ &= \int_G\int_G \delta(y^{-1})f(x)\,d\nu(y)\,{}^tTh(x)\,dm_G(x)\\ &= \int_G\int_G \delta(y^{-1})f(x)\,({}^tTh)^\natural(x)\,dm_G(x)\,d\nu(y)\\ &= \int_G\int_G \big(\delta(y^{-1})f\big)^\natural(x)\,{}^tTh(x)\,dm_G(x)\,d\nu(y)\\ &= \int_G\int_G \big(\delta(y^{-1})Tf\big)^\natural(x)\,h(x)\,dm_G(x)\,d\nu(y)\\ &= \int_G (T(f)*\nu)h\,dm_G(x).\end{aligned}$$

This completes the proof. ■

Let $Z=Z(G/K)$ be the locally compact topological space of the all spherical functions on G with respect to K that are of positive type and let $\mathscr{F}'$ be the inverse of the Plancherel transformation (Godement [29], Dieudonné [12]). Define

$$A(K\diagdown G/K)=\mathscr{F}'L^1(Z).$$

We shall establish the following characterization of the space $A(K\diagdown G/K)$ by a L^2-factorization in the spirit of Eymard [24].

THEOREM 2. *The identity*

$$A(K\diagdown G/K)=L^2(K\diagdown G/K)*L^2(K\diagdown G/K)$$

holds.

Proof. If $f\in L^2(K\diagdown G/K)*L^2(K\diagdown G/K)$, then $f=g*\check{\bar h}$, where $g,h\in L^2(K\diagdown G/K)$. Thus

$$\begin{aligned} f(x) &= \int_G g(y)\check{\bar h}(y^{-1}x)\,dm_G(y)=\int_G g(y)\bar h(x^{-1}y)\,dm_G(y)\\ &= \big(g|\gamma(x)h\big)_{L^2(G)}=\big(\overline{\mathscr{F}}g|\overline{\mathscr{F}}(\gamma(x)h)\big)_{L^2(Z)}\\ &= \int_Z \overline{\mathscr{F}}g(\omega)\overline{\overline{\mathscr{F}}h(\omega)}\,\overline{\omega}(x)\,dm_Z(\omega)=\int_Z F(\omega)\overline{\omega}(x)\,dm_Z(\omega)\qquad (x\in G),\end{aligned}$$

where $F=\overline{\mathscr{F}}g.\overline{\overline{\mathscr{F}}h}\in L^1(Z)$. Consequently, we have $f=\mathscr{F}'(F)$ and therefore $L^2(K\diagdown G/K)*L^2(K\diagdown G/K)\subseteq A(K\diagdown G/K)$.

Conversely, let $F\in L^1(Z)$. Then we can write $F=F_1F_2$, where $F_j\in L^2(Z)$ $(j=1,2)$. Choose $f_j\in L^2(K\diagdown G/K)$ such that $F_j=\mathscr{F}f_j$ $(j=1,2)$. Let $f=f_1*\check{f_2}\in L^2(K\diagdown G/K)*L^2(K\diagdown G/K)$. We show that $f=\mathscr{F}'(F)$. Indeed,

$$\mathscr{F}'F(x) = \int_Z F(\omega)\overline{\omega}(x)dm_Z(\omega) = \int_Z F_1(\omega)\overline{\overline{F}_2(\omega)\omega(x)}\,dm_Z(\omega)$$
$$= \int_Z \mathscr{F}f_1(\omega)\overline{\overline{\mathscr{F}f_2(\omega)}\omega(x)}\,dm_Z(\omega) = \int_G f_1(y)\big(\gamma(x)\overline{f}_2(y)\big)\,dm_G(y)$$
$$= f_1 * \check{\overline{f}}_2(x) = f(x) \quad (x \in G).$$

Consequently, $A(K\diagdown G/K) \subseteq L^2(K\diagdown G/K) * L^2(K\diagdown G/K)$ and the proof is complete. ■

Let $A(G)$ denote the Fourier algebra of G in the sense of Eymard [24].

COROLLARY. *$A(K\diagdown G/K)$ is a vector subspace of the commutative complex Banach algebra $A(G) \cap \mathscr{C}^0(K\diagdown G/K)$.*

The corollary shows that Theorem 2 supra is of the Riemann–Lebesgue type for the space $L^1(Z)$.

THEOREM 3 (Bochner–Schoenberg–Eberlein). *Let $\varphi\colon Z \to \boldsymbol{C}$ be a m_Z-measurable function and $M \geqslant 0$. Consider the following four properties for φ and M.*

(i) *There exists a measure $\mu \in \mathscr{M}^1(K\diagdown G/K)$ such that $\overline{\mathscr{F}}\mu = \varphi$ and $||\mu|| \leqslant M$.*

(ii) *For every $h \in L^1(Z)$, $h\varphi \in L^1(Z)$ and the inequality*

$$\Big|\int_Z h\varphi\, dm_Z\Big| \leqslant M||\mathscr{F}'h||_\infty$$

holds.

(iii) *There exists a measure $\mu \in \mathscr{M}^1(K\diagdown G/K)$ such that $\overline{\mathscr{F}}\mu = \varphi$, locally m_Z-almost everywhere on Z and $||\mu|| \leqslant M$.*

(iv) *For every $p = \sum_{1\leqslant j\leqslant m} \alpha_j \omega_j \in \mathscr{C}^b(K\diagdown G/K)$, where $a_j \in \boldsymbol{C}$, $\omega_j \in Z$ $(1 \leqslant j \leqslant m)$, the inequality*

$$\Big|\Big|\sum_{1\leqslant j\leqslant m} \alpha_j \varphi(\omega_j)\Big|\Big| \leqslant M\,.\,||p||_\infty$$

holds.

Then we have: (ii) *is equivalent to* (iii). *For continuous φ,* (i) *is equivalent to* (iii). *If G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G, then* (i), (ii), (iii), (iv) *are equivalent.*

Proof. (iii) ⇒ (ii): Let $h \in L^1(Z)$. Then $\mathscr{F}'h \in \mathscr{C}^0(K\diagdown G/K)$, by the Corollary of Theorem 2. Thus

$$\Big|\int_Z h(\omega)\varphi(\omega)\,dm_Z(\omega)\Big| = \Big|\int_Z h(\omega)\overline{\mathscr{F}}\mu(\omega)\,dm_Z(\omega)\Big|$$
$$= \Big|\int_G \check{\overline{\mathscr{F}'h}}(x)\,d\mu(x)\Big|$$
$$\leqslant ||\mu||\,.\,||\mathscr{F}'h||_\infty \leqslant M\,.\,||\mathscr{F}'h||_\infty .$$

Therefore (ii) holds.

(ii) ⇒ (iii): The linear form $\Psi\colon A(K\diagdown G/K) \to \boldsymbol{C}$ defined by

$$\Psi\colon \check{\overline{\mathscr{F}'h}} \rightsquigarrow \int_Z h(\omega)\varphi(\omega)\,dm_Z(\omega)$$

is continuous with respect to the topology of uniform convergence on $K\diagdown G/K$. The form Ψ admits a continuous linear extension $\mu \in \mathscr{M}^1(K\diagdown G/K)$ such that $||\mu|| \leqslant M$. We find, for all $h \in L^1(Z)$, that

$$\Big|\int_Z h(\omega)\varphi(\omega)\,dm_Z(\omega)\Big| = |\Psi(\check{\overline{\mathscr{F}'h}})| = \Big|\int_Z h(\omega)\overline{\mathscr{F}}\mu(\omega)\,dm_Z(\omega)\Big|.$$

Consequently, $\varphi = \overline{\mathscr{F}}\mu$ locally m_Z-almost everywhere on Z. Thus (iii) holds.

It is clear that for continuous φ, (i) and (iii) are equivalent.

If G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G, then Theorem 2 of Harish-Chandra [30] implies $Z \subseteq \mathscr{C}^0(K\diagdown G/K)$. The rest of the proof now follows by the results of [22]. ■

COROLLARY. *Let $(\mu_\iota)_{\iota\in I}$ be a family of measures of the space $\mathscr{M}^1(K\diagdown G/K)$ such that $\sup_{\iota\in I} ||\mu_\iota|| \leqslant M$. Suppose that the section filter of the directed set I admits a countable base, and that*

$$\lim_{\iota\in I} \overline{\mathscr{F}}\mu_\iota(\omega) = \varphi(\omega)$$

holds m_Z-almost everywhere in Z. Then there is a measure $\mu \in \mathscr{M}^1(K\diagdown G/K)$ such that

$$\overline{\mathscr{F}}\mu(\omega) = \varphi(\omega),$$

locally m_Z-almost everywhere in Z and $||\mu|| \leqslant M$.

Proof. For every $h \in L^1(Z)$, we have $h\,.\,\overline{\mathscr{F}}\mu_\iota \in L^1(Z)$ and

$$\Big|\int_Z h\overline{\mathscr{F}}\mu_\iota dm_Z\Big| \leqslant M||\mathscr{F}'h||_\infty ,$$

since $\sup_{\iota\in I}||\mu_\iota|| \leqslant M$. The estimate $\sup_{\iota\in I}||\overline{\mathscr{F}}\mu_\iota||_\infty \leqslant M$ shows that the Lebesgue theorem of dominated convergence is applicable. It follows $h\varphi \in L^1(Z)$ and $\Big|\int_Z h\varphi\, dm_Z\Big| \leqslant M||\mathscr{F}'h||_\infty$. This completes the proof. ■

3. Saturation

Let A be a semisimple commutative complex Banach algebra with spectrum $X(A)$ and Gelfand transformation $\mathscr{G}$. Then the equation $\mathscr{G}(Tx) = \hat{T}\,.\,\mathscr{G}x$ assigns to any multiplier $T \in \mathrm{Mult}(A)$ a complex-valued function $\hat{T} \in \mathscr{C}^b(X(A))$ such that $\hat{T}\,.\,\mathscr{G}(A) \subseteq \mathscr{G}(A)$, and the linear mapping $T \rightsquigarrow \hat{T}$ defines a continuous isomorphism of $\mathrm{Mult}(A)$ onto this vector subspace.

Let $(I, \leqslant)$ be a directed set such that the section filter of I admits a countable base. A family $(T)_{\iota\in I}$ of multipliers of A is called a *prosaturation family of type* $(\varphi;\psi)$ if there exist a mapping $\varphi\colon I \to \boldsymbol{R}_+^*$ satisfying $\lim_{\iota\in I}\varphi(\iota) = 0$ and a mapping $\psi\colon X(A) \to \boldsymbol{C}$ such that

$$\lim_{\iota\in I} \frac{\hat{T}_\iota(\omega)-1}{\varphi(\iota)} = \psi(\omega)$$

holds pointwise for all $\omega \in X(A)$.

A prosaturation family $(T_\iota)_{\iota\in I}$ of type $(\varphi;\psi)$ is called a *saturation family of type* $(\varphi;\psi)$ if there exists a family $(S_\iota)_{\iota\in I}$ of multipliers of A such that $\sup_{\iota\in I}\|S_\iota\| \leqslant M_0$ and

(1) $$\frac{\hat{T}_\iota - 1}{\varphi(\iota)} = \psi\,.\,\hat{S}_\iota$$

holds, for all $\iota\in I$.

Remark 1. Based on a lemma by P. Malliavin, Sunouchi [40] has shown that, in the case $A = L^1(T)$, there are prosaturation families which are not saturation families. In this connection also see Igari [34].

A family $(T_\iota)_{\iota\in I}$ of multipliers of A is said to *admit a saturation structure* $(\varphi; V, N)$ if the following conditions are satisfied:

(I) There exists a mapping $\varphi\colon I\to \boldsymbol{R}_+^*$ satisfying $\lim_{\iota\in I}\varphi(\iota) = 0$ such that $f\in A$ and

$$\|T_\iota(f)-f\| = o(\varphi(\iota)) \quad (\iota\in I)$$

if and only if $f\in N\subseteq A$.

(II) If $f\in A$ and

$$\|T_\iota(f)-f\| = O(\varphi(\iota)) \quad (\iota\in I),$$

then $f\in V\subseteq A$.

(III) If $f\in V$, then

$$\|T_\iota(f)-f\| = O(\varphi(\iota)) \quad (\iota\in I).$$

In this case, V is said to be the *Favard space* and N the *trivial space* of the saturation structure $(\varphi; V, N)$.

Remark 2. Let the saturation family $(T_\iota)_{\iota\in I}$ of type $(\varphi;\psi)$ be given. Suppose that the family $(S_\iota)_{\iota\in I}$ in $\mathrm{Mult}(A)$, occuring in (1), satisfies $\lim_{\iota\in I}\|S_\iota(x)-x\| = 0$, for all $x\in A$. Then we can show that $(T_\iota)_{\iota\in I}$ admits a saturation structure $(\varphi; V, N)$. In this case, the Favard space coincides with the relative completion (Butzer–Berens [8], Berens [2]) of

$$V_\psi = \{x\in A \mid \psi\,.\,\mathscr{G}x\in\mathscr{G}(A)\},$$

with respect to the norm of A, and the trivial space N equals to the kernel of the (densely defined) linear operator M_ψ of factor sequence type associated to V_ψ.

In the sequel, let A be the semisimple commutative Banach algebra $L^1(K\diagdown G/K)$. By Theorem 1, $\mathrm{Mult}(L^1(K\diagdown G/K))$ can be identified with $\mathscr{M}^1(K\diagdown G/K)$. For any function $\psi\colon Z\to \boldsymbol{C}$ define

$$V_\psi^1(K\diagdown G/K) = \{f\in L^1(K\diagdown G/K) \mid \psi\,.\,\overline{\mathscr{F}}f \doteq \overline{\mathscr{F}}\mu,\ \mu\in\mathscr{M}^1(K\diagdown G/K)\}$$

where $\doteq$ denotes equality m_Z-locally almost everywhere in Z.

THEOREM 4. *Let $(\mu_\iota)_{\iota\in I}$ be a prosaturation family of type $(\varphi;\psi)$ in $\mathscr{M}^1(K\diagdown G/K)$. Then $(\mu_\iota)_{\iota\in I}$ admits a saturation structure $(\varphi; V, N)$, where*

$$V\subseteq V_\psi^1(K\diagdown G/K), \qquad N\subseteq \mathrm{Ker}\,M_\psi.$$

Proof. (I) Since the Fourier–Stieltjes cotransformation $\overline{\mathscr{F}}\colon \mathscr{M}^1(K\diagdown G/K)\to \mathscr{C}^b(Z)$ is a continuous homomorphism of commutative complex Banach algebras and $(\mu_\iota)_{\iota\in I}$ is a prosaturation family of type $(\varphi;\psi)$, we have

$$\lim_{\iota\in I}\left\|\overline{\mathscr{F}}f\,.\,\frac{\overline{\mathscr{F}}\mu_\iota-1}{\varphi(\iota)}\right\|_\infty = 0,$$

for all $f\in N$. Therefore

$$\psi(\omega)\,.\,\overline{\mathscr{F}}f(\omega) = 0 \quad (\omega\in Z),$$

i.e., $f\in \mathrm{Ker}\,M_\psi$ as we wished to prove.

(II) Let $f\in V$ be given. There exist $M_0\geqslant 0$ and $\iota_0\in I$ such that

$$\frac{\|\mu_\iota * f - f\|_1}{\varphi(\iota)} \leqslant M_0,$$

for all $\iota\in J$, $\iota_0\leqslant\iota$. Since $(\mu_\iota)_{\iota\in I}$ is a prosaturation family of type $(\varphi;\psi)$, the Corollary of Theorem 3 guarantees the existence of a measure $\mu\in\mathscr{M}^1(K\diagdown G/K)$ such that

$$\psi(\omega)\overline{\mathscr{F}}f(\omega) \doteq \overline{\mathscr{F}}\mu(\omega).$$

Thus $f\in V_\psi^1(K\diagdown G/K)$. ∎

THEOREM 5. *Let $(\mu_\iota)_{\iota\in I}$ be a saturation family of type $(\varphi;\psi)$ in $\mathscr{M}^1(K\diagdown G/K)$. Then $(\mu_\iota)_{\iota\in I}$ admits the saturation structure*

$$(\varphi; V_\psi^1(K\diagdown G/K), \mathrm{Ker}\,M_\psi).$$

Proof. It suffices to prove

$$V_\psi^1(K\diagdown G/K)\subseteq V, \qquad \mathrm{Ker}\,M_\psi\subseteq N.$$

Both inclusions are direct consequences of the fact that the Fourier–Stieltjes transformation $\mathscr{F}\colon \mathscr{M}^1(K\diagdown G/K)\to \mathscr{C}^b(K\diagdown G/K)$ is an injective homomorphism of commutative complex Banach algebras. ∎

Let the Plancherel measure be expressed uniquely in the form

$$m_Z = m_Z^1 + m_Z^2,$$

where $m_Z^1\in\mathscr{M}(Z)$ is a diffuse measure and $m_Z^2\in\mathscr{M}(Z)$ an atomic measure concentrated on the set $C_Z\subseteq Z$.

THEOREM 6. *Let $(\mu_\iota)_{\iota\in I}$ be a prosaturation family of type $(\varphi;\psi)$ in $\mathscr{M}^1(K\diagdown G/K)$. Let $N_\psi = \{\omega\in Z \mid \psi(\omega) = 0\}$ be m_Z^1-negligible and suppose $m_Z^2(N_\psi) < +\infty$. Then $(\mu_\iota)_{\iota\in I}$ admits a saturation structure $(\varphi; V, N)$, where*

$$V\subseteq V_\psi^1(K\diagdown G/K)$$

and N is the vector subspace of $L^1(K\diagdown G/K)$ spanned by the set $N_\psi\cap C_Z$.

Proof. An application of the Fourier inversion formula shows that $f \in \operatorname{Ker} M_\psi$ if and only if f admits the form

$$f = \sum_{\omega \in N_\psi \cap C_Z} \alpha_\omega \omega \qquad (\alpha_\omega \in C).$$

Thus, $\operatorname{Ker} M_\psi = \operatorname{Span}(N_\psi \cap C_Z)$. The injectivity of the Fourier transformation shows $\operatorname{Ker} M_\psi \subseteq N$ and the theorem follows. ∎

4. Applications

(1) Let $\mathfrak{a} = (a_n)_{n \in Z}$ be a sequence of integers $a_n \geqslant 2$ and $G = T_\mathfrak{a}$ the $\mathfrak{a}$-adic solenoid (Hewitt–Ross [33], Hawley [31]). If $Z_\mathfrak{a}$ denotes the group of $\mathfrak{a}$-adic integers and B the subgroup of $R \times Z_\mathfrak{a}$ formed by the pairs (n, n), then $T_\mathfrak{a} = (R \times Z_\mathfrak{a})/B$ is a compact connected topological group. The monothetic group $Z_\mathfrak{a}$ admits a basis $(\Lambda_n)_{n \in N}$ of open and closed subgroups $\Lambda_n = \{(b_m)_{m \in N} \in Z_\mathfrak{a} |\ b_m = 0 \text{ for } m < n\}$ about the neutral element. Let $(\alpha_n)_{n \in N}$ be a nonincreasing sequence in the interval $]0, \frac{1}{2}[$ tending to 0 and

$$U_{\alpha_n} =]-\tfrac{1}{2}\alpha_n, +\tfrac{1}{2}\alpha_n[\times \Lambda_{n+1} \qquad (n \in N).$$

Denoting by $\xi_{U_{\alpha_n}}$ the characteristic function of U_{α_n} in $T_\mathfrak{a}$, define

$$H_{\alpha_n} = \frac{1}{(m_{T_\mathfrak{a}}(U_{\alpha_n}))^2} \xi_{U_{\alpha_n}} * \xi_{U_{\alpha_n}} \qquad (n \in N).$$

For each character $\chi_{\frac{m}{A_k}} \in \hat{T}_\mathfrak{a}$ $(A_k = \prod_{0 \leqslant n \leqslant k} a_n,\ m \in Z)$ we obtain

$$\mathscr{F} H_{\alpha_n}(\chi_{-\frac{m}{A_k}}) = \begin{cases} \dfrac{A_k^2}{2m^2\alpha_n^2\pi^2}\left(1 - \cos\left(\dfrac{2\pi m \alpha_n}{A_k}\right)\right) & \text{if } n \geqslant k, \\ 0 & \text{if } n < k \end{cases}$$

and

$$\lim_{n \to \infty} \frac{\mathscr{F} H_{\alpha_n}(\chi_{-\frac{m}{A_k}}) - 1}{\alpha_n^2} = \frac{1}{3}\pi^2 \left(\frac{m}{A_k}\right)^2.$$

Consequently, $(H_{\alpha_n} . m_{T_\mathfrak{a}})_{n \in N}$ is a prosaturation family in $\mathscr{M}(T_\mathfrak{a})$ of type $(\varphi; \psi)$, where

$$\varphi\colon N \ni n \rightsquigarrow \alpha_n^2, \qquad \psi\colon \hat{T}_\mathfrak{a} \ni \chi_{\frac{m}{A_k}} \rightsquigarrow \frac{1}{3}\pi^2\left(\frac{m}{A_k}\right)^2.$$

By Theorem 6, it admits the saturation structure $(\varphi; V, N)$, where $V \subseteq V_\psi^1(T_\mathfrak{a})$ and $N = C$.

A second example can be obtained by letting

$$E_{\alpha_n} = \frac{A_n}{2\alpha_n} \xi_{]-\alpha_n, +\alpha_n[\times \Lambda_{n+1}} \qquad (n \in N).$$

It follows

$$\mathscr{F} E_{\alpha_n}(\chi_{-\frac{m}{A_k}}) = \begin{cases} \dfrac{A_k}{2\alpha_n \pi m} \sin\left(\dfrac{2\pi m \alpha_n}{A_k}\right) & \text{if } n \geqslant k, \\ 0 & \text{if } n < k. \end{cases}$$

Hence, $(E_{\alpha_n} . m_{T_\mathfrak{a}})_{n \in N}$ defines a prosaturation family in $\mathscr{M}(T_\mathfrak{a})$ of type $(\varphi; \psi)$, where

$$\varphi\colon N \ni n \rightsquigarrow \alpha_n^2, \qquad \psi\colon \hat{T}_\mathfrak{a} \ni \chi_{\frac{m}{A_k}} \rightsquigarrow \frac{2}{3}\pi^2\left(\frac{m}{A_k}\right)^2.$$

(2) The case where G is a locally compact Abelian topological group and K denotes its subgroup consisting of the neutral element was studied by Buchwalter [7] and by the authors [17], [18], [38]. For the case $G = R$, also see Butzer–Nessel [9]. We add the following example that is well known in literature for special choices of G.

Let $I = R_+^*$ under its natural ordering and $(\mu_t)_{t>0}$ a vaguely continuous convolution semigroup on G (Berg–Forst [5], [6]). Then $(\mu_t)_{t>0}$ forms a prosaturation family of type $(\varphi; \psi)$ in $\mathscr{M}^1(G)$, where $\varphi = \mathrm{id}_{R_+^*}$, $\psi = -q$, and q is the uniquely determined continuous, negative definite function on G such that

$$\mathscr{F}\mu_t = e^{-tq} \qquad (t \in R_+^*).$$

Consequently, $(\mu_t)_{t>0}$ admits, by Theorem 4, the saturation structure $(\varphi; V, N)$, where $V \subseteq V_\psi^1(G)$, $N \subseteq \operatorname{Ker} M_\psi^1$ and M_ψ^1 denotes the linear operator of factor sequence type associated with $V_\psi^1(G)$. An application of the Fourier transformation shows that the identity

$$\mu_t * f - f = \int_0^t \mu_s * (M_\psi^1(f))\, ds \qquad (t > 0)$$

holds. It follows in the present case

$$V = V_\psi^1(G), \qquad N = \operatorname{Ker} M_\psi^1.$$

(3) Let G_0 be an arbitrary compact topological group, $G = G_0 \times G_0$ and K the diagonal of G. Then we may identify the complex Banach space $L^1(K \diagdown G/K)$ with the subspace of $L^1(G_0)$ formed by the central functions on G_0. The saturation structure of approximation processes acting in $L^1(G_0)$ is studied in [13], [14], [21].

If G_0 denotes a compact Lie group of dimension n, then any radial saturation family of type $(\varphi; \psi)$ on the Lie algebra $\mathfrak{g}_0$ of G_0 (considered as a real Euclidean n-space) induces in a natural way a "radial" saturation family of type $(\varphi; \psi)$ in $\mathscr{M}(G_0)$. Details can be found in paper [15].

(4) Let (G, K) be a Gelfand pair. Consider in analogy with Example (2) a vaguely continuous convolution semigroup $(\mu_t)_{t>0}$ on G of positive measures in $\mathscr{M}^1(K \diagdown G/K)$ (Berg [3]). In this case, again there exists a continuous negative definite function $q\colon Z \to C$ such that

$$\mathscr{F}\mu_t = e^{-tq} \qquad (t \in R_+^*).$$

Thus $(\mu_t)_{t>0}$ admits the saturation structure $(\mathrm{id}_{R_+^*}, V^1_{-q}(K\diagdown G/K), \mathrm{Ker}\, M^1_{-q})$.

(5) Let (G, K) be a Riemannian symmetric pair and $X = G/K$ a Riemannian globally symmetric manifold (with respect to a G-invariant Riemannian structure) of rank $l \geqslant 1$.

(a) Let G be compact. Then our results apply to approximation processes defined by expansions in K-spherical functions on X in l variables. If $l = 1$, the Wang classification shows that only the following cases are possible: $X = S_n = SO(n+1)/ /SO(n)$, $X = \boldsymbol{P}_n(k)$, where $k = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ and $X = \boldsymbol{P}$ (Cay). The associated spherical functions are the Jacobi polynomials $P_m^{(\alpha,\beta)}$, for certain indices $\alpha > -1, \beta > -1$. See Bavinck [1] and papers [13], [14], [19]. Also see Koornwinder [35], for the case $l = 2$.

(b) Let G be noncompact and let X be of the Euclidean type. For an investigation of the Poisson approximation process in this setting by means of the Hankel–Stieltjes transformation in the sense of Schwartz [39], see the paper [20].

(c) Let X be of noncompact type and have rank $l = 1$. In this case, the Fourier transformation $\mathscr{F}$ is just the Jacobi transformation investigated by Flensted–Jensen and Koornwinder [26]. For an approach that covers both the cases (b) and (c), see Chébli [10], [11].

(d) Let G be a connected noncompact semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. Then $X = G/K$ is a symmetric manifold of noncompact type and rank $l \geqslant 1$. Let ϱ denote the half-sum of the positive roots of (G, K) and define, for $t > 0$, the Gauss kernel

$$g_t\colon\ G \ni x \leadsto \frac{1}{w}\int\limits_{\Lambda_0} \exp -\big((\lambda|\lambda)+(\varrho|\varrho)\big)t\,.\,\omega_\lambda(x)|c(\lambda)|^{-2}d\lambda.$$

Here w denotes the order of a Weyl group W of (G, K) acting on the real Euclidean space Λ_0 of dimension l and c denotes the Harish-Chandra function (Gangolli [27], [28]). It follows that $(g_t\,.\,m_G)_{t>0}$ forms a vaguely continuous convolution semigroup of positive measures in $\mathscr{M}^1(K\diagdown G/K)$ having the Laplace–Beltrami operator Δ of X as its infinitesimal generator.

Since we have

$$\mathscr{F}g_t\colon\ \lambda \leadsto \exp -\big((\lambda|\lambda)+(\varrho|\varrho)\big)t \qquad (t > 0),$$

Example (4) shows that $(g_t\,.\,m_G)_{t>0}$ admits the saturation structure

$$(\mathrm{id}_{R_+^*}, V^1_{-q}(K\diagdown X), 0), \quad \text{where} \quad q\colon\ \Lambda_0 \ni \lambda \leadsto (\lambda|\lambda)+(\varrho|\varrho).$$

In the case $G = SO_0(1, n)$, $K = SO(n)$, we obtain for X the hyperbolic space of dimension $n \geqslant 2$ (Berg–Faraut [4]). The Laplace–Beltrami operator of X takes the form

$$\Delta = \frac{1}{(\sinh r)^{n-1}}\frac{d}{dr}\left((\sinh r)^{n-1}\frac{d}{dr}\right),$$

where r denotes the geodesic distance from the pole of X and we have

$$q.\ \boldsymbol{R} \ni \lambda \leadsto \lambda^2 + \left(\frac{n-1}{2}\right)^2$$

in the present case.

5. Concluding remarks

An important tool for the proof of our saturation theorem (Theorem 4) is our version of the Bochner–Schoenberg–Eberlein theorem (Theorem 3 and its Corollary). For the connection of the theorems of this type with the extension principle, cf. [22], [23]. A generalization of the Bochner–Schoenberg–Eberlein theorem for arbitrary unimodular locally compact topological groups can be found in a forthcoming paper by E. Siebers.

In addition, we refer to paper [16] that deals with the characterization of L^p-multipliers and continuity theorems of the Lévy type.

Parts of this paper were presented by the junior author at the "Oesterreichischer Mathematikerkongress" held in Vienna, 17–21 September, 1973.

References

[1] H. Bavinck, *Approximation processes for Fourier–Jacobi expansions*, Math. Centrum, Amsterdam, Afd. toegepaste Wisk. TW 126/1971.

[2] H. Berens, *Interpolationsmethoden zur Behandlung von Approximationsprozessen auf Banachräumen*, Lecture Notes in Math., Vol. 64, Springer, Berlin–Heidelberg–New York 1968.

[3] Ch. Berg, *Dirichlet forms on symmetric spaces*, Ann. Inst. Fourier (Grenoble) 23 (1973), pp. 135–156.

[4] Ch. Berg, J. Faraut, *Semi-groupes de Feller invariants sur les espaces homogènes non moyennables*, Matematisk Institut, Københavns Universitet, Preprint Series 21 (1973).

[5] Ch. Berg, G. Forst, *Non-symmetric translation invariant Dirichlet forms*, Invent. Math. 21 (1973), pp. 199–212.

[6] —, —, *Potential theory on locally compact abelian groups*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 87, Springer, Berlin–Heidelberg–New York 1975.

[7] H. Buchwalter, *Saturation sur un groupe abélien localement compact*, C.R. Acad. Sci. Paris 250 (1960), pp. 808–810.

[8] P. L. Butzer, H. Berens, *Semi-groups of operators and approximation*, Die Grundlehren der math. Wissenschaften in Einzeldarstellungen, Band 145, Springer, Berlin–Heidelberg–New York 1967.

[9] P. L. Butzer, R. J. Nessel, *Fourier analysis and approximation*, Vol. 1: *One-dimensional theory*, Birkhäuser, Basel und Stuttgart 1971.

[10] H. Chébli, *Opérateurs de translation généralisée et semi-groupes de convolution*, in: *Théorie du potentiel et analyse harmonique* (Ed. J. Faraut), pp. 35–59, Lecture Notes in Mathematics, Vol. 404, Springer, Berlin–Heidelberg–New York 1974.

[11] —, *Sur un théorème de Paley–Wiener associé à la décomposition spectrale d'un operateur de Sturm–Liouville sur*]0, ∞[, J. Functional Analysis 17 (1974), pp. 447–461.

[12] J. Dieudonné, *Éléments d'analyse, tome* VI: *Analyse harmonique*, Gauthier–Villars, Paris 1975.

[13] B. Dreseler, *Saturation auf kompakten topologischen Gruppen und homogenen Räumen*, Dissertation, Universität Mannheim 1972.

[14] —, *Saturationstheorie auf kompakten topologischen Gruppen*, in: *Approximation theory*, Proc. Conference at Poznań, 22–26 August, 1972 (Ed. Z. Ciesielski and J. Musielak), pp. 55–61, D. Reidel Publishing Company, Dordrecht–Holland 1975.

[15] —, *Zu Entwicklungen nach sphärischen Funktionen gehörende Approximationsverfahren auf kompakten symmetrischen Mannigfaltigkeiten*, Habilitations schrift, Universität Siegen, 1976.

[16] —, *Charakterisierungen von L^p-Multiplikatoren und Stetigkeitssätze vom Lévy-Typ auf unimodularen lokalkompakten Gruppen*, (to appear).

[17] B. Dreseler, W. Schempp, *Saturation on locally compact abelian groups*, Manuscripta math. 7 (1972), pp. 141–174.

18] —, —, *Saturation on locally compact abelian groups: An extended theorem*, ibid. 8 (1973), pp. 271–286.

[19] —, —, *Zonal approximation processes*, Math. Z. 133 (1973), pp. 81–92.

[20] —, —, *Spezielle Funktionen der mathematischen Physik und Saturation*, Methoden und Verfahren der mathematischen Physik 11 (1974), pp. 1–17.

[21] —, —, *Central approximation processes*, Tôhoku Math. J. 27 (1975), pp. 153–173.

[22] —, —, *Spaces of test functions and theorems of the Bochner–Schoenberg–Eberlein type*, Math. Z. 148 (1976), pp. 207–213.

[23] —, —, *On some applications of Bochner–Schoenberg–Eberlein type theorems*, this volume, pp. 83–91.

[24] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), pp. 181–236.

[25] —, *Algèbres A_p et convoluteurs de L^p*, Séminaire Bourbaki, 22e année, 1969/70, 367, Lecture Notes in Math., Vol. 180, Springer, Berlin–Heidelberg–New York 1971.

[26] M. Flensted-Jensen, T. Koornwinder, *The convolution structure for Jacobi function expansions*, Ark. Mat. 11 (1973), pp. 245–262.

[27] R. Gangolli, *Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces*, Acta math. 121 (1968), pp. 151–192.

[28] —, *Spherical functions on semisimple Lie groups. Spectra of discrete uniform subgroups of semisimple Lie groups*, in: *Symmetric spaces* (Ed. W. M. Boothby and G. L. Weiss), pp. 41–117, Marcel Dekker, New York 1972.

[29] R. Godement, *Introduction aux travaux de A. Selberg*, Séminaire Bourbaki, Exposé 144, Paris 1957.

[30] Harish-Chandra, *Spherical functions on a semi-simple Lie group II*, Amer. J. Math. 80 (1958), pp. 553–613.

[31] D. N. Hawley, *Fourier analysis on $\mathfrak{a}$-adic solenoids*, Dissertation, University of Oregon. 1969.

[32] S. Helgason, *Differential geometry and symmetric spaces*, Pure and applied Mathematics, Vol. 12, Academic Press, New York–London 1962.

[33] A. Hewitt, K. A. Ross, *Abstract harmonic analysis I*, Die Grundlehren der math. Wissenschaften in Einzeldarstellungen, Band 115, Springer, Berlin–Göttingen–Heidelberg 1963.

[34] S. Igari, *On the (L^p, L^p)-multipliers*, in: *Functional analysis and its applications* (Ed H. G. Garnir, K. R. Unni and J. H. Williamson), pp. 254–257, Lecture Notes in Math., Vol. 399, Springer, Berlin–Heidelberg–New York 1974.

[35] T. Koornwinder, *Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators*, I, II, Nederl. Akad. Wetensch. Proc. Ser. A 77, (1974), pp. 48–66.

[36] R. Larsen, *The multiplier problem*, Lecture Notes in Math., Vol. 105, Springer, Berlin–Heidelberg–New York 1969.

[37] —, *An introduction to the theory of multipliers*, Die Grundlehren der math. Wissenschaften in Einzeldarstellungen, Band 175, Springer, Berlin–Heidelberg–New York 1971.

[38] W. Schempp, *Zur Theorie der saturierten Approximationsverfahren auf lokalkompakten abelschen Gruppen*, in: *Approximation theory. Proc. Conference at Poznań, 22–26 August, 1972* (Ed. Z. Ciesielski and J. Musielak), pp. 181–187, D. Reidel Publishing Company, Dordrecht–Holland 1975.

[39] A. Schwartz, *The structure of the algebra of Hankel transforms and the algebra of Hankel–Stieltjes transforms*, Canad. J. Math. 23 (1971), pp. 236–246.

[40] G. Sunouchi, *Direct theorems in the theory of approximation*, Acta Math. Acad. Sci. Hungar. 20 (1969), pp. 409–420.

Presented to the Semester
Approximation Theory
September 17–December 17, 1975