Finally, let us reproduce Theorem 4.5 of [7], which shows why the process Z defined by (6) intervenes.

THEOREM 2.5. (a) *Let q be a nonnegative $\mathscr{F}_0^0$-measurable random variable such that $E(q) = 1$. Assume that $E(qZ_\infty) = 1$ and that $qZ_S = 0$ P-a.s. on the set $\bigcup_{(n)} \{S_n = S < \infty\}$. Then X admits $\mathscr{C}'$ for $\hat{P}$-local characteristics if $\hat{P} = (qZ_\infty)\cdot P$.*

(b) *Assume $P' \ll P$ and let $q = \dfrac{dQ'}{dQ}$. If S_n-uniqueness holds for $(\mathscr{C}', Q')$ for each n, we have $E(qZ_\infty) = 1$, $qZ_S = 0$ on the set $\bigcup_{(n)} \{S_n = S < \infty\}$ and qZ is a version of the martingale $E\left(\dfrac{dP'}{dP}|\mathscr{F}_t\right)$.*

(For statement (b) above, we recall that P' is supposed to be given a priori, with P'-local characteristics $\mathscr{C}'$ for X; statement (b) remains true of we replace S-uniqueness by the "property of representation for martingales" with respect to X, for P; cf. [7], [8].)

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SEMI-STABLE MEASURES

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1

We shall deal with the theory of infinitely divisible measures. One of the most important and most interesting problems in this theory is to describe some natural subclasses of the class of all infinitely divisible distributions. By a natural class we mean here a class of measures which coincides with the set of all possible limit laws for some more or less standard array of random variables. Some natural subclasses of infinitely divisible measures are well known and have been examined in detail, for example: stable measures, self-decomposable measures, all infinitely divisible measures.

The class of stable measures will play some role in the sequel, so we recall now that it can be defined as the class of all limit laws for normed sums of random variables. Namely, for a sequence of independent, identically distributed random variables $\xi_1, \xi_2, \ldots$ we consider normed sums of the form $\eta_n = A_n \sum_{k=1}^{n} \xi_k + B_n$, where $A_n > 0$ and B_n are arbitrary real numbers. The class of stable measures consists of all limit laws for the sums η_n. Of course, we can consider random variables ξ_k taking their values in the linear vector space Y. Then B_n are vectors from Y and we obtain the definition of stable measures in Y. In this case the sums may be normed by linear operators, that is, the numbers A_n may be replaced by linear operators acting in Y. In particular, if we consider $\boldsymbol{R}^N$-valued random variables and A_n are non-singular linear operators acting in the N-dimensional Euclidean space, then we obtain the class of operator-stable measures. That interesting class has been introduced and examined by M. Sharpe [12]. More precisely, M. Sharpe described the class of full operator-stable measures. Recall that a measure in $\boldsymbol{R}^N$ is said to be *full* if its support is not contained in any $(N-1)$-dimensional hyperplane.

We shall now quote the theorem of Sharpe [12].

THEOREM 1. *A full measure μ in $\boldsymbol{R}^N$ is operator-stable if and only if it is infinitely divisible and there is a non-singular linear operator B in $\boldsymbol{R}^N$ such that $\mu^t = t^B\mu * \delta_{b(t)}$, $t > 0$, for some $b(t) \in \boldsymbol{R}^N$. We put here by definition $t^B = \exp\{\lg t \cdot B\}$.*

The operators B which can occur in the above representation can be characterized by their spectral properties. Namely, (i) the spectrum of B is in the half-plane $\operatorname{Re} z \geqslant 1/2$, and (ii) the eigenvalues lying on the line $\operatorname{Re} z = 1/2$ are simple.

Recently V. M. Kruglov [9], [10] introduced an interesting subclass of infinitely divisible measures. This class is a natural extension of the class of stable measures. For this reason, it will be called in the sequel a class of *semi-stable* measures (and denoted by S). A probability measure μ is called *semi-stable* if it is the limit law for a sequence of normed sums $A_n \sum_{k=1}^{k_n} \xi_k + B_n$, where ξ_j are independent identically distributed random variables, $A_n > 0$, $B_n \in \boldsymbol{R}$ and $\{k_j\}$ is an increasing sequence of positive integers such that $k_{n+1} k_n^{-1} \to r < \infty$.

We now quote theorem of Kruglov for the case of a separable real Hilbert space [10].

THEOREM 2. *A measure μ in a separable real Hilbert space H is semi-stable if and only if its Fourier transform is either of the form*

$$\hat{\mu}(x) = \exp\{i(a, x) - 1/2(Sx, x)\} \qquad (\textit{Gaussian case})$$

or of the form

$$\hat{\mu}(x) = \exp\left\{i(a, x) + \int_H \left(e^{i(u,x)} - 1 - \frac{i(u, x)}{1+||u||^2}\right) M(du)\right\},$$

where $a \in H$, and M is a semi-finite measure such that the following conditions are satisfied:

(α) $\int_H \min(1, ||u||^2) M(du) < \infty$;

(β) *there exist $\alpha \in (0, 2)$ and $0 < a \neq 1$ such that $T_a M = a^\alpha M$.*

In comparison with the characterization of stable measures given in [5] we can see that there is only one difference in condition (β). Namely, if instead of (β) we take the condition

(β′) *there exists a number $\alpha \in (0, 2)$ such that $T_a M = a^\alpha M$ for all $a > 0$,*

then we get the description of the class of stable measures in H.

2

Let us denote by G the group of all non-singular linear operators in $\boldsymbol{R}^N$. Our next purpose is to describe the class of all full measures in $\boldsymbol{R}^N$ which are the limit laws for sums of the form

$$(*) \qquad A_n(\xi_1 + \ldots + \xi_{k_n}) + c_n,$$

where $\{\xi_n\}$ is a sequence of independent, identically distributed $\boldsymbol{R}^N$-valued random variables, $A_n \in G$, $c_n \in \boldsymbol{R}^N$ and $\{k_n\}$ is an increasing sequence of positive integers such that $k_{n+1} k_n^{-1} \to r < \infty$.

Limit laws for the sums $(*)$ will be called *operator semi-stable.*

THEOREM 3. *A full probability measure in $\boldsymbol{R}^N$ is operator semi-stable if and only if it is infinitely divisible and there exist a number $0 < c < 1$, a vector $b \in \boldsymbol{R}^N$ and an operator $B \in G$ such that*

$$(**) \qquad \mu^c = B\mu * \delta(b)$$

holds. The spectrum of B is contained in the disc $\{|z|^2 \leqslant c\}$. Eigenvalues of B satisfying $|\lambda|^2 = c$ are simple, i.e. the elementary divisors of B associated with these eigenvalues are one-dimensional.

Furthermore, the measure μ can be decomposed into a product $\mu = \mu_1 * \mu_2$ of two measures μ_1 and μ_2, concentrated on B-invariant subspaces X_1 and X_2, respectively, and such that $\boldsymbol{R}^N = X_1 \oplus X_2$, μ_1 is a full semi-stable measure on X_1 of the Poisson-type (having no Gaussian component) and μ_2 is a full Gaussian measure on X_2. The spectrum of $B|X_1$ is then contained in the disc $\{|z|^2 < c\}$ and for the eigenvalues of $B|X_2$ the equality $|z|^2 = c$ holds.

The proof of our theorem is rather long and we will give it only in outline. Some details can be found in [4]. We start with some lemmas. In the proofs of the following two lemmas the technique developed by K. Urbanik [15] is used. We omit their proofs.

LEMMA 1. *Let μ be a full measure for which the formula*

$$(1) \qquad \mu = \lim_{n\to\infty} A_n \nu^{k_n} * \delta(b_n)$$

holds, where $\nu \in M$, $A_n \in G$, $b_n \in \boldsymbol{R}^N$ and $k_n^{-1} k_{n+1} \to \gamma < \infty$. Then $A_n \to \theta$.

LEMMA 2. *If μ is a full measure for which formula* (1) *holds, then the sequence of operators*

$$(2) \qquad \{(A_k^*)^{-1} A_{k+1}^*\}$$

is precompact in G. Moreover, if C is a limit point of sequence (2), *then the formula*

$$(3) \qquad \hat{\mu}(y) = [\hat{\mu}(Cy)]^\gamma e^{i(b,y)}$$

holds.

In the sequel it will be convenient to adopt the following definition. A Borel measure M in $\boldsymbol{R}^N$ is called a *Lévy–Khintchine spectral measure* (or briefly LK-measure) if M is a semi-finite measure concentrated on $\boldsymbol{R}^N \setminus \{0\}$ and such that

$$(\text{a}) \qquad \int \min(1, ||u||^2) M(du) < \infty.$$

Denote by Π_x the orbit $\Pi_x = \{B^k x,\ k \text{ runs over all integers}\}$. The following lemma is crucial for the proof of our theorem.

LEMMA 3. *Let M be a non-trivial* LK-*measure concentrated on the orbit Π_x, where $B \in G$, and let the formula $BM = cM$ hold for a certain* $0 < c < 1$. If *V_x is the complex Euclidean space spanned over Π_x, then the spectrum of the operator $B|V_x$ is contained in the open disc $\{|z|^2 < c\}$.*

Sketch of the proof of the lemma. In our case condition (a) is equivalent to

$$\sum_{n=1}^{\infty} \|B^n x\|^2 c^{-n} < \infty, \tag{b}$$

where $\|\cdot\|$ is an arbitrary norm in V_x. To establish a suitable norm in (b) we take the Jordan basis in V_x with respect to B, say $\{z_1, z_2, \ldots, z_n\}$. Let us choose the norm by putting

$$\|y\| = \sum_{s=1}^{n} |\alpha_s| \quad \text{for} \quad y = \sum_{s=1}^{n} \alpha_s z_s.$$

Then from the properties of the Jordan basis it can easily be deduced that each eigenvalue λ of $B|V_x$ satisfies

$$\|B^n x\| \geqslant d|\lambda|^n, \quad d > 0 \text{ for all } n.$$

Comparing this with (b), we obtain $|\lambda|^2 < c$.

From Lemma 3, by standard reasoning, we can get the following

LEMMA 4. *If M is an* LK-*measure and $BM = cM$ for a certain $0 < c < 1$, $B \in G$, then M is concentrated on a B-invariant subspace $X \subset \boldsymbol{R}^N$ such that the spectrum $B|X$ is contained in $\{|z|^2 < c\}$.*

LEMMA 5. *If a measure μ is full and μ is a weak limit of a sequence of the form*

$$\mu = \lim A_n \nu^{k_n} * \delta(b_n),$$

where $A_n \in G$, $b_n \in \boldsymbol{R}^N$, ν is a probability measure in $\boldsymbol{R}^N$, $k_n \to \infty$, $k_{n+1} \cdot k_n^{-1} \to 1$, then μ is operator-stable in the sense of Sharpe.

Sketch of the proof of the lemma. By Lemma 1, μ is infinitely divisible. Let $\alpha \in (0, 1)$. Fix a sequence $l(n)$ of integers such that $k_{l(n)} k_n^{-1} \to \alpha$. After some computations we get

$$\mu^\alpha = \lim C_n P_n * \delta(a_n),$$

where $P_n = A_{l(n)} \nu^{k_{l(n)}} * \delta(b_{l(n)})$ and $C_n \in G$.

Then $P_n \to \mu$ and $C_n P_n * \delta(a_n) \to \mu^\alpha$ (full measure!). By the Compactness Lemma of Sharpe ([12], p. 55), the sequence $\{C_n\}$ is precompact in G, and $\{a_n\}$ is precompact in $\boldsymbol{R}^N$. Denoting by $C_{1/\alpha}$ and a_α the limit points of these sequences, we get $\mu^\alpha = C_{1/\alpha} \mu * \delta(c_n)$, $c_n \in \boldsymbol{R}^N$. Putting $\alpha = n^{-1}$, we obtain $\mu = C_n \mu^n * \delta(c_n)$, which means the operator-stability of μ in the sense of Sharpe.

Sketch of the proof of Theorem 3. *Sufficiency.* Let μ be an infinitely divisible measure satisfying

$$\mu^c = B\mu * \delta(b) \tag{$*$}$$

for a certain $0 < c < 1$, $B \in G$ and $b \in \boldsymbol{R}^N$. From $(*)$ it easily follows that there exists a sequence of vectors $\{b_n\}$ of $\boldsymbol{R}^N$ such that

$$\mu = (B^n \mu)^{r^n} * \delta(b_n), \quad \text{where} \quad r = 1/c > 1,$$

holds. Putting $k_n = \operatorname{Entier}(r^n)$, we obtain

$$\mu = \lim_{n\to\infty} B^n \mu^{k_n} * \delta(b_n),$$

where, obviously, $k_n \nearrow \infty$ and $k_n^{-1} k_{n+1} \to r > 1$, which proves the semistability of μ.

Necessity. Infinite divisibility of μ follows immediately from Lemma 1. Condition $(*)$ in the case where $r > 1$ follows from Lemma 2. If $r = 1$, then μ is stable by Lemma 5 and Sharpe's formula holds.

$$\mu^t = t^A \mu * \delta(b_t), \quad t > 0.$$

Taking an arbitrary $0 < c < 1$ and putting $B = c^A$, $b_c = b$, we get

$$\mu^c = B\mu * \delta(b) \tag{$*$}$$

also in the case where $r = 1$.

Let us now write the Lévy–Khintchine representation of $\hat{\mu}$,

$$\hat{\mu}(y) = \exp\Big\{i(x_0, y) - \tfrac{1}{2}(Dy, y) + \int_{R^N \setminus \{0\}} K(x, y) M(dx)\Big\}, \tag{c}$$

where $x_0 \in \boldsymbol{R}^N$, D is a symmetric non-negative linear operator in $\boldsymbol{R}^N$, M is a Lévy–Khintchine spectral measure and the kernel K is defined by

$$K(x, y) = e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|^2}.$$

Writing $(*)$ in terms of characteristic functions, we get, by the uniqueness of representation (c), the following conditions:

$$BM = cM, \tag{4}$$

$$BDB^* = cD. \tag{5}$$

Let $\boldsymbol{R}^N = X \oplus Y$ be a decomposition of $\boldsymbol{R}^N$ into a direct sum of B-invariant subspaces such that

$$\operatorname{spectrum} B|X \subset \{|z|^2 < c\},$$

$$\operatorname{spectrum} B|Y \subset \{|z|^2 \geqslant c\}.$$

By virtue of Lemma 4 we have $M(Y) = 0$. In particular, if $X = \boldsymbol{R}^N$, then the measure μ is a full measure of the Poisson-type (without a Gaussian component). To simplify the notation we assume for a moment that $Y = \boldsymbol{R}^N$. In this case μ reduces to a Gaussian measure with the characteristic function

$$\hat{\mu}(y) = \exp\{i(x_0, y) - \tfrac{1}{2}(Dy, y)\},$$

where the operator D satisfies (5). This immediately implies that the spectrum of $B|Y$ lies in fact on the circle $\{|z|^2 = c\}$. By $\tilde{X}$ (or $\tilde{A}$) we denote a natural linear extension of X (or A acting in X) to the complex case. The sign "$\sim$" will be omitted if it is clear which case we deal with. When talking about spectral properties of operators, we always mean the properties of their natural complex extensions.

One can now show that all the eigenvalues $\tilde{B}|\tilde{Y}$ are simple.

Summing up, there exists a decomposition of $\boldsymbol{R}^N$ into a direct sum $\boldsymbol{R}^N = X \oplus Y$ such that μ can be represented as a product $\mu = \mu_1 * \mu_2$, where μ_1 is a semi-stable full measure on X without a Gaussian component and μ_2 is a full Gaussian measure on Y. The spectrum of B is contained in the disc $\{|z|^2 \leqslant c\}$. Moreover, spectrum $B|X \subset \{|z|^2 < c\}$ and spectrum $B|Y \subset \{|z|^2 = c\}$. This ends the proof of necessity.

3

It is well known that the class L of Lévy's (self-decomposable) measures contains the class of all stable measures. It is not difficult to notice that there exist semi-stable measures which are not self-decomposable and conversely. Our next aim is to describe the intersection LS of the classes L (of self-decomposable measures) and S (of semi-stable measures). We will confine ourselves to the case of the real line. The characterization of the class LS is given by the following

THEOREM 4. *A function φ is the characteristic function of a distribution from LS if and only if either*

$$\varphi(t) = \exp(it\gamma - \tfrac{1}{2}\sigma^2 t^2),$$

where $\gamma \in \boldsymbol{R}$, $\sigma > 0$ (i.e. φ is the characteristic function of Gaussian measure), or there exist $\alpha \in (0, 2)$ and $a \in (0, 1)$ such that

$$\varphi(t) = \exp\left\{i\gamma t + \sum_{k=-\infty}^{\infty} a^{k\alpha} \int\limits_{[c,|c|]\setminus\{0\}} \left[\int\limits_{R\setminus\{0\}} \left(e^{ita^{-k}z} - 1 - \frac{ita^k z}{a^{2k}+z^2}\right) \frac{\psi_s(z)}{|z|}\, dz\right] \nu(ds)\right\},$$

where $c = \ln a < 0$, ν is a finite Borel measure on $[c, |c|]\setminus\{0\}$ and ψ_s is given by the formula

$$\psi_s(z) = 1_{\operatorname{sgn} s[1, e^{|s|})}(z) + a^{\alpha} \cdot 1_{\operatorname{sgn} s[e^{|s|}, a^{-1})}(z), \qquad z \in \boldsymbol{R}.$$

This description of LS is rather complicated but it does not seem possible to simplify it.

The proof of the Theorem is long but the general idea is simple and we will present it here. The proof is based on the extreme-point method (Krein–Milman–Choquet Theorem [8], [2]), adapted to the theory of infinitely divisible distributions by Kendall [7], Johansen [6] and Urbanik [13], [14]. Some details can be found in [3].

Let $\boldsymbol{R}$ denote the set of all real numbers, $\boldsymbol{R}_0 = \boldsymbol{R}\setminus\{0\}$, $\boldsymbol{R}^+ = (0, \infty)$, $\boldsymbol{R}^- = (-\infty, 0)$ and $\overline{\boldsymbol{R}} = [-\infty, \infty]$.

Denote by P the space of all Borel probability measures on $\overline{\boldsymbol{R}}$ with the topology of weak convergence of distributions. P is a metrizable compact space.

We consider for $\mu \in P$ the following functions:

$$\begin{aligned} Q_\mu^1(x) &= \int\limits_{[-\infty, -e^{-x})} (1+y^2)y^{-2}\mu(dy), \\ Q_\mu^2 x &= \int\limits_{(e^x, \infty]} (1+y^2)y^{-2}\mu(dy), \end{aligned} \qquad x \in R. \tag{6}$$

For a characteristic function φ of infinitely divisible distribution let us write the Lévy–Khinchine formula

$$\varphi(t) = \exp\left\{it\gamma + \int\limits_{\boldsymbol{R}} [e^{itx} - 1 - itx(1+x^2)^{-1}](1+x^2)x^{-2}\mu(dx)\right\}, \qquad t \in \boldsymbol{R}, \tag{7}$$

where $\gamma \in \boldsymbol{R}$ and μ is a finite Borel measure on $\boldsymbol{R}$.

It is known ([11]) that the class L coincides with the class of all infinitely divisible distributions for which both functions in (6) are convex.

Here μ denotes the Lévy–Khinchine spectral measure (concentrated on $\boldsymbol{R}$) which corresponds to the infinitely divisible distribution in representation (7).

On the other hand, the class S consists of all distributions for which in (7) either μ is concentrated on $\{0\}$ (the Gaussian case) or μ is concentrated on $\boldsymbol{R}_0$, and there exist $\alpha \in (0, 2)$ and $a \in (0, 1)$ such that

$$\int\limits_{a^{-1}E} (1+y^2)y^{-2}\mu(dy) = a^{\alpha} \int\limits_{E} (1+y^2)y^{-2}\mu(dy) \tag{8}$$

for every $E \in \operatorname{Bor}(\boldsymbol{R}_0)$ (comp. [10]).

Let $LS_{a,\alpha}$ be the class of all self-decomposable distributions for which the spectral measure in representation (7) satisfies (8). Let $N_{a,\alpha}$ denote the set of all finite Borel measures μ on $\overline{\boldsymbol{R}}$ such that the functions in (6) are convex and (8) holds. By $N_{a,\alpha}^0$ ($N_{a,\alpha}^+$, $N_{a,\alpha}^-$) we shall mean the set of those measures μ from $N_{a,\alpha}$ which are concentrated on $\boldsymbol{R}_0$ ($\boldsymbol{R}^+$, $\boldsymbol{R}^-$, resp.).

Finally, $K_{a,\alpha}$ denotes the intersection of the sets $N_{a,\alpha}$ and P. $K_{a,\alpha}^0$, $K_{a,\alpha}^+$, $K_{a,\alpha}^-$ are defined in an analogous manner. The sets K with suitable indices are endowed with the topologies induced by P.

Notice that μ is the Lévy–Khinchine spectral measure in representation (7) of some distribution from $LS_{a,\alpha}$ if and only if μ belongs to $N_{a,\alpha}^0$.

Obviously, the set $K_{a,\alpha}$ is convex. One can prove that it is compact.

To use the Choquet Theorem we have to find the set $e(K_{a,\alpha})$ of extreme points of $K_{a,\alpha}$.

Let us put $\mu^-(E) = \mu(-E)$, $E \in \operatorname{Bor}(\boldsymbol{R}_0)$.

Notice that the one-point measures δ_0, $\delta_{-\infty}$, δ_∞ belong to $K_{a,\alpha}$ and, moreover, the measure μ belongs to $N_{a,\alpha}$ if and only if both of its restrictions to $\boldsymbol{R}^+$ and $\boldsymbol{R}^-$ do.

Hence we can see that the extreme points of $K_{a,\alpha}$ are measures concentrated on one of the following sets: $\{-\infty\}$, $\{0\}$, $\{\infty\}$, $\boldsymbol{R}^+$ and $\boldsymbol{R}^-$.

The one-point measures δ_0, δ_∞, $\delta_{-\infty}$ are evidently extreme points of $K_{a,\alpha}$. If μ is an extremal measure concentrated on $\boldsymbol{R}^+$, then μ^- is such a measure on $\boldsymbol{R}^-$ and conversely. Consequently, it suffices to find those extreme points of $K_{a,\alpha}$ which are probability measures concentrated on $\boldsymbol{R}^-$. Notice that all of these extreme points are extreme points of $K_{a,\alpha}^-$.

Consider an arbitrary μ from $K_{a,\alpha}^-$. The suitable function

$$Q_\mu(x) = \int\limits_{(-\infty, -e^{-x})} 1(1+y^2)y^{-2}\mu(dy), \qquad x \in \boldsymbol{R}$$

is non-negative, monotone non-decreasing, convex and $Q_\mu(-\infty) = 0$. Moreover, Q_μ satisfies the following condition of quasi-periodicity:

$$Q_\mu(x+c) = a^\alpha Q_\mu(x), \quad x \in \boldsymbol{R},$$

where $c = \ln a < 0$.

Thus the function Q may be written in the form

$$Q_\mu(x) = \int_{-\infty}^{x} q_\mu(u)\,du, \quad x \in \boldsymbol{R},$$

where q_μ is non-negative, monotone non-decreasing and such that

$$q_\mu(x+c) = a^\alpha q_\mu(x) \tag{9}$$

for almost all $x \in \boldsymbol{R}$, and

$$\int_{-\infty}^{0} q_\mu(-\ln|x|)|x|(1+x^2)^{-1}dx = 1.$$

One can assume that q_μ is continuous from the left. Then q_μ is uniquely determined by μ and condition (9) holds for all $x \in \boldsymbol{R}$.

Conversely, if q is non-negative, monotone non-decreasing, continuous from the left and satisfies the conditions

$$q(x+c) = a^\alpha q(x), \quad x \in \boldsymbol{R},\ c = \ln a, \tag{10}$$

$$\int_{-\infty}^{0} q(-\ln|x|)|x|(1+x^2)^{-1}dx = 1, \tag{11}$$

then the formula

$$\mu(E) = \int_E q(-\ln|x|)|x|(1+x^2)^{-1}dx, \quad E \in \mathrm{Bor}(\boldsymbol{R}^-) \tag{12}$$

defines a measure $\mu \in K_{a,\alpha}^-$ such that $q_\mu = q$.

In other words, the above formulas establish a one-to-one correspondence between the set $K_{a,\alpha}^-$ and the class Q of all those non-negative, monotone non-decreasing functions continuous from the left which satisfy (10) and (11). The correspondence in question preserves convex combinations and hence extreme points of $K_{a,\alpha}^-$ are transformed onto extreme functions (non-decomposable in the class Q). The problem has been reduced to finding all non-decomposable functions in the class Q.

One can prove that the class of functions non-decomposable in Q coincides with the one-parameter family of functions

$$q_s(u) = \begin{cases} \beta_s, & c < u \leqslant s, \\ \gamma_s, & s < u \leqslant 0, \end{cases}$$

where $s \in [c, 0)$ and the constants β_s and γ_s are uniquely determined by the re-

lations

$$\beta_s = a^\alpha \gamma_s,$$

$$\beta_s \int_{(-a^{-1}, -e^{-s}]} \Big[\sum_{k=-\infty}^{\infty} a^{\alpha k}|z|(a^{2k}+z^2)^{-1}\Big] dz + \gamma_s \int_{(-e^{-s}, -1]} \Big[\sum_{k=-\infty}^{\infty} a^{2k}|z|(a^{2k}+z^2)^{-1}\Big] dz = 1.$$

The limit case $s = c$ means that

$$\gamma_c = \Big\{\int_{(-a^{-1}, -1]} \Big[\sum_{k=-\infty}^{\infty} a^{\alpha k}|z|(a^{2k}+z^2)^{-1}\Big] dz\Big\}^{-1}$$

and β_c simply disappears.

The one-to-one mapping m from $S = [c, |c|] \cup \{\pm\infty\}$ onto $e(K_{a,\alpha})$ given by $m(s) = \mu_s$ is a Borel automorphism. By the Choquet Theorem for an arbitrary $\mu \in K_{a,\alpha}$ there exists a probability measure $\bar{\nu}$ on S such that

$$\int_{\bar{\boldsymbol{R}}} f(x)\mu(dx) = \int_S \Big[\int_{\bar{\boldsymbol{R}}} f(x)\mu_s(dx)\Big] \bar{\nu}(ds)$$

for every function f continuous on $\bar{\boldsymbol{R}}$. Notice that for $\mu \in K_{a,\alpha}^0$ the suitable measure $\tilde{\nu}$ is concentrated on $[c, |c|] \setminus \{0\}$. Thus

$$\int_{\boldsymbol{R}_0} f(x)\mu(dx) = \int_{[c,|c|]\setminus\{0\}} \Big\{\int_{\boldsymbol{R}_0} f(x)\mu_s(dx)\Big\} \tilde{\nu}(ds)$$

holds for every continuous and bounded function f on $\boldsymbol{R}_0$. Putting

$$f_t(x) = [e^{itx} - 1 - itx(1+x^2)^{-1}](1+x^2)x^{-2}$$

and using (12) we obtain after calculations

$$\int_{\boldsymbol{R}_0} f_t(x)\mu_s(dx) = \gamma_s \sum_{k=-\infty}^{\infty} a^{\alpha k} \int_{\boldsymbol{R}_0} \Big| e^{ita^{-k}z} - 1 - \frac{ita^k z}{a^{2k}+z^2}\Big| \frac{\psi_s(z)}{|z|}\,dz.$$

Now, we observe that γ_s, as a function of $s \in [c, |c|] \setminus \{0\}$, is symmetric and on the interval $[c, 0)$ it increases from γ_c to $\gamma^* = a^{-\alpha}\gamma_c$. Hence the function γ is positive bounded and Borel measurable.

If now $\mu \in N_{a,\alpha}^0$, that is, if μ is a spectral measure of probability distribution from $LS_{a,\alpha}$, then there exists a finite Borel measure on $[c, |c|] \setminus \{0\}$ such that

$$\int_{\boldsymbol{R}_0} f_t(x)\mu(dx) = \sum_{k=-\infty}^{\infty} a^{\alpha k} \int_{[c,|c|]\setminus\{0\}} \Big[\int_{\boldsymbol{R}_0} \Big(e^{ita^{-1}z} - 1 - \frac{ita^k z}{a^{2k}+z^2}\Big) \frac{\psi_s(z)}{|z|}\,dz\Big] \nu(ds),$$

which completes the proof.

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A REPRESENTATION OF THE CHARACTERISTIC FUNCTIONS OF STABLE DISTRIBUTIONS IN HILBERT SPACES

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1

Let H be a separable, real Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $||\cdot||$. A countable additive and normed measure μ defined on the σ-field $\mathscr{B}$ of Borel subsets of H is called a *probability distribution* in H. By $\mu*\nu$ we denote the convolution of the distributions μ and ν (see [5], p. 57), and by δ_x, where $x \in H$, we denote the one-point distribution concentrated at the point x. The characteristic function $\hat{\mu}$ of a distribution μ is defined by the formula

$$\hat{\mu}(y) = \int_H e^{i(y,x)}\mu(dx),$$

where $y \in H$. A distribution μ is uniquely defined by the characteristic function $\hat{\mu}$ (see [5], p. 152).

For every non-negative number a we define the mapping T_a from H into itself by means of the formula $T_a x = ax$. Further, if μ is a distribution, and a is a positive number, then $T_a\mu$ denotes the distribution defined by the formula

$$(T_a\mu)(E) = \mu(a^{-1}E)$$

for all $E \in \mathscr{B}$. For $a = 0$ we put $T_0\mu = \delta_0$. A distribution μ is said to be *stable* if for every pair of positive numbers a and b there exist a positive number c and an element x of the space H such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x.$$

In [2] the following theorem has been proved:

THEOREM 1. *A function φ defined on H is the characteristic function of a stable distribution in H if and only if either*

$$\varphi(y) = \exp[i(y, x_0) - \tfrac{1}{2}(Dy, y)],$$

where $x_0 \in H$ and D is an S-operator, or

$$\varphi(y) = \exp\left\{i(y, x_0) + \int_{H\setminus\{0\}} \left[e^{i(y,x)} - 1 - \frac{i(y, x)}{1+||x||^2}\right] M(dx)\right\},$$