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HOMOGENIZATION AND ERGODIC THEORY

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1. Introduction

Homogenization deals with the following general phenomenon. Let us consider a model describing some physical system, with a periodic spatial structure. More precisely, the model (in general, a partial differential equation or an integro-differential equation) involves coefficients depending on the space variable in a periodic way, but with a very small period ε (to simplify the same in all directions). Such a situation occurs in many concrete applications, especially in the field of composite materials, or nuclear reactors. More generally, one can think of rapidly varying coefficients not only in the space variable, but also in the time variable, in which case the terminology "averaging" is more standard than "homogenization".

The problem concerns the behaviour of the model as $\varepsilon \to 0$. A reasonable intuition is that the ε-model can be approximated with a model describing the same type of physical phenomenon, but with homogenized coefficients. In other words, the homogenized coefficients will be some mean of the original coefficients. Such a statement turns out to be true in many cases. However, the computation of the right mean does not correspond in general to what can be guessed a priori, and intuition can be very misleading.

Although homogenization problems are not problems arising in probability theory, and can be dealt with using only analytical techniques, it turns out that since many models of interest have a probabilistic interpretation, there is a probabilistic approach to homogenization. It uses namely results of ergodic theory. In this article, we will restrict ourselves to the probabilistic approach. We refer to our forthcoming book [4] for details concerning analytical approaches (see also [2], [3]).

Homogenization has been studied by several authors in various fields (Analysis. Numerical Analysis. Probability theory). Let us mention, in particular, Babuska [1], de Giorgi–Spagnolo [6], Freidlin [7], Spagnolo [10], Stroock–Varadhan [11], Tartar [13].

2. Homogenization for diffusions

We will restrict ourselves to diffusions, although the same methodology can be applied in many other situations.

Let $a_{ij}(x)$, $b_i(x)$, $c_i(x)$, $i, j = 1, \ldots, n$ be functions on R^n such that

$$a_{ij}, b_i, c_i \in C^1(R^n); \quad \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} \in L^\infty(R^n), \quad \sum a_{ij}\xi_i\xi_j \geqslant \beta \sum \xi_i^2, \quad \beta > 0, \ \forall \xi_1, \ldots, \xi_n \in R, \tag{2.1}$$

$$a_{ij}, b_i, c_i \text{ are periodic in all variables with period } 1. \tag{2.2}$$

Let $\mathcal{O}$ be an open bounded regular subset of R^n, whose boundary is denoted by Γ, and $f \in C^0(\overline{\mathcal{O}})$, $\alpha > 0$. Let $u_\varepsilon(x)$ be the solution of the Dirichlet problem

$$-a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} - \frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_i} - c_i\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_i} + \alpha u_\varepsilon = f, \quad u_\varepsilon|_\Gamma = 0. \tag{2.3}$$

In (2.3) we have not written the signs of summation. The general convention which will be used hereafter is that when indices are repeated there must be a summation over that index (Einstein's convention).

We are interested in the behaviour of $u_\varepsilon(x)$ as $\varepsilon \to 0$. We will use the probabilistic interpretation of $u_\varepsilon(x)$, which we now describe.

Let us set $\Omega = C^0([0, \infty); R^n)$ equipped with the topology of Fréchet space of uniform convergence on compact subsets of $[0, \infty)$. Let

$$x(t, \omega) = \omega(t)$$

be the canonical process and $\mathcal{F}^t = \sigma(x(s), 0 \leqslant s \leqslant t)$. Then $\mathcal{F}^\infty$ coincides with the Borel σ-algebra on Ω.

By changing a_{ij} into $\frac{1}{2}(a_{ij}+a_{ji})$ which does not affect the solution u_ε of (2.3), we may without loss of generality assume that

$$a_{ij} = a_{ji}. \tag{2.4}$$

Let then σ be such that

$$\frac{\sigma\sigma^*}{2} = a, \tag{2.5}$$

where a is the matrix a_{ij}. Such a factorization exists.

There exists one and only one measure P_x^ε on Ω such that $x(t)$ is the solution of the Ito equation

$$dx = \frac{1}{\varepsilon} b\left(\frac{x}{\varepsilon}\right)dt + c\left(\frac{x}{\varepsilon}\right)dt + \sigma\left(\frac{x}{\varepsilon}\right)dw_\varepsilon(t), \quad x(0) = x, \tag{2.6}$$

where $w_\varepsilon(t)$ is a standard Wiener process and an $\mathcal{F}^t$ martingale. Let τ be the exit time of $x(t)$ from $\mathcal{O}$, i.e.

$$\tau = \inf\{s \geqslant 0 | \ x(s) \notin \mathcal{O}\};$$

then it is well known that the following relation holds true

$$u_\varepsilon(x) = E_x^\varepsilon \int_0^\tau e^{-\alpha t} f(x(t))\,dt. \tag{2.7}$$

Moreover, we can write

$$\int_0^\tau e^{-\alpha t} f(x(t))\,dt = F(\omega)$$

which defines a functional on Ω.

The following result is standard:

$$P_x^\varepsilon \text{ a.s. } F \text{ is continuous and bounded.} \tag{2.8}$$

We can rewrite (2.7) as follows:

$$u_\varepsilon(x) = \int_\Omega F(\omega)\,dP_x^\varepsilon(\omega). \tag{2.9}$$

It is clear from (2.9) that the convergence of $u_\varepsilon(x)$ is related to the weak limit of P_x^ε in the space μ_+^1 of probabilities measures on Ω equipped with the weak topology

$$\mu_n \to \mu \Leftrightarrow \mu_n(\varphi) \to \mu_n(\varphi)$$

for any φ continuous and bounded on Ω. If we prove that

$$P_x^\varepsilon \to P_x \text{ in the weak topology} \tag{2.10}$$

and if

$$F(\omega) \text{ is } P_x \text{ a.s. continuous,} \tag{2.11}$$

then it is well known that (see Gihman–Skorohod [8])

$$u_\varepsilon(x) \to u(x) = \int_\Omega F(\omega)\,dP_x(\omega). \tag{2.12}$$

We will prove (2.10) and P_x will be a nondegenerate diffusion for which (2.11) is true, hence (2.12) follows.

3. Some technical results from ergodic theory

Let us consider on an arbitrary probability space the process

$$dy = b(y)dt + \sigma(y)dw(t), \quad y(0) = x, \tag{3.1}$$

where $w(t)$ is a standard Wiener process. Such a construction is possible, by virtue of the regularity assumptions on b, σ.

Let $\Pi = (0, 1)^n$ and let $\tilde{\Pi}$ be the torus obtained by identifying the opposite faces of Π. We also can write

$$\tilde{\Pi} = R^n/Z^n$$

and $\tilde{\Pi}$ is a compact space. The Borel σ-algebra on $\tilde{\Pi}$ can be identified with the sub-σ-algebra of Borel periodic sets of $\boldsymbol{R}^n$ (i.e. whose characteristic function is periodic), by the formula

$$E = \bigcup_{k_1,\ldots,k_n \in \boldsymbol{Z}} \dot{E}+k_i e_i, \tag{3.2}$$

where $\dot{E}$ is a Borel subset of $\tilde{\Pi}$, and E is a periodic Borel subset of $\boldsymbol{R}^n$ ($e_1, \ldots, e_n$ are the unit coordinate vectors). Since b and σ are periodic, (3.1) defines a process on the torus and, moreover, a Markov process on the torus.

We can easily compute its transition function. Indeed, if we set

$$A = -a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - b_i \frac{\partial}{\partial x_i}, \tag{3.3}$$

there exists a unique Green functions $p(x, t, y) = \boldsymbol{R}^n \times [0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$ such that

$$\begin{aligned} & p(x, t, y) > 0, \quad p \text{ is continuous on } \{t > 0,\ x \in \boldsymbol{R}^n, y \in \boldsymbol{R}^n\}, \\ & p \text{ is } C^2 \text{ in } x,\ C^1 \text{ in } t, \end{aligned} \tag{3.4}$$

as a function of x, t, the function p satisfies

$$\begin{aligned} & \frac{\partial p}{\partial t} + Ap = 0, \\ & \forall x,\ \int p(x, t, y) f(y)\, dy \to f(x) \text{ as } t \to 0, \quad \forall f \text{ continuous and bounded.} \end{aligned} \tag{3.5}$$

Now, if f is periodic, $p(x, t, y)$ is periodic in x. If $\dot{x} \in \tilde{\Pi}$, $\dot{E}$ is a Borel subset of $\tilde{\Pi}$, then we can set

$$P(\dot{x}, t, \dot{E}) = \sum_{k_1,\ldots,k_n \in \boldsymbol{Z}} \int_{\dot{E}} p\Big(x, t, y+\sum_i k_i e_i\Big) dy = \int_E p(x, t, y)\, dy \tag{3.6}$$

and the right hand side of (3.6) does not depend on the particular choice of the representative x of $\dot{x}$. Then $P(\dot{x}, t, \dot{E})$ is the transition probability function of the Markov process $y(t)$ on the torus.

Since for any $t > 0$, $p(x, t, y) \geqslant C > 0$ $\forall x, y \in \bar{\Pi}$ (the closure of Π) and since $\sum_{k_1,\ldots,k_n \in \boldsymbol{Z}} p(x, t, y+\sum_i k_i e_i)$ is convergent (x, y fixed, $t > 0$), then $P(\dot{x}, t, \dot{E})$ has a density with respect to the Lebesgue measure (which is a probability on $\tilde{\Pi}$)

$$P(\dot{x}, t, \dot{E}) = \int_{\dot{E}} p_0(\dot{x}, t, \dot{z})\, d\dot{z}$$

such that $\forall \dot{E} \in \tilde{\Pi}$ of positive Lebesgue measure, and $\forall \dot{x} \in \tilde{\Pi}$, $t > 0$, there exists $\delta > 0$ such that $p_0(\dot{x}, t, \dot{z}) \geqslant \delta$, $\forall \dot{z} \in \dot{E}$.

This is much more that what is sufficient to insure that $y(t)$ is a strong ergodic process on the torus.

By standard ergodic theory we can thus state

THEOREM 3.1. *Under assumptions* (2.1), (2.2) *there exists one and only one invariant measure on* $\tilde{\Pi}$, $\bar{P}$ *such that* $\forall f$ *Borel, periodic and bounded function on* $\boldsymbol{R}^n$ (*i.e., f is Borel and bounded on the torus* $\tilde{\Pi}$) *one has*

$$\sup_x \Big| Ef(y_x(t)) - \int_{\tilde{\Pi}} f d\bar{P} \Big| \leqslant \|f\| \gamma e^{-\delta t}, \tag{3.7}$$

where γ, δ *are positive constants depending only on* β *and the bounds of* a_{ij}, b_i. ■

Remark 3.1. If one can solve the adjoint equation

$$A^* m = 0, \quad \int_{\tilde{\Pi}} m(x)\, dx = 1, \tag{3.8}$$

where

$$A^* = -\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\cdot) + \sum_i \frac{\partial}{\partial x_i}(b_i \cdot), \tag{3.9}$$

then it can be shown that $\bar{P}$ has a density with respect to the Lebesgue measure, which is m. In particular, if

$$A = -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j},$$

i.e.

$$b_j = \frac{\partial a_{ij}}{\partial x_i},$$

then $A^* = A$, and $m = 1$, i.e. P is the Lebesgue measure on $\tilde{\Pi}$ ($\int_{\tilde{\Pi}} f d\bar{P} = \int_{\Pi} f(x)\, dx$). ■

THEOREM 3.2. *The assumptions are those of Theorem* 3.1. *Let* $\varphi(x)$ *be a Borel periodic bounded function such that*

$$\int_{\tilde{\Pi}} \varphi\, d\bar{P} = 0. \tag{3.10}$$

Then there exists one and only one (up to a constant) solution of

$$\begin{aligned} & -a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} - b_i \frac{\partial z}{\partial x_i} = \varphi, \\ & \qquad z \in W^{2,p,\mu}(\boldsymbol{R}^n),^{(1)}\ \forall p \geqslant 1,\ p < \infty,\ z \text{ periodic.} \end{aligned} \tag{3.11}$$

Proof. Without assuming (3.10), but only that φ is $L^p(\boldsymbol{R}^n)$ one can solve for $\alpha > 0$,

$$-a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} - b_i \frac{\partial z_\alpha}{\partial x_i} + \alpha z_\alpha = \varphi, \quad z \in W^{2,p,\mu}(\boldsymbol{R}^n). \tag{3.12}$$

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(1) Sobolev space with weights; $z \in W^{2,p,\mu} \Leftrightarrow z, \dfrac{\partial z}{\partial x_i}, \dfrac{\partial^2 z}{\partial x_i \partial x_j} \in L^{p,\mu}$ where $L^{p,\mu} = \{\varphi \mid \int_{\boldsymbol{R}^n} |\varphi(x)|^p \exp - p\mu |x|\, dx < \infty\}$.

There exists one and only one solution of (3.12). This can be proved by using variational techniques and the iterative scheme

$$(3.13)\quad -a_{ij}\frac{\partial^2 z_\alpha^{n+1}}{\partial x_i \partial x_j} - b_i \frac{\partial^2 z_\alpha^{n+1}}{\partial x_i} + \alpha z_\alpha^{n+1} + \lambda z_\alpha^{n+1} = \lambda z_\alpha^n + \varphi, \qquad z_\alpha^{n+1} \in W^{2,p,\mu}(R^n)$$

as in Bensoussan–Lions [5]. Details are omitted. Furthermore, $z_\alpha(x)$ has a probabilistic representation, namely

$$(3.14)\qquad z_\alpha(x) = E\int_0^\infty e^{-\alpha t}\varphi(y_x(t))dt.$$

It is clear that if φ is periodic, then $z_\alpha(x)$ is periodic. Now, if (3.10) holds true, then according to (3.7), one has

$$|E\varphi(y_x(t))| \leqslant ||\varphi||\gamma e^{-\delta t}$$

from which it easily follows that $|z_\alpha(x)| \leqslant C$ as $\alpha \to 0$. This estimate and the equation insures that z_α remains in a bounded subset of $W^{2,p,\mu}(R^n)$. One can then let $\alpha \to 0$, and obtain a solution of (3.11).

The uniqueness is obtained as follows. Take $\varphi = 0$; then by Ito's formula,

$$z(x) = Ez(y_x(t)) \qquad \forall t \geqslant 0.$$

Since z is periodic, it follows from (3.7), letting $t \to \infty$, that

$$z(x) = \int_{\tilde{\Pi}} z\, d\bar{P};$$

hence z is constant. ■

4. Main convergence result

We can now state our main convergence result.

THEOREM 4.1. *Under assumptions* (2.1), (2.2) *and if*

$$(4.1)\qquad \int_{\tilde{\Pi}} b(x)d\bar{P}(x) = 0,$$

then P_x^ε converges weakly towards the diffusion P with constant coefficients r and q given by formulas (4.13) *below.*

Proof. Let us first prove that the family P_x^ε remains in a compact subset of μ_+^1 (for the weak topology) as $\varepsilon \to 0$. Indeed, let $\chi(y)$ be the vector function solution of

$$(4.2)\qquad -a_{ij}(y)\frac{\partial^2\chi}{\partial y_i \partial y_j} - b_i\frac{\partial \chi}{\partial y_i} = +b(y).$$

The components $\chi^i(y)$ of $\chi(y)$ are $W^{2,p,\mu}(R^n)$ functions and periodic. This follows from Theorem 3.2 and (4.1).

Let us set

$$(4.3)\qquad z_\varepsilon(t) = x(t) + \varepsilon\chi\left(\frac{x(t)}{\varepsilon}\right);$$

hence

$$(4.4)\qquad |z_\varepsilon(t,\omega) - x(t,\omega)| \leqslant C\varepsilon \qquad \forall t, \omega.$$

By Ito's formula, we have

$$(4.5)\qquad z_\varepsilon(t) = x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right) + \int_0^t \frac{1}{\varepsilon}\left[b\left(x\left(\frac{s}{\varepsilon}\right)\right) + \frac{\partial\chi}{\partial y}b\left(x\left(\frac{s}{\varepsilon}\right)\right) + \frac{\partial^2\chi}{\partial y_i \partial y_j}a_{ij}\left(\frac{x(s)}{\varepsilon}\right)\right]ds + \int_0^t\left(I + \frac{\partial\chi}{\partial y}\right)\sigma\left(\frac{x(s)}{\varepsilon}\right)dw_\varepsilon(s) + \int_0^t\left(I + \frac{\partial\chi}{\partial y}\right)c\left(\frac{x(s)}{\varepsilon}\right)ds.$$

By the choice of $\chi(y)$ it follows that

$$(4.6)\quad z_\varepsilon(t) = x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right) + \int_0^t\left(I + \frac{\partial\chi}{\partial y}\right)\sigma\left(\frac{x(s)}{\varepsilon}\right)dw_\varepsilon(s) + \int_0^t\left(I + \frac{\partial\chi}{\partial y}\right)c\left(\frac{x(s)}{\varepsilon}\right)ds.$$

Since $\dfrac{\partial\chi}{\partial y}$ is periodic and bounded, we have

$$(4.7)\qquad E_x^\varepsilon|z_\varepsilon(t_2) - z_\varepsilon(t_1)|^4 \leqslant C(t_2 - t_1)^2 \qquad \forall t_1 \leqslant t_2.$$

From (4.7) it follows that if Q_x^ε denotes the probability measure on Ω associated with $z_\varepsilon(t)$ (which is clearly a continuous process), then Q_x^ε remains in a compact subset of μ_+^1. But from (4.7) and (4.3) we also have

$$E_x^\varepsilon \sup_{|t-s|<\varrho}|x(t) - x(s)|^2 \leqslant 2E_x^\varepsilon \sup_{|t-s|<\varrho}|z_\varepsilon(t) - z_\varepsilon(s)|^2 + 2C_\varepsilon \leqslant 2C'\varrho + 2\varepsilon C;$$

hence

$$(4.8)\qquad \limsup_{\substack{\varepsilon\to 0\\ \varrho\to 0}} E_x^\varepsilon \sup_{|t-s|<\varrho}|x(t) - x(s)|^2 = 0.$$

Also

$$(4.9)\qquad \limsup_{\substack{\varepsilon\to 0\\ \varrho\to 0}} P_x^\varepsilon[\sup_{|t-s|<\varrho}|x(t) - x(s)| \geqslant \delta] = 0 \qquad \forall \delta > 0,$$

$$(4.10)\qquad \limsup_{\substack{l\uparrow +\infty\\ \varepsilon\to 0}} P_x^\varepsilon[|x(0)| \geqslant l] = 0.$$

But (4.9) and (4.10) insure, applying Parthasarthy [9], that P_x^ε remains in a compact subset of μ_+^1.

Let $\varphi \in \mathscr{D}(R^n)$. By Ito's formula,

$$\varphi(z_\varepsilon(t)) = \varphi(z_\varepsilon(s)) + \int_s^t \frac{\partial\varphi}{\partial x}(z_\varepsilon(\lambda))\cdot\left(I+\frac{\partial\chi}{\partial y}\right)\sigma\left(\frac{x(\lambda)}{\varepsilon}\right)dw_\varepsilon + \tag{4.11}$$
$$+\int_s^t \operatorname{tr}\frac{\partial^2\varphi}{\partial x^2}(z_\varepsilon(\lambda))\left(I+\frac{\partial\chi}{\partial y}\right)a\left(I+\frac{\partial\chi}{\partial y}\right)^*\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda+$$
$$+\int_s^t \frac{\partial\varphi}{\partial x}(z_\varepsilon(\lambda))\cdot\left(I+\frac{\partial\chi}{\partial y}\right)c\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda.$$

Hence

$$E_x^\varepsilon\left[\varphi(x(t))-\int_s^t \operatorname{tr} q\frac{\partial^2\varphi}{\partial x^2}(x(\lambda))d\lambda-\int_s^t\frac{\partial\varphi}{\partial x}(x(\lambda))\cdot r\,d\lambda|\mathscr{F}^s\right] \tag{4.12}$$
$$= \varphi(x(s))+E_x^\varepsilon[\varphi(x(t))-\varphi(z_\varepsilon(t))|\mathscr{F}^s]+\varphi(z_\varepsilon(s))-\varphi(x(s))+$$
$$+E_x^\varepsilon\left[\int_s^t \operatorname{tr}\left(\frac{\partial^2\varphi}{\partial x^2}(z_\varepsilon(\lambda))-\frac{\partial^2\varphi}{\partial x^2}(x(\lambda))\right)\left(I+\frac{\partial\chi}{\partial y}\right)a\left(I+\frac{\partial\chi}{\partial y}\right)^*\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right]+$$
$$+E_x^\varepsilon\left[\int_s^t\left(\frac{\partial\varphi}{\partial x}(z_\varepsilon(\lambda))-\frac{\partial\varphi}{\partial x}(x(\lambda))\right)\cdot\left(I+\frac{\partial\chi}{\partial y}\right)c\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right]+$$
$$+E_x^\varepsilon\left[\int_s^t \operatorname{tr}\frac{\partial^2\varphi}{\partial x^2}(x(\lambda))\tilde q\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda+\int_s^t\frac{\partial\varphi}{\partial x}(x(\lambda))\cdot\tilde r\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right],$$

where we have set

$$q = \int_{\tilde\Pi}\left(I+\frac{\partial\chi}{\partial y}\right)a\left(I+\frac{\partial\chi}{\partial y}\right)^* d\bar P(y), \qquad r = \int_{\tilde\Pi}\left(I+\frac{\partial\chi}{\partial y}\right)c\,d\bar P(y) \tag{4.13}$$

and

$$\tilde q(y) = \left(I+\frac{\partial\chi}{\partial y}\right)a\left(I+\frac{\partial\chi}{\partial y}\right)^*(y)-q, \qquad \tilde r(y) = \left(I+\frac{\partial\chi}{\partial y}\right)c(y)-r. \tag{4.14}$$

Using (4.3) and the fact that $\varphi\in\mathscr{D}(\boldsymbol{R}^n)$, we can check that

$$\left|E_x^\varepsilon\left[\varphi(x(t))-\varphi(z_\varepsilon(t))+\varphi(z_\varepsilon(s))-\varphi(x(s))+\right.\right. \tag{4.15}$$

$$+\int_s^t \operatorname{tr}\left(\frac{\partial^2\varphi}{\partial x^2}(z_\varepsilon(\lambda))-\frac{\partial^2\varphi}{\partial x^2}(x(\lambda))\right)\left(I+\frac{\partial\chi}{\partial y}\right)a\left(I+\frac{\partial\chi}{\partial y}\right)^*\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda+$$
$$\left.\left.+\int_s^t\left(\frac{\partial\varphi}{\partial x}(z_\varepsilon(\lambda))-\frac{\partial\varphi}{\partial x}(x(\lambda))\right)\cdot\left(I+\frac{\partial\chi}{\partial y}\right)c\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right]\right| \leqslant C\varepsilon.$$

We are going to prove that

$$\left|E_x^\varepsilon\left[\int_s^t \operatorname{tr}\frac{\partial^2\varphi}{\partial x^2}(x(\lambda))\tilde q\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda+\int_s^t\frac{\partial\varphi}{\partial x}(x(\lambda)).\tilde r\left(\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right]\right| \leqslant C_\varepsilon. \tag{4.16}$$

Assuming for a while that (4.16) holds true and using (4.15), it follows from (4.12) that any measure P_x, which is a weak limit of some subsequence of P_x^ε, will satisfy

$$E_x\left[\varphi(x(t))-\int_s^t \operatorname{tr} q\frac{\partial^2\varphi}{\partial x^2}(x(y))\,d\lambda-\int_s^t\frac{\partial\varphi}{\partial x}(x(\lambda))\cdot r\,d\lambda|\mathscr{F}^s\right] = \varphi(x(s)) \tag{4.17}$$
$$\forall s\leqslant t;\ \forall\varphi\in\mathscr{D}(\boldsymbol{R}^n).$$

But by the theory of Stroock–Varadhan [12], there exists one and only one probability measure P_x on Ω satisfying (4.17) and such that

$$P_x[x(0) = x] = 1.$$

Hence the whole sequence P_x^ε converges towards P_x and the proof of the theorem will be complete, provided we prove (4.16).

To prove (4.16), let $\gamma(x,y)$ be the solution of

$$-a_{ij}\frac{\partial^2\gamma}{\partial y_i\partial y_j}-b_i\frac{\partial\gamma}{\partial y_i} = \frac{\partial\varphi}{\partial x}(x).\ \tilde r(y)+\operatorname{tr}\frac{\partial^2\varphi}{\partial x^2}(x)\tilde q(y). \tag{4.18}$$

For fixed x, (4.18) is an equation of the type (3.11) since $\tilde r(y)$ and $\tilde q(y)$ have 0 mean with respect to the invariant measure $\bar P$. Hence as a function of y, $\gamma(x,y)\in W^{2,p,\mu}(\boldsymbol{R}^n)$ and is periodic. Clearly from the equation $\chi(x,y)$ is smooth in x. From Ito's formula, one has

$$E_x^\varepsilon\left[\gamma\left(x(t),\frac{x(t)}{\varepsilon}\right)-\right. \tag{4.19}$$
$$-\int_s^t\left(\frac{\partial\gamma}{\partial x}\left(x(\lambda),\frac{x(\lambda)}{\varepsilon}\right)+\frac{1}{\varepsilon}\frac{\partial\gamma}{\partial y}\left(x(\lambda),\frac{x(\lambda)}{\varepsilon}\right)\right).\left(c\left(\frac{x(\lambda)}{\varepsilon}\right)+\frac{1}{\varepsilon}b\left(\frac{x(\lambda)}{\varepsilon}\right)\right)d\lambda-$$
$$-\int_s^t a_{ij}\left(\frac{x(\lambda)}{\varepsilon}\right)\left(\frac{\partial^2\gamma}{\partial x_i\partial x_j}+\frac{1}{\varepsilon}\frac{\partial^2\gamma}{\partial x_i\partial x_j}+\frac{1}{\varepsilon}\frac{\partial^2\gamma}{\partial y_i\partial y_j}+\frac{1}{\varepsilon^2}\frac{\partial^2\gamma}{\partial y_i\partial y_j}\right).$$
$$\left.\left(x(\lambda),\frac{x(\lambda)}{\varepsilon}\right)d\lambda|\mathscr{F}^s\right] = \gamma\left(x(s),\frac{x(s)}{\varepsilon}\right).$$

Using (4.18) and the fact that γ, $\frac{\partial\gamma}{\partial x}$, $\frac{\partial\gamma}{\partial y}$, $\frac{\partial^2\gamma}{\partial x_i\partial x_j}$, $\frac{\partial^2\gamma}{\partial x_i\partial x_j}$, are bounded, one easily checks (4.16). The proof of the theorem is now complete. ∎

Remark 4.1. The matrix a_{ij} is positive definite. Indeed,

$$q_{ij} = \int_{\Pi}\left(\delta_{ik}+\frac{\partial\chi^i}{\partial y_k}\right)a_{kl}\left(\delta_{lj}+\frac{\partial\chi^j}{\partial y_l}\right)d\bar{P}(y) = \int_{\Pi} a_{kl}\frac{\partial}{\partial y_k}(y_i+\chi^i)\frac{\partial}{\partial y_l}(y_j+\chi^j)\,d\bar{P}(y).$$

Hence if $\xi_1, \ldots, \xi_n$ are reals, one has

$$q_{ij}\xi_i\xi_j = \int_{\Pi} a_{kl}\frac{\partial}{\partial y_k}\big(\xi_i(y_i+\chi^i)\big)\frac{\partial}{\partial y_l}\big(\xi_j(y_j+\chi^j)\big)\,d\bar{P}$$
$$\geqslant \beta\sum_k\int_{\Pi}\left(\frac{\partial}{\partial y_k}\big(\xi_i(y_i+\chi^i)\big)\right)^2 d\bar{P}(y).$$

At least, in the case when $\bar{P}$ has a density with respect to the Lebesgue measure, then $q_{ij}\xi_i\xi_j = 0$ implies

$$\xi_i\big(y_i+\chi^i(y)\big) = C.$$

Since χ^i is bounded, such an equality is impossible when one of the ξ_i is non 0.

The existence of this density follows from Remark 3.1 and Fredholm's alternative applied to the operator

$$A = -\sum_{ij} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j} - \sum_i b_i\frac{\partial}{\partial x_i}.$$

According to Theorem 3.2, there is one and only one periodic solution in, say, $W^{2,2,\mu}$ (up to a constant) of

$$Az = 0.$$

This solution is obviously $z = 1$. One can also consider the operator A in the subspace of $H^1(\Pi)$ defined by the periodic functions (rewriting

$$A = -\sum_j \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - \sum_i\left(b_i - \sum_j\frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial}{\partial x_i}$$

and using variational formulation). It is well known then that the Fredholm's alternative applies. Therefore the equation

$$A^*m = 0$$

has one and only one solution which is periodic (up to a constant). The constant is fixed by the condition $\int_{\Pi} m\,dx = 1$.

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