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ON A NON-LINEAR SEMI-GROUP ASSOCIATED WITH STOCHASTIC OPTIMAL CONTROL AND ITS EXCESSIVE MAJORANT

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§ 1. Introduction

In [8] we introduced a non-linear semi-group attached to the Bellman principle of stochastic optimal controls in the following way: Let Γ be a σ-compact subset of $\boldsymbol{R}^k$, called a *control region*. Let $\Omega = (\Omega, F, F_t, P)$ be a probability space and B an n-dimensional F_t-Wiener Martingale. Let U be a Γ-valued F_t-progressively measurable bounded process. We call U an *admissible control*, or, to be more precise, the triple $A = (\Omega, B, U)$ is called an *admissible system*. By $\mathfrak{A}$ we denote the set of all admissible systems. For $A = (\Omega, B, U)$ we consider the following n-dimensional stochastic differential equation:

$$dX(t) = \alpha(X(t), U(t))dB(t)+\gamma(X(t), U(t))dt, \tag{1.1}$$

where $\alpha(x, u)$ is a symmetric and non-negative definite $n\times n$-matrix valued Borel function on $\boldsymbol{R}^n\times\Gamma$ and $\gamma(x, u)$ a $\boldsymbol{R}^n$-valued Borel function. We introduce the following conditions of boundedness and continuity:

$$|F(x, u)| \leqslant b, \tag{1.2}$$

$$|F(x, u)-F(y, u)| \leqslant \mu|x-y|, \quad \forall u\in\Gamma, \tag{1.3}$$

where μ is a positive constant. Applying the usual successive approximation, we can easily see that (1.1) has a unique solution X, which is called the *response* for A. The problem is to maximize the mean of the following gain:

$$V(t,x,A,\varphi) \equiv E_x\Big[\int_0^t e^{-\int_0^s c(X(\theta), U(\theta))d\theta} f(X(s), U(t))ds + e^{-\int_0^t c(X(\theta), U(\theta))d\theta}\varphi(X(t))\Big], \tag{1.4}$$

where c and f satisfy (1.2) and (1.3), c is non-negative and X is the response for A with $X(0) = x$. Put

$$V(t, x, \varphi) = \sup_{A\in\mathfrak{A}} V(t, x, A, \varphi). \tag{1.5}$$

The Bellman principle gives us the following two-stage optimization:

$$(1.6)\quad V(t,x,\varphi)=\sup_{A\in\mathfrak{A}}E_x\Big[\int_0^s e^{-\int_0^\theta c(X,U)} f\big(X(\theta),U(\theta)\big)\,d\theta+e^{-\int_0^s c(X,U)}\,V\big(t-s,X(s),\varphi\big)\Big].$$

Hence, denoting $V(t,x,\varphi)$ by $Q_t\varphi(x)$, we have

$$Q_t\varphi(x)=Q_s(Q_{t-s}\varphi)(x);$$

namely, the Bellman principle is nothing but the semi-group property of the operator Q_t. We give a precise definition of the operator Q_t and the following theorem, Theorem 1, is proved in § 3.

Let C be a Banach lattice of all bounded and uniformly continuous functions on $\boldsymbol{R}^n$. Define Q_t by

$$(1.7)\quad Q_t\varphi(x)=\sup_{A\in\mathfrak{A}}E_x\Big[\int_0^t e^{-\int_0^s c(X,U)} f\big(X(s),U(s)\big)\,ds+e^{-\int_0^t c(X,U)}\,\varphi\big(X(t)\big)\Big],\qquad \varphi\in C,$$

where X is the response starting at $X(0)=x$. Then we have

THEOREM 1. *Let α, γ, c and f satisfy* (1.2) *and* (1.3) *and $c\geqslant 0$. Then Q_t is a strongly continuous non-linear semi-group on C which is monotone and contractive. Moreover, the generator G of Q_t is expressed by*

$$(1.8)\qquad G\varphi(x)=\sup_{u\in\Gamma}L^u\varphi(x)+f(x,u)\quad \text{for}\quad \varphi\in C^2,$$

where

$$(1.9)\quad L^u\varphi(x)=\frac{1}{2}\sum_{i,j=1}^n\alpha_{ij}^2(x,u)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}(x)+\sum_{i=1}^n\gamma_i(x,u)\frac{\partial\varphi}{\partial x_i}(x)-c(x,u)\varphi(x)$$

and

$$C^2=\Big\{\varphi\in C;\ \frac{\partial\varphi}{\partial x_i}\ \text{and}\ \frac{\partial^2\varphi}{\partial x_i\,\partial x_j}\in C,\quad ij=1,\dots,n\Big\}.$$

The right side of (1.8) can be found in the Bellman equation [1], [4], [5].

In § 4 we shall review a similar problem in a more general set-up [9], [10], i.e. our general problem is the following: Let P_t^u, $t\geqslant 0$, be a positive and contractive linear semi-group on C with generator A^u. We seek a semi-group acting on C whose generator is an extension of $G\varphi=\sup\limits_{u\in\Gamma}(A^u\varphi+f^u)$. Such a semi-group will be obtained as the envelope of the semi-groups T_t^u, $u\in\Gamma$

$$(1.10)\qquad T_t^u\varphi=P_t^u\varphi+\int_0^t P_\theta^u f^u\,d\theta$$

whose generators are

$$(1.11)\qquad G^u\varphi=A^u\varphi+f^u,\qquad u\in\Gamma,$$

as we can infer from the fact that G is the envelope of G^u, $u\in\Gamma$. We have Theorem 2, [10].

THEOREM 2. *Suppose that the following conditions,* (A1)–(A4), *hold:*

(A1) *If $\varphi_n\in C$ is an increasing sequence tending to $\varphi\in C$ at each point, then $P_t^u\varphi_n$ increases and tends to $P_t^u\varphi$ at each point for any $u\in\Gamma$ and $t\geqslant 0$.*

(A2) *Let $D(A^u)$ be the domain of the generator A^u.*

$$D(A^u)\supset C^2,\qquad u\in\Gamma$$

and

$$(1.12)\qquad \sup_u\|A^u\varphi\|<\infty\quad \text{for}\quad \varphi\in C^2.$$

(A3) *With a positive constant h,*

$$(1.13)\qquad \sup_u\|f^u\|\leqslant h\quad\text{and}\quad \sup_u|f^u(x)-f^u(y)|\leqslant h|x-y|.$$

(A4) *For any positive T there exists a constant $q=q(T)$ such that*

$$(1.14)\qquad \sup_u|P_t^u\varphi(x)-P_t^u\varphi(y)|\leqslant e^{qt}|x-y|\qquad \forall t\leqslant T,$$

whenever $|\varphi(x)-\varphi(y)|\leqslant|x-y|$ and $\|\varphi\|\leqslant 1$.

Then there exists a unique strongly continuous monotone and contractive semi-group S_t on C which satisfies the following three conditions:

(i) *For any $t\geqslant 0$ and $u\in\Gamma$*

$$(1.15)\qquad P_t^u\varphi+\int_0^t P_\theta^u f^u\,d\theta\leqslant S_t\varphi.$$

(ii) *The weak generator G_w of S_t is expressed by*

$$(1.16)\qquad G_w\varphi=\sup_u(A^u\varphi+f^u)\quad\text{for}\quad \varphi\in C^2.$$

(iii) *If $\tilde{S}_t$ is a strongly continuous monotone and contractive semi-group on C with* (i), *then*

$$(1.17)\qquad S_t\varphi\leqslant\tilde{S}_t\varphi;$$

namely, S_t is the envelope of semi-groups T_t^u, $u\in\Gamma$.

Let A^u be the second order elliptic operator L^u expressed by (1.9). Then we have the following theorem [10]:

THEOREM 3. *Suppose that Γ is a convex σ-compact subset of $\boldsymbol{R}^k$ and the coefficients of L^u, α, γ, c and f, satisfy* (1.2) *and* (1.18).

$$(1.18)\qquad |F(x,u)-F(y,v)|\leqslant\mu|x-y|+\varrho(|u-v|),$$

where ϱ is a concave and strictly increasing continuous function on $[0,\infty)$ with $\varrho(0)=0$. Then the linear semi-group P_t^u satisfies conditions (A1)–(A4) *and the two semi-groups Q_t and S_t coincide.*

In § 5 we deal with the optimal stopping problem and an excessive majorant. This is a generalization of the optimal stopping problem of Markov processes [ref. [2]]; namely, we can control not only the stopping time of the motion but also the motion itself. For the stochastic motion described by the stochastic differential equation (1.1) this problem has been considered in [4], [6] and [8]. We review the following theorem, Theorem 4, [8].

For an admissible system $A = (\Omega, B, U)$ a $[0, \infty]$-valued random variable τ on Ω is called *A-stopping time* if

$$\{\omega;\ \tau(\omega) \leqslant t\} \in \bigcup_{\theta > t} \sigma_\theta(B, U) \equiv \sigma_{t+}(B, U), \qquad t \geqslant 0,$$

where $\sigma(\xi)$ is the σ-algebra spanned by $\{\xi(x),\ s \leqslant \theta\}$. By $\mathscr{T}(A)$ we denote the set of all A-stopping times. We assume (1.19)

$$\inf_{x,u} c(x, u) > 0. \tag{1.19}$$

Define $v(x)$ by

$$v(x) = \sup_{A \in \mathfrak{A}} \sup_{\tau \in \mathscr{T}(A)} E_x \Big[\int_0^\tau e^{-\int_0^t c(X, U)} f(X, U)\, dt + e^{-\int_0^\tau c(X, U)} \varphi(X(\tau)) \Big], \tag{1.20}$$

where X is the response for A with $X(0) = x$.

THEOREM 4. *Under the conditions of Theorem* 1 *and* (1.19), $v \in C$ *and* v *is the least* Q_t*-excessive majorant of* φ, *i.e.*

$$Q_t v \leqslant v \qquad \forall t \geqslant 0, \tag{1.21}$$

$$\varphi \leqslant v; \tag{1.22}$$

moreover, if $\tilde{v} \in C$ *satisfies* (1.21) *and* (1.22), *then* $v \leqslant \tilde{v}$.

In the general problem we can prove Theorem 5.

THEOREM 5. *Suppose that* (A1)–(A4) *are satisfied and assume*

(A5) *There exists a positive c such that*

$$\|P_t^u\| \leqslant e^{-ct} \quad \textit{for any } t \geqslant 0 \textit{ and } u \in \Gamma.$$

Let g be a bounded and Lipschitz continuous function. Then there exists a unique least S_t*-excessive majorant of g.*

Although the method of construction of an excessive majorant is analytic, we will discuss an example of a probabilistic method in § 6 according to [6], [13].

§ 2. Preliminaries

Let C be the set of all bounded and uniformly continuous functions on R^n. C becomes a Banach lattice by the usual norm and partial order, [12], i.e. $\|\varphi\| = \sup_{x \in R^n} |\varphi(x)|$ and "$\varphi \leqslant \psi$" is defined by "$\varphi(x) \leqslant \psi(x)\ \forall x$". When $\varphi_n \in C$ is increasing to $\varphi \in C$ at each point, we say that $\varphi = 0_l - \lim \varphi_n$. If a subset $\{\varphi_\alpha\}$ of C is uniformly bounded and equi-uniformly continuous, then $\sup \varphi_\alpha$ and $\inf \varphi_\alpha$ exist in C. Hence for any positive constant M and positive function δ on $(0, \infty)$, the set $H_{M,\delta}$ defined by

$$H_{M,\delta} = \{\varphi \in C;\ \|\varphi\| \leqslant M,\ |\varphi(x) - \varphi(y)| < \varepsilon \text{ for } |x-y| < \delta(\varepsilon)\}$$

is a complete lattice. But C is not complete as a lattice. According to the definitions of $\sup \varphi_\alpha$ and $\inf \varphi_\alpha$, the following inequalities are clear:

$$\inf(\varphi_\alpha - \psi_\alpha) \leqslant \sup \varphi_\alpha - \sup \psi_\alpha \leqslant \sup(\varphi_\alpha - \psi_\alpha), \tag{2.1}$$

$$\|\sup \varphi_\alpha - \sup \psi_\alpha\| \leqslant \sup \|\varphi_\alpha - \psi_\alpha\|. \tag{2.2}$$

For $\delta(\theta) = \theta/K$, where K is a positive constant, we denote $H_{M,\delta}$ by $\Sigma_{M,K}$ and $\Sigma_{K,K}$ by Σ_K.

PROPOSITION 1. *Suppose* (A1)–(A5) *hold. For any* $g \in \Sigma_p$ *and* $\lambda \in [0, \infty)$ *we define* $T_t = T_t^{u\lambda g}$ *by*

$$T_t \varphi = e^{-\lambda t} P_t^u \varphi + \int_0^t e^{-\lambda\theta} P_\theta^u (f^u + \lambda g)\, d\theta, \qquad \varphi \in C. \tag{2.3}$$

Then T_t is a monotone contractive and strongly continuous semigroup on C, whose generator G is given by

$$G\varphi = A^u \varphi - \lambda\varphi + \lambda g + f^u \tag{2.4}$$

and $D(A^u) = D(G)$. *Moreover, the following evaluations hold for* $\varphi \in \Sigma_K$ *and* $t \leqslant T$:

$$|T_t\varphi(x) - T_t\varphi(y)| \leqslant |x-y| \Big(e^{qt} K + \frac{h}{q}(e^{qt} - 1) + p + qte^{qt} p \Big), \tag{2.5}$$

where $q = q(T)$ *and*

$$\begin{aligned} \|T_t \varphi\| &\leqslant e^{-(\lambda+c)t} \|\varphi\| + \frac{h}{\lambda + c}(1 - e^{-(\lambda+c)t}) + \frac{\|g\|\lambda}{\lambda+c}(1 - e^{-(\lambda+c)t}) \\ &\leqslant e^{-ct}\|\varphi\| + \frac{h}{c}(1 - e^{-ct}) + \|g\|. \end{aligned} \tag{2.6}$$

We only show (2.5).

$$\begin{aligned} T_t\varphi(x) - T_t\varphi(y) = e^{-\lambda t}\big(P_t^u\varphi(x) - P_t^u\varphi(y)\big) + \int_0^t e^{-\lambda\theta}\big(P_\theta^u f^u(x) - P_\theta^u f^u(y)\big)\, d\theta + \\ + \lambda \int_0^t e^{-\lambda\theta}\big(P_\theta^u g(x) - P_\theta^u g(y)\big)\, d\theta. \end{aligned} \tag{2.7}$$

By virtue of (A4) we have the following evaluations:

$$|\text{1st term}| \leqslant e^{(q\lambda)t} K|x-y|,$$

$$|\text{2nd term}| \leqslant h \int_0^t |x-y| e^{q\theta} d\theta = h|x-y|(e^{qt} - 1)/q$$

and

$$|\text{3rd term}| \leqslant \lambda p|x-y|(e^{(q-\lambda)t} - 1)/q - \lambda.$$

Using the inequality "$(1 - e^{-xt})/x \leqslant t$ for $x > 0$ and $t \geqslant 0$", we get

$$\frac{\lambda(e^{(q-\lambda)t} - 1)}{q - \lambda} \leqslant e^{(q-\lambda)t} \lambda t \leqslant e^{qt} qt \quad \text{for} \quad 0 \leqslant \lambda < q,$$

and

$$\frac{\lambda(e^{(q-\lambda)t}-1)}{q-\lambda} = 1-e^{-(\lambda-q)t}+q\,\frac{1-e^{-(\lambda-q)t}}{\lambda-q},$$

$$\leqslant 1+qt \leqslant 1+qte^{qt} \quad \text{for} \quad q \leqslant \lambda.$$

Combining these evaluations with (2.7) we have (2.5).

COROLLARY. *For any $\varphi \in C$ and $T > 0$ there exists a positive function $\delta^* = \delta^*(\varphi, T, g)$ on $(0, \infty)$ and a constant M such that*

(2.8) $$T_t\varphi \in H_{M,\delta^*}, \quad t \leqslant T,\ u \in \Gamma,\ \lambda \geqslant 0.$$

Moreover, for the set $H_{M,\delta}$ there exist a constant $\tilde{M}$ and a positive function $\tilde{\delta}$ such that

(2.9) $$T_t\varphi \in H_{\tilde{M},\tilde{\delta}}, \quad t \leqslant 1,\ \varphi \in H_{M,\delta},\ u \in \Gamma,\ \lambda \in [0, \infty).$$

Proof. Since $\Sigma \equiv \bigcup_{K>0} \Sigma_K$ is strongly dense in C, there exists an approximation ψ of φ in Σ. Hence we have

(2.10) $$|T_t\varphi(x)-T_t\varphi(y)| \leqslant |T_t\varphi(x)-T_t\psi(x)|+|T_t\psi(x)-T_t\psi(y)|+|T_t\psi(y)-T_t\varphi(y)|$$
$$\leqslant 2\|\varphi-\psi\|+|T_t\psi(x)-T_t\psi(y)|.$$

Applying the evaluation of (2.5) and (2.6) to the last term of (2.10), we can find out the required function δ^* and the constant M of (2.8).

To prove the second part we remark that for $\varepsilon > 0$ there exists a positive K such that for any $\varphi \in H_{M,\delta}$ we can take an ε-approximation ψ in Σ_K, i.e. $\|\varphi-\psi\| < \varepsilon$. Using the same calculation as (2.10), we have (2.9).

PROPOSITION 2. *Let $\varphi_n \in C$ be an increasing sequence. Suppose that $0_l-\lim\varphi_n$ and $0_l-\lim_n\sup_{u\lambda} T_t^{u\lambda g}\varphi_n$ exist. Then*

(2.11) $$\sup_{u\lambda} T_t^{u\lambda g}(0_l-\lim_n\varphi_n) = 0_l-\lim_n\sup_{u\lambda} T_t^{u\lambda g}\varphi_n.$$

Proof. Put $\varphi = 0_l-\lim\varphi_n$ and $J(\psi) = \sup_{u\lambda} T_t^{u\lambda g}\psi$. Since J is monotone, we have

$$J(\varphi_n) \leqslant J(\varphi_{n+1}) \leqslant J(\varphi).$$

Hence

(2.12) $$0_l-\lim_n J(\varphi_n) \leqslant J(\varphi).$$

Recalling (A1), we can see that

(2.13) $$T_t^{u\lambda g}\varphi = e^{-\lambda t}(0_l-\lim P_t^u\varphi_n)+\int_0^t e^{-\lambda\theta}P_\theta^u(f^u+\lambda g)\,d\theta$$
$$= 0_l-\lim\Big(e^{-\lambda t}P_t^u\varphi_n+\int_0^t e^{-\lambda\theta}P_\theta^u(f^u+\lambda g)\,d\theta\Big)$$
$$= 0_l-\lim T_t^{u\lambda g}\varphi_n \leqslant 0_l-\lim J(\varphi_n) \qquad \forall u, \lambda.$$

Taking the supremum w.r. to (u, λ), we get the opposite inequality to (2.12). This completes the proof of Proposition 2.

§ 3. Proof of Theorem 1

We proved Theorem 1 under the stronger conditions in [8]. Thus Theorem 1 is here a little improved. We can apply the same method as in [8], although a little modification is required in order to get the semi-group property of Q_t.

Let X and Y be the responses for A starting at x and y, respectively. Then the following inequalities hold:

(3.1) $$E|X(t)|^2 \leqslant K_1(t+t^2)+|x|,$$

(3.2) $$E|X(t)-Y(t)|^2 \leqslant |x-y|^2e^{2K_2t},$$

(3.3) $$E|X(t)-X(s)|^2 \leqslant K_3(|t-s|+|t-s|^2),$$

where K_i stands for a constant independent of: the admissible system, starting point and time t. Hence for $\varepsilon > 0$ there exists a positive $\delta = \delta(T, \varphi)$ such that

(3.4) $$|Q_t\varphi(x)-Q_t\varphi(y)| \leqslant \sup_{A\in\mathfrak{A}}|E_xI(t, A, \varphi)-E_yI(t, A, \varphi)| < \varepsilon$$

whenever $|x-y| < \delta$ and $t \leqslant T$, where

(3.5) $$I(t, A, \varphi) = \int_0^t e^{-\int_0^s c(X(\theta)U(\theta))\,d\theta} f(X(s)U(s))\,ds+e^{-\int_0^t c(X(\theta)U(\theta))\,d\theta}\varphi(X(t)).$$

Moreover, we have

(3.6) $$\|Q_t\varphi\| \leqslant \|\varphi\|+bt.$$

(3.7) $$|Q_t\varphi(x)-Q_s\varphi(x)| \leqslant \sup_{A\in\mathfrak{A}}|E_xI(t, A, \varphi)-E_xI(s, A, \varphi)|$$
$$\leqslant b|t-s|+\|\varphi\|b|t-s|+\sup_A E_x|\varphi(X(t))-\varphi(X(s))|.$$

By (3.3) this evaluation (3.7) proves the strong continuity of $Q_t\varphi$ w.r. to t.

Using the following fact (3.8):

(3.8) $$|I(t, A, \varphi)-I(t, A, \psi)| \leqslant \|\varphi-\psi\|,$$

we can see that

$$|Q_t\varphi(x)-Q_t\psi(x)| \leqslant \sup_A|E_xI(t, A, \varphi)-E_xI(t, A, \psi)| \leqslant \|\varphi-\psi\|;$$

namely, Q_t is contractive.

(3.9) $$I(t, A, \varphi) \leqslant I(t, A, \psi) \quad \text{whenever} \quad \varphi \leqslant \psi.$$

Therefore $Q_t\varphi \leqslant Q_t\psi$; namely, Q_t is monotone.

In order to prove the semi-group property, we shall note the following lemma.

LEMMA. *If two admissible systems, $A = (\Omega, B, U)$ and $\tilde{A} = (\Omega, B, \tilde{U})$, satisfy the condition "$U(t, \omega) = \tilde{U}(t, \omega)$ $\forall(t, \omega)$", then their responses X and $\tilde{X}$ coincide whenever $X(0) = \tilde{X}(0) = x$, i.e. "$X(t, \omega) = \tilde{X}(t, \omega)$ $\forall t$" with probability 1.*

Proof. Put $\Omega = (\Omega, F, F_t, P)$. Since X, $\tilde{X}$, U and $\tilde{U}$ are F_t-adapted processes and B is an F_t-Wiener Martingale, we have

$$E\left|\int_0^t \alpha_{ij}(X(s), U(s))\,dB_j(s) - \int_0^t \alpha_{ij}(X(s), \tilde{U}(s))\,dB_j(s)\right|^2 = E\int_0^t |\alpha_{ij}(X(s), U(s)) - \alpha_{ij}(X(s), \tilde{U}(s))|^2 ds = 0.$$

Hence with probability 1

$$\int_0^t \alpha(X(s), U(s))\,dB(s) = \int_0^t \alpha(X(s), \tilde{U}(s))\,dB(s) \quad \forall t.$$

Therefore with probability 1

$$X(t) = \int_0^t \alpha(X(s), \tilde{U}(s))\,dB(s) + \int_0^t \gamma(X(s), \tilde{U}(s))\,ds \quad \forall t.$$

This means that X is the response for $\tilde{A}$. Thus we complete the proof of the lemma.

Put $W = C([0, \infty) \to R^n) \times L^{2,\gamma}_{\mathrm{loc}}[0, \infty)$. Then W is a complete separable metric space with the metric ϱ,

$$\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_n}{1+\|f-g\|_n},$$

where $\|h\|_n = \max_{0 \leqslant t \leqslant n} |h_1(t)| + \left(\int_0^n |h_2(t)|^2 dt\right)^{1/2}$ for $h = (h_1, h_2)$. By the lemma (B, U) can be regarded as a W-valued random variable. Let F_s be the σ-algebra spanned by coordinate $w(\theta), \theta \leqslant s$, and $F = \bigvee_{s>0} F_s$. Let ν denote the probability law of (B, U) on W. For almost all $\xi \in C([0, s] \to R^n) \times L^2[0, s]$, there exists a regular conditional probability $\nu(\cdot, \xi)$ on F which gives a nice version of the conditional probability $\nu(\cdot / F_s)$, [11]. On the regular probability space $(W, F, \nu(\cdot, \xi))$, the system after s $\{dB(\theta), U(\theta), \theta \geqslant s\}$ is again an admissible system if we take s as the origin of time. Moreover, $X(\theta), \theta \geqslant s$, is the response starting at $X(s)$. Hence, recalling the definition of gain V, we have

$$(3.10) \qquad \int_W \left[\int_s^t e^{-\int_s^\theta c(X,U)} f(X(\theta), U(\theta))\,d\theta + e^{-\int_s^t c(X,U)} \varphi(X(t))\right] d\nu(w, \xi) = V(t-s, X(s), A^\xi, \varphi)$$

with some admissible system A^ξ. Thus we can see the following

LEMMA. $Q_{t+s}\varphi \leqslant Q_s(Q_t\varphi)$.

Proof. By virtue of (3.10) we have

$$E_x I(t, A, \varphi) = E_x\left[\int_0^s e^{-\int_0^\theta c(X,U)} f(X, U)\,d\theta + e^{-\int_0^s c(X,U)} E_x\left(\int_s^t e^{-\int_s^\theta c(X,U)} f(X, U)\,d\theta + e^{-\int_s^t c(X,U)} \varphi(X(t))/F_s\right)\right] \leqslant E_x I(s, A, Q_{t-s}\varphi).$$

Taking the supremum w.r. to A, we obtain the lemma.

In order to obtain the semi-group property we shall derive the inverse inequality. In view of (3.4) there exists for $\varepsilon > 0$ a positive δ such that

$$(3.11) \qquad |\varphi(x) - \varphi(y)| < \varepsilon \quad \text{for} \quad |x-y| < \delta$$

and

$$(3.12) \qquad |E_x I(t, A, \varphi) - E_y I(t, A, \varphi)| < \varepsilon \quad \text{for} \quad |x-y| < \delta \quad \forall A \in \mathfrak{A}.$$

We apply a measurable partition $\{\Delta_i, i = 1, 2, \ldots\}$ of R^n such that $\operatorname{dia}(\Delta_i) < \delta$ and any compact subset of R^n can be covered by finitely many Δ_i. Fix $x_k \in \Delta_k$ arbitrarily and take an ε-optimal system $A_k = (\Omega_k, B_k, U_k)$, i.e.

$$(3.13) \qquad E_{x_k} I(t, A_k, \varphi) \geqslant Q_t\varphi(x_k) - \varepsilon, \quad k = 1, 2, \ldots$$

From (3.11) and (3.12) we can see that

$$(3.14) \qquad E_y I(t, A_k, \varphi) \geqslant E_{x_k} I(t, A_k, \varphi) - \varepsilon \geqslant Q_t\varphi(x_k) - 2\varepsilon \geqslant Q_t\varphi(y) - 3\varepsilon \quad \text{for} \quad y \in \Delta_k;$$

namely, A_k is 3ε-optimal system for $y \in \Delta_k$. Fix $A_0 \in \mathfrak{A}$ arbitratrily and take a compact set Δ such that

$$(3.15) \qquad P_x(X_0(s) \in \Delta) > 1 - \varepsilon.$$

Define $A = (\Omega, B, U)$ as follows:

$$\Omega = \Omega_0 \times \Omega_1 \times \ldots \quad \text{(product probability space)},$$

$$U(\theta) = \begin{cases} U_0(\theta), & \theta < s, \\ \sum_{k=1}^N U_k(\theta - s)\chi_{\Delta_k}(X_0(s)) + U_{N+1}(\theta - s)\chi_{\Delta^*}(X_0(s)), & \theta \geqslant s, \end{cases}$$

$$dB(\theta) = \begin{cases} dB_0(\theta), & \theta < s, \\ \sum_{k=1}^N dB_k(\theta - s)\chi_{k\Delta}(X_0(s)) + dB_{N+1}(\theta - s)\chi_{\Delta^*}(X_0(s)), & \theta \geqslant s, \end{cases}$$

where $\Delta \subset \bigcup_{k=1}^N \Delta_k$, $\Delta^* = \bigcup_{k=N+1}^{\infty} \Delta_k$ and χ_Δ is the indicator function of Δ. Then A is an admissible system and its response X is written by

$$X(\theta) = \begin{cases} X_0(\theta), & \theta < s, \\ \sum_{k=1}^N X_k(\theta - s)\chi_{\Delta_k}(X_0(s)) + X_{N+1}(\theta - s)\chi_{\Delta^*}(X_0(s)), & \theta \geqslant s, \end{cases}$$

where X_k is the response for A_k starting at $X_0(s)$. Moreover, A is nearly optimal after s and we have

$$Q_t\varphi(x) \geqslant E_x I(t, A, \varphi) \geqslant E_x I(s, A_0, Q_{t-s}\varphi) - K_4\varepsilon.$$

Since A_0 and ε are arbitrary, we have

$$Q_t\varphi \geqslant Q_s(Q_{t-s}\varphi).$$

This implies the semi-group property of Q_t by virtue of the lemma.

Now we shall calculate the generator G of Q_t. For $A \in \mathfrak{A}$ and $\varphi \in C^2$ Ito's formula gives

$$E_x e^{-\int_0^t c(X,U)} \varphi(X(t)) - \varphi(x) = E_x \int_0^t e^{-\int_0^s c(X,U)} L^{U(s)}\varphi(X(s))\,ds.$$

Hence we have

$$Q_t\varphi(x) - \varphi(x) = \sup_A E_x \int_0^t e^{-\int_0^s c(X,U)} \big(L^{U(s)}\varphi(X(s)) + f(X(s), U(s))\big). \tag{3.16}$$

On the other hand,

$$\sup_A E_x \int_0^t \big(L^{U(s)}\varphi(x) + f(x, U(s))\big)\,ds \leqslant E_x \int_0^t \sup_{u\in\Gamma} \big(L^u\varphi(x) + f(x,u)\big)\,ds$$

$$= \sup_{u\in\Gamma} \big(L^u\varphi(x) + f(x,u)\big)t = \sup_{u\in\Gamma} \int_0^t \big(L^u\varphi(x) + f(x,u)\big)\,ds \tag{3.17}$$

$$\leqslant \sup_A E_x \int_0^t \big(L^{U(s)}\varphi(x) + f(x, U(s))\big)\,ds.$$

Hence the inequalities in (3.17) turn into equalities. By virtue of (3.3) we can obtain (1.8) from (3.16) and (3.17).

§ 4. Envelope of Markovian semi-groups

This section is a review of [10]. In order to construct the envelope of semi-groups T_t^u, $u \in \Gamma$, it is convenient that the basic space where T_t^u acts should be a complete lattice endowed with the supremum norm, [9]. Since C is not complete as a lattice, condition (A4) is required. We sketch an outline of the construction of S_t. Define $J = J(N)$ by

$$J\varphi = \sup_u T^u_{2^{-N}}\varphi, \quad \varphi \in \Sigma\ (\equiv \bigcup_{K>0} \Sigma_K). \tag{4.1}$$

Then J is a mapping from Σ into Σ by (2.5) and (2.6). So we can define J^k successively, i.e.

$$J^{k+1}\varphi = J(J^k\varphi), \quad \varphi \in \Sigma. \tag{4.2}$$

Define an approximate envelope $S_t^{(N)}$ by

$$S_t^{(N)}\varphi = J^k(N)\varphi \quad \text{for} \quad t = k2^{-N},\ \varphi \in \Sigma. \tag{4.3}$$

From conditions (A1)–(A4) and Proposition 1 we can see the following lemma:

LEMMA 1. *For binary time $t = k2^{-N}$ and $\varphi \in \Sigma$, $S_t^{(N)}$ has the following properties*:

(0) $S_{t+\theta}^{(N)}\varphi = S_t^{(N)}(S_\theta^{(N)}\varphi) = S_\theta^{(N)}(S_t^{(N)}\varphi)$,

(i) $S_t^{(N)}\varphi \leqslant S_t^{(N)}\psi$ *whenever* $\varphi \leqslant \psi$,

(ii) $\|S_t^{(N)}\varphi - S_t^{(N)}\psi\| \leqslant \|\varphi - \psi\|$,

(iii) $\|S_t^{(N)}\varphi - S_\theta^{(N)}\varphi\| \leqslant |t-\theta|(\sup_u\|A^u\varphi\| + \sup_u\|f^u\|)$ *for* $\varphi \in C^2$,

(iv) $T_t^u\varphi \leqslant S_t^{(N)}\varphi$ *for* $u \in \Gamma$,

(v) $S_t^{(N)}\varphi = 0_l - \lim_n S_t^{(N)}\varphi_n$ *when* $\varphi = 0_l - \lim_n \varphi_n$ *and* $\varphi_n \in \Sigma_K, n = 1, 2, \ldots$,

(vi) $|S_t^{(N)}\varphi(x) - S_t^{(N)}\varphi(y)| \leqslant |x-y|e^{qt}\Big(K + \dfrac{h}{\lambda}(1-e^{-qt})\Big)$ $\varphi \in \Sigma_K$, $t \leqslant T$,

(vii) $\|S_t^{(N)}\varphi\| \leqslant K + ht$ *for* $\varphi \in \Sigma_K$.

By definition $S_t^{(N)}\varphi$ is increasing as $N \to \infty$, i.e.

$$S_t^{(N)}\varphi \leqslant S_t^{(N+1)}\varphi \quad \text{for} \quad t = k2^{-N},\ \varphi \in \Sigma. \tag{4.4}$$

Moreover, (vi) and (vii) imply the existence of $0_l - \lim_N S_t^{(N)}\varphi$ in Σ for binary t. Define

$$S_t\varphi = 0_l - \lim_N S_t^{(N)}\varphi \quad \text{for binary } t \text{ and } \varphi \in \Sigma. \tag{4.5}$$

Then S_t has properties (i)–(vii) and we can extend S_t in the following way:

$$S_t\varphi = \lim_{l\to\infty} S_{t_l}\varphi_l \quad \text{for} \quad t \geqslant 0,\ \varphi \in C, \tag{4.6}$$

where t_l is a binary approximation to t and $\varphi_l \in \Sigma$ is an approximation to φ. Thus we can see from Lemma 1 that S_t maps Σ into Σ and is a monotone contractive and strongly continuous operator on C.

LEMMA 2. *$S_{t+\theta}\varphi = S_t(S_\theta\varphi)$ for any t, θ and $\varphi \in C$.*

Proof. For $t = i2^{-l}$ and $\theta = j2^{-l}$ we have from (0) and (i)

$$S_{t+\theta}^{(N)}\varphi = S_\theta^{(N)}(S_t^{(N)}\varphi) \leqslant S_\theta^{(N)}(S_t\varphi) \leqslant S_\theta(S_t\varphi), \quad \varphi \in C^2,\ N \geqslant l. \tag{4.7}$$

On the other hand,

$$S_\theta(S_t\varphi) = 0_l - \lim S_\theta^{(N)}(S_t\varphi), \tag{4.8}$$

$$S_{t+\theta}\varphi = 0_l - \lim S_{t+\theta}^{(N)}\varphi. \tag{4.9}$$

Hence by (4.7) and (4.9) we get

$$S_{t+\theta}\varphi \leqslant S_\theta(S_t\varphi). \tag{4.10}$$

Moreover, for $l \leqslant n \leqslant N$ we have

$$S_\theta^{(n)}(S_t^{(N)}\varphi) \leqslant S_\theta^{(N)}(S_t^{(N)}\varphi) = S_{\theta+t}^{(N)}\varphi \leqslant S_{\theta+t}\varphi \tag{4.11}$$

and from (v), (vi) and (vii) of Lemma 1 we see that

$$S_{\theta}^{(n)}(S_t\varphi) = 0_t\text{-}\lim_{N} S_\theta^{(n)}(S_t^{(N)}\varphi). \tag{4.12}$$

Therefore by (4.8) and (4.11) we have

$$S_\theta(S_t\varphi) \leqslant S_{\theta+t}\varphi; \tag{4.13}$$

namely, Lemma 2 holds for $\varphi \in C^2$ and binary t and θ. Since S_t is contractive on C and continuous in t, we complete the proof.

We shall calculate the weak generator G_w of S_t. We denote S_t by Λ_t when $f^u \equiv 0$ for any u. Put $\Delta = 2^{-N}$ and $A\varphi = \sup(A^u\varphi + f^u)$. For $\varphi \in C^2$ we have $A\varphi \in C$ and

$$S_\Delta^{(N)}\varphi - \varphi = \sup_u (T_\Delta^u\varphi - \varphi) = \sup_u \int_0^\Delta P_\theta^u(A^u\varphi + f^u)\,d\theta$$

$$\leqslant \sup_u \int_0^\Delta P_\theta^u(A\varphi)\,d\theta \leqslant \int_0^\Delta \Lambda_\theta(A\varphi)\,d\theta.$$

Moreover, we get

$$S_{(k+1)\Delta}^{(N)}\varphi - S_k^{(N)}\varphi \leqslant \int_{k\Delta}^{(k+1)\Delta} \Lambda_\theta A\varphi\,d\theta. \tag{4.14}$$

Taking the summation w.r. to k, we have

$$S_t^{(N)}\varphi - \varphi \leqslant \int_0^t \Lambda_\theta A\varphi\,d\theta \quad \text{for} \quad t = j2^{-N}. \tag{4.15}$$

As N tends to ∞, we have

$$S_t\varphi - \varphi \leqslant \int_0^t \Lambda_\theta A\varphi\,d\theta \quad \text{for} \quad \varphi \in C^2 \text{ and binary } t. \tag{4.16}$$

Since both sides of (4.16) are continuous in t, (4.16) holds for any t. Consequently

$$\overline{\lim_{t\downarrow 0}}\, \frac{1}{t}\big(S_t\varphi(x) - \varphi(x)\big) \leqslant A\varphi(x) \quad \text{for any } x. \tag{4.17}$$

On the other hand, using the inequality

$$\frac{1}{t}(S_t\varphi - \varphi) \geqslant \frac{1}{t}(T_u^u\varphi - \varphi) \quad \forall u,$$

we get

$$\underline{\lim_{t\downarrow 0}}\, \frac{1}{t}\big(S_t\varphi(x) - \varphi(x)\big) \geqslant \lim_{t\downarrow 0} \frac{1}{t}\big(T_t^u\varphi(x) - \varphi(x)\big) = A^u\varphi(x) + f^u(x), \quad u \in \Gamma. \tag{4.18}$$

Therefore

$$\underline{\lim_{t\downarrow 0}}\, \frac{1}{t}\big(S_t\varphi(x) - \varphi(x)\big) \geqslant A\varphi(x). \tag{4.19}$$

By (4.17) and (4.18) we have (1.16).

The proof of (iii) is easy. Put $\Delta = 2^{-N}$. Then

$$T_\Delta^u\varphi \leqslant \tilde{S}_\Delta\varphi \quad \text{for any } u.$$

Taking the supremum w.r. to u, we have

$$S_\Delta^{(N)}\varphi \leqslant \tilde{S}_\Delta\varphi.$$

Hence

$$S_t^{(N)}\varphi \leqslant \tilde{S}_t\varphi \quad \text{for} \quad t = k2^{-N}. \tag{4.20}$$

This implies

$$S_t\varphi \leqslant \tilde{S}_t\varphi \quad \text{for binary } t. \tag{4.21}$$

Since both sides of (4.21) are continuous in t, we infer (1.17). This completes the proof of Theorem 2.

We sketch the proof of Theorem 3. Let P_t^u be the transition operator of diffusion whose generator is L^u. Recalling conditions (1.2) and (1.18), we can easily see that P_t^u maps C into C and (A1), (A2) and (A4) hold by virtue of (3.2). Hence there exist two semi-groups, Q_t and S_t. From (iii) of Theorem 2 we see that

$$S_t\varphi \leqslant Q_t\varphi. \tag{4.22}$$

We shall show the converse to (4.22) under the conditions of Theorem 3. Define $\mathfrak{A}_N$ by

$$\mathfrak{A}_N = \{A = (\Omega, B, U) \in \mathfrak{A};\ U(t) = U(k2^{-N}),\ t \in [k2^{-N}, (k+1)2^{-N})\}. \tag{4.23}$$

Then we have the following approximation lemma:

LEMMA. *For any $A \in \mathfrak{A}$ there exists an $A_k \in \bigcup_{N=1}^{\infty} \mathfrak{A}_N$ such that $V(t, x, A_k, \varphi)$ converges to $V(t, x, A, \varphi)$ at each t, x and φ.*

Proof. Put $W_k(t) = 2^k \int_{t-2^{-k}}^{t} U(s)\,ds$, where $U(s) = U(0)$ for $s \leqslant 0$, and $W_{k,l}(t) = W_k(2^{-l}[2^l t])$ where $[\cdot]$ is the integer part. Then we can see that

$$\lim_{k\uparrow\infty} \lim_{l\uparrow\infty} E\int_0^T |W_{k,l}(t) - U(t)|^2\,dt = 0 \quad \forall T > 0,$$

and for a suitable subsequence (k_p, l_p), $p = 1, 2, \ldots$ the required U_p is defined by $U_p = W_{k_p, l_p}$. Put $V_N(t, x, \varphi) = \sup_{A \in \mathfrak{A}_N} V(t, x, A, \varphi)$. Then the lemma tells us

$$Q_t\varphi(x) = V(t, x, \varphi) = \lim_{N\uparrow\infty} V_N(t, x, \varphi). \tag{4.24}$$

For $A \in \mathfrak{A}_N$ we have

$$\begin{aligned} V(\Delta, x, A, \varphi) &= E_x E\big(I(\Delta, A, \varphi)/U(0)\big) \\ &= E_x T_\Delta^{U(0)}\varphi\big(X(0)\big) \leqslant S_\Delta^{(N)}\varphi(x) \leqslant S_\Delta\varphi(x), \end{aligned} \tag{4.25}$$

where $\Delta = 2^{-N}$. Furthermore

$$V(k\Delta, x, A, \varphi) \leqslant S_{k\Delta}\varphi(x), \quad A \in \mathfrak{A}_N; \tag{4.26}$$

namely,

(4.27) $$V_N(t,x,\varphi) \leqslant S_t\varphi(x) \quad \text{for} \quad t = k2^{-N}.$$

From (4.24) and (4.27) we see that

(4.28) $$Q_t\varphi(x) \leqslant S_t\varphi(x) \quad \text{for binary } t.$$

Since both sides are continuous in t, we obtain the converse to (4.22).

§ 5. Optimal stopping and excessive majorant

First we review the proof of Theorem 4. By routine we see that

(5.1) $$E|X(\tau\wedge t)-Y(\tau\wedge t)|^2 \leqslant |x-y|^2 e^{2K_2 t}, \quad A\in\mathfrak{A},\ \tau\in\mathcal{T}(A),$$

where X and Y are the responses for A starting at x and y, respectively. Hence v is in C. We reduce the optimal stopping problem to the control problem without stopping by virtue of the randomized stopping [4], [6]. The method is the following: Put $\gamma_{n,i}(\theta) = n\chi_{[t-i/n,t]}(\theta)$. Then for any fixed i we have

(5.2) $$\int_0^\infty e^{-\int_0^s c(X,U)+\gamma_{n,i}} f(X,U)\,ds + \int_0^\infty e^{-\int_0^s c(X,U)+\gamma_{n,i}} \gamma_{n,i}(s) v(X(s))\,ds$$
$$\to \int_0^t e^{-\int_0^s c(X,U)} f(X,U)\,ds + e^{-i}\int_t^\infty e^{-\int_0^s c(X,U)} f(X,U)\,ds + (1-e^{-i}) e^{-\int_0^t c(X,U)} v(X(t))$$

as $n\to\infty$. As i tends to ∞, the right side of (5.2) becomes $I(t,A,v)$. By $\mathcal{R}(A)$ we denote the set of all bounded processes γ such that $\gamma(t)$ is σ_t-(B, U)-measurable. Appealing to (5.2), we can see that

(5.3) $$v(x) = \sup_{A\in\mathfrak{A}} \sup_{\gamma\in R(A)} E_x \int_0^\infty e^{-\int_0^t c(X,U)+\gamma} \big(f(X,U)+\gamma(\theta)v(X(t))\big)\,dt.$$

Therefore

(5.4) $$Q_t v \leqslant v,$$

i.e. v is Q_t-excessive. On the other hand, "$\varphi \leqslant v$" is clear from the definition of v.

Let $\tilde v$ be a Q_t-excessive majorant of ξ. Then the process ξ defined by

(5.5) $$\xi(t) = \int_0^t e^{-\int_0^s c(X,U)} f(X,U)\,ds + e^{-\int_0^s c(X,U)} \tilde v(X(t))$$

is $\sigma_{t+}(B, U)$-super martingale. Hence we have

(5.6) $$E_x\xi(\tau) \leqslant \tilde v(x) \quad \text{for any } \tau\in\mathcal{T}(A).$$

By virtue of "$\varphi\leqslant\tilde v$" we have

(5.7) $$I(t,A,\varphi) \leqslant \xi(t).$$

Thus

(5.8) $$E_x I(\tau,A,\varphi) \leqslant E_x\xi(\tau) \leqslant \tilde v(x), \quad A\in\mathfrak{A},\ \tau\in\mathcal{T}(A).$$

This derives "$v \leqslant \tilde v$" and completes the proof of Theorem 4.

We shall prove Theorem 5. In order to construct the least S_t-excessive majorant, we apply a similar method to that used in [9]; namely, we use the randomized stopping. Take $\Gamma\times[0,\infty)$ as the new control region and define $J = J(N)$ by

(5.9) $$J\varphi = \sup_{u\lambda} T^{u\lambda g}_{2^{-N}}\varphi, \quad \varphi\in C,$$

where $T^{u\lambda g}_t$ is defined by (2.3). From Corollary in § 2 we can see that J maps C into C. Hence we can define J^k, $k = 1, 2\ldots$ successively by $J^{k+1}\varphi = J(J^k\varphi)$.

Lemma 1. *Putting $\Delta = 2^{-N}$, we have*

(J0) $J^{k+l}\varphi = J^k(J^l\varphi) = J^l(J^k\varphi)$, $\quad k, l = 1, 2, \ldots$;

(J1) $J^k\varphi \leqslant J^k\psi$ *whenever* $\varphi\leqslant\psi$;

(J2) $\|J^k\varphi - J^k\psi\| \leqslant e^{-ck\Delta}\|\varphi-\psi\|$;

(J3) $\|J^k\varphi\| \leqslant e^{-ck\Delta}\|\varphi\| + \frac{h}{c}(1-e^{-ck\Delta}) + \|g\|$;

(J4) $J^k\varphi = 0_l\text{-}\lim_n J^k\varphi_n$, $k = 1,\ldots,l$ *when* $\varphi = 0_l\text{-}\lim\varphi_n$ *and* $0_l\text{-}\lim J^k\varphi_n$, $k = 1,\ldots,l$, *exist;*

(J5) $g\leqslant J\varphi$;

(J6) $|J^k\varphi(x) - J^k\varphi(y)| \leqslant |x-y|e^{-qk\Delta}\Big[K + \frac{h}{q}(1-e^{-qk\Delta}) + p(qk\Delta + e^{-qk\Delta})\Big]$

for $k\Delta\leqslant T$ and $\varphi\in\Sigma_K$ where $q = q(T)$.

Hereafter, the time parameter stands for positive binary number until the end of Lemma 4. We define $\mathcal{S}^{(N)}_t\varphi$ by

(5.10) $$\mathcal{S}^{(N)}_t\varphi = J^k(N)\varphi \quad \text{for} \quad t = k2^{-N} \text{ and } \varphi\in C.$$

Then we have the following lemma:

Lemma 2.

(i) *For T and $\varphi\in C$ there exist M and δ^* such that $\mathcal{S}^{(N)}_t\varphi\in H_{M,\delta^*}$ for any $t = k2^{-N}\leqslant T$, $N = 1, 2, \ldots$*

(ii) *$\mathcal{S}^{(N)}_t\varphi$ is increasing as $N\nearrow\infty$.*

(iii) *For fixed N, $\mathcal{S}^{(N)}_t\varphi$ is increasing as $t\nearrow\infty$, whenever $\varphi\leqslant g$.*

Proof. The first part is nothing but (2.8), since $H_{M,\delta}$ is a complete lattice. Put $\Delta = 2^{-(N+1)}$. Then

$$\mathcal{S}^{(N)}_{2\Delta}\varphi = \sup_{u\lambda} T^{u\lambda g}_{2\Delta}\varphi = \sup_{u\lambda} T^{u\lambda g}_{\Delta}(T^{u\lambda g}_{\Delta}\varphi)$$
$$\leqslant \sup_{u\lambda} T^{u\lambda g}_{\Delta}(J(N+1)\varphi) = J^2(N+1)\varphi = \mathcal{S}^{(N+1)}_{2\Delta}\varphi.$$

Suppose $\mathcal{S}^{(N)}_{2k\Delta}\varphi \leqslant \mathcal{S}^{(N+1)}_{2k\Delta}\varphi$. Then

$$\mathcal{S}^{(N)}_{2\Delta+2k\Delta}\varphi = \mathcal{S}^{(N)}_{2\Delta}(\mathcal{S}^{(N)}_{2k\Delta}\varphi) \leqslant \mathcal{S}^{(N)}_{2\Delta}(\mathcal{S}^{(N+1)}_{2k\Delta}\varphi) \leqslant \mathcal{S}^{(N+1)}_{2\Delta}(\mathcal{S}^{(N+1)}_{2k\Delta}\varphi) = \mathcal{S}^{(N+1)}_{2\Delta+2k\Delta}\varphi.$$

This means the second part of the lemma.

For the third part we have by $(J5)$

$$\varphi \leqslant g \leqslant J\varphi.$$

Hence from $(J1)$ we can see that

$$\varphi \leqslant J\varphi \leqslant J^2\varphi \leqslant \ldots$$

This completes the proof of Lemma 2.

By Lemma 2 we can define $\mathcal{S}_t\varphi$ by

$$\mathcal{S}_t\varphi = 0_l-\lim_n \mathcal{S}^{(N)}_t\varphi \quad \text{for binary } t \text{ and } \varphi \in C. \tag{5.11}$$

Then $\mathcal{S}_t$ has the following properties:

($\mathcal{S}1$) $\mathcal{S}_t\varphi \leqslant \mathcal{S}_t\psi$ whenever $\varphi \leqslant \psi$;

($\mathcal{S}2$) $\|\mathcal{S}_t\psi - \mathcal{S}_t\varphi\| \leqslant e^{-ct}\|\varphi-\psi\|$;

($\mathcal{S}3$) $\|\mathcal{S}_t\varphi\| \leqslant e^{-ct}\|\varphi\| + \dfrac{h}{c}(1-e^{-ct}) + \|g\| \leqslant \|\varphi\| + \dfrac{h}{c} + \|g\| \equiv \tilde M$;

($\mathcal{S}4$) $\mathcal{S}_t\varphi = 0_l-\lim\limits_n \mathcal{S}_t\varphi_n$ when $\varphi_n \in H_{M,\delta}$, $n = 1, 2 \ldots$ and $\varphi = 0_l-\lim\limits_n \varphi_n$;

($\mathcal{S}5$) $\mathcal{S}_{t+\theta}\varphi = \mathcal{S}_t(\mathcal{S}_\theta\varphi) = \mathcal{S}_\theta(\mathcal{S}_t\varphi)$;

($\mathcal{S}6$) $g \leqslant \mathcal{S}_t\varphi$.

We prove ($\mathcal{S}4$) and ($\mathcal{S}5$) since the other properties are clear from Lemma 1.

($\mathcal{S}4$). Put $t = i2^{-l}$. By Corollary in § 2 we have a constant $\tilde M$ and a positive function $\tilde\delta$ such that

$$\mathcal{S}^{(N)}_t\varphi_n \in H_{\tilde M,\tilde\delta}, \quad n = 1, 2 \ldots, \quad N = l, l+1, \ldots \tag{5.12}$$

Since $\mathcal{S}^{(N)}_t\varphi_n$ is increasing as $n \to \infty$, $0_l-\lim\limits_n \mathcal{S}^{(N)}_t\varphi_n$ exists in $H_{\tilde M,\tilde\delta}$. On the other hand,

$$\mathcal{S}_t\varphi_n \in H_{\tilde M,\tilde\delta}, \quad n = 1, 2 \ldots \tag{5.13}$$

by the definition of $\mathcal{S}_t$. Recalling ($\mathcal{S}1$), we can see that $0_l-\lim \mathcal{S}_t\varphi_n$ exists in $H_{\tilde M,\tilde\delta}$. Thus by virtue of $(J4)$ we have

$$\mathcal{S}^{(N)}_t\varphi = 0_l-\lim_n \mathcal{S}^{(N)}_t\varphi_n \leqslant 0_l-\lim_n \mathcal{S}_t\varphi_n \leqslant \mathcal{S}_t\varphi. \tag{5.14}$$

As N tends to ∞, we get

$$\mathcal{S}_t\varphi \leqslant 0_l-\lim_n \mathcal{S}_t\varphi_n \leqslant \mathcal{S}_t\varphi. \tag{5.15}$$

This proves ($\mathcal{S}4$).

($\mathcal{S}5$). From $(J1)$ and the definition of $\mathcal{S}_t$ we have

$$\mathcal{S}^{(N)}_{t+\theta}\varphi = \mathcal{S}^{(N)}_t(\mathcal{S}^{(N)}_\theta\varphi) \leqslant \mathcal{S}^{(N)}_t(\mathcal{S}_\theta\varphi) \leqslant \mathcal{S}_t(\mathcal{S}_\theta\varphi). \tag{5.16}$$

As N tends to ∞, we have

$$\mathcal{S}_{t+\theta}\varphi \leqslant \mathcal{S}_t(\mathcal{S}_\theta\varphi). \tag{5.17}$$

On the other hand, for $k \leqslant N$, we have

$$\mathcal{S}^{(N)}_t(\mathcal{S}^{(k)}_\theta\varphi) \leqslant \mathcal{S}^{(N)}_t(\mathcal{S}^{(N)}_\theta\varphi) = \mathcal{S}^{(N)}_{t+\theta}\varphi \leqslant \mathcal{S}_{t+\theta}\varphi. \tag{5.18}$$

As N tends to ∞, we have

$$\mathcal{S}_t(\mathcal{S}^{(k)}_\theta\varphi) \leqslant \mathcal{S}_{t+\theta}\varphi. \tag{5.19}$$

Since $\mathcal{S}^{(k)}_\theta\varphi$ is increasing to $\mathcal{S}_\theta\varphi$ as $k \nearrow \infty$ and by Lemma 2 $\mathcal{S}^{(k)}_\theta\varphi \in H_{M,\delta}$, for large k, ($\mathcal{S}4$) implies

$$\mathcal{S}_t(\mathcal{S}_\theta\varphi) = 0_l-\lim_k \mathcal{S}_t(\mathcal{S}^{(k)}_\theta\varphi). \tag{5.20}$$

Combining (5.20) with (5.19), we have

$$\mathcal{S}_t(\mathcal{S}_\theta\varphi) \leqslant \mathcal{S}_{t+\theta}\varphi; \tag{5.21}$$

namely, (5.17) and (5.21) complete the proof of ($\mathcal{S}5$).

LEMMA 3. *There exists a positive function $\tilde\delta$ on $(0, \infty)$ such that*

$$\mathcal{S}_t\varphi \in H_{\tilde M,\tilde\delta} \quad \textit{for any positive binary } t. \tag{5.22}$$

Moreover, if $\varphi \leqslant g$, then $0_l-\lim\limits_{t\nearrow\infty} \mathcal{S}_t\varphi$ exists.

Proof. From Lemma 2 we can take a positive function δ^T, so that

$$\mathcal{S}_t\varphi \in H_{\tilde M,\delta^T} \quad \text{for} \quad t \leqslant T, \tag{5.23}$$

where $\tilde M$ is given in ($\mathcal{S}3$). By ($\mathcal{S}2$) and ($\mathcal{S}3$)

$$\|\mathcal{S}_{t+\theta}\varphi - \mathcal{S}_t\varphi\| = \|\mathcal{S}_t(\mathcal{S}_\theta\varphi) - \mathcal{S}_t\varphi\| \leqslant e^{-ct}\|\mathcal{S}_\theta\varphi - \varphi\| \leqslant 2\tilde M e^{-ct}. \tag{5.24}$$

Hence, for $\varepsilon > 0$ there exists a large binary $T = T(\varepsilon)$ such that

$$\|\mathcal{S}_{t+\theta}\varphi - \mathcal{S}_t\varphi\| < \varepsilon/4 \quad \text{for} \quad t \geqslant T \text{ and } \theta \geqslant 0. \tag{5.25}$$

Thus for $|x-y| < \delta^T(\varepsilon/2)$ we have

$$\begin{aligned} &|\mathcal{S}_{T+\theta}\varphi(x) - \mathcal{S}_{T+\theta}\varphi(y)| \\ &\quad \leqslant 2\|\mathcal{S}_{T+\theta}\varphi - \mathcal{S}_T\varphi\| + |\mathcal{S}_T\varphi(x) - \mathcal{S}_T\varphi(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\theta \geqslant 0. \end{aligned} \tag{5.26}$$

Therefore, putting $\tilde\delta(\varepsilon) = \delta^{T(\varepsilon)}(\varepsilon/2)$, we see that

$$|\mathcal{S}_t\varphi(x) - \mathcal{S}_t\varphi(y)| < \varepsilon \quad \text{for} \quad t > 0 \text{ and } |x-y| < \tilde\delta(\varepsilon). \tag{5.27}$$

This means that $\mathcal{S}_t\varphi \in H_{\tilde M,\tilde\delta}$ for any $t > 0$.

By Lemma 2 the second part is clear. This completes the proof.

We denote $0_l-\lim\limits_t \mathcal{S}_t\varphi$ by v_φ when $\varphi \leqslant g$. From ($\mathcal{S}6$) we have

$$g \leqslant v_\varphi. \tag{5.28}$$

For simplicity we put $v = v_\varphi$, if no confusion arises.

LEMMA 4. *v is $\mathcal{S}_t$-invariant, i.e.*

$$\mathcal{S}_t v = v \quad \textit{for positive binary } t. \tag{5.29}$$

Proof. By the definition of v and Lemma 3 we see that

$$\mathscr{S}_t v = \mathscr{S}_t(0_i - \lim_\theta \mathscr{S}_\theta \varphi) = 0_i - \lim_\theta \mathscr{S}_t(\mathscr{S}_\theta \varphi) = 0_i - \lim_\theta \mathscr{S}_{t+\theta}\varphi = v.$$

For the proof of Theorem 5, we show that v_φ is the least S_t-excessive majorant of g by the following two propositions:

PROPOSITION 3. *v is an S_t-excessive majorant of g, i.e.*

(5.30) $$S_t v \leqslant v \quad \textit{for} \quad t \geqslant 0 \textit{ and } g \leqslant v.$$

Proof. From the definitions of $\mathscr{S}_t$ and S_t we see that

(5.31) $$S_t\varphi \leqslant \mathscr{S}_t\varphi \quad \text{for positive binary } t \text{ and } \varphi \in C.$$

Hence by Lemma 4 we have

(5.32) $$S_t v \leqslant \mathscr{S}_t v = v \quad \text{for positive binary } t.$$

Since $S_t v$ is continuous in t, $S_t v \leqslant v$ for any $t \geqslant 0$. This proves Proposition 3.

PROPOSITION 4. *For any $\varphi \leqslant g$, v_φ is the least S_t-excessive majorant of g.*

Proof. Let V be an S_t-excessive majorant of g. Recalling the definitions of $T_t^{u\lambda g}$ and T_t^u, we have

(5.33) $$T_t^{u\lambda g}\varphi = e^{-\lambda t}T_t^u\varphi + \lambda\int_0^t e^{-\lambda\theta}T_\theta^u g\,d\theta \quad \text{for} \quad \varphi \in C.$$

Since V is S_t-excessive, we get

(5.34) $$T_t^u V \leqslant S_t V \leqslant V.$$

Thus from (5.33) and (5.34) we can see that

(5.35) $$T_t^{u\lambda g}V \leqslant e^{-\lambda t}V + \lambda\int_0^t e^{-\lambda\theta}T_\theta^u g\,d\theta.$$

Recalling "$g \leqslant V$", we have

(5.36) $$T_\theta^u g \leqslant T_\theta^u V \leqslant V$$

by virtue of (5.34). Combining (5.36) with (5.35), we have

(5.37) $$T_t^{u\lambda g}V \leqslant e^{-\lambda t}V + (1-e^{-\lambda t})V = V \quad \forall u\lambda.$$

Taking the supremum w.r.t. to u and λ, (5.37) turns into

(5.38) $$J(N)V = \sup_{u\lambda} T_{2^{-N}}^{u\lambda g}V \leqslant V.$$

Since $J(N)$ is monotone, we obtain (5.39) from (5.38):

(5.39) $$J^k(N)V \leqslant J^{k-1}(N)V \leqslant \ldots \leqslant J(N)V \leqslant V;$$

namely,

$$\mathscr{S}_t^{(N)}V \leqslant V \quad \text{for} \quad t = k2^{-N},\ N = 1, 2 \ldots$$

As N tends to ∞, we have

(5.40) $$\mathscr{S}_t V \leqslant V \quad \text{for binary } t.$$

Recalling "$\varphi \leqslant g \leqslant V$", we can see from (5.40) that

(5.41) $$\mathscr{S}_t\varphi \leqslant \mathscr{S}_t V \leqslant V.$$

As t tends to ∞, we have

$$v_\varphi \leqslant V.$$

This completes the proof of Proposition 4.

Since the least S_t-excessive majorant is unique, v_φ does not depend on φ. In usual optimal stopping problems, the construction of the least excessive majorant corresponds to v_g, [ref. § 6].

Remark. Recalling Lemma 2, we can define $v_\varphi^{(N)}(x)$ by

(5.42) $$v_\varphi^{(N)}(x) = \lim_k \mathscr{S}_{k2^{-N}}^{(N)}\varphi(x) = \lim_k \mathscr{S}_k^{(N)}\varphi(x),$$

where k runs through integers. Since $v_\varphi^{(N)}(x)$ is increasing as $N \nearrow \infty$, we put $V_\varphi(x) = \lim_N v_\varphi^{(N)}(x)$. On the other hand, $\mathscr{S}_k^{(N)}\varphi(x)$ increases to $\mathscr{S}_k\varphi(x)$ as $N \nearrow \infty$ and $\mathscr{S}_k\varphi(x)$ increases to $v_\varphi(x)$. Thus we have, for $\varepsilon > 0$,

$$v_\varphi(x) - \varepsilon \leqslant \mathscr{S}_k\varphi(x) \leqslant \mathscr{S}_k^{(N)}\varphi(x) + \varepsilon$$

with some k and N. Furthermore,

$$\mathscr{S}_k^{(N)}\varphi(x) \leqslant v_\varphi^{(N)}(x) \leqslant V_\varphi(x).$$

Therefore we have

$$v_\varphi(x) \leqslant V_\varphi(x).$$

On the other hand, "$\mathscr{S}_k^{(N)}\varphi(x) \leqslant \mathscr{S}_k\varphi(x) \leqslant v_\varphi(x)$" implies "$V_\varphi(x) \leqslant v_\varphi(x)$". Hence we get

(5.43) $$V_\varphi = v_\varphi.$$

§ 6. Optimal stopping of the stochastic control of switchings

Let W be the path space, namely the set of all right-continuous R^n-valued functions on $[0, \infty)$ with left limits. Let F_t and F be σ-algebras on W generated by $\{w(s), s \leqslant t\}$ and $\{w(s), s < \infty\}$, respectively. By P_x^u we denote the probability law of the path starting at x with transition probability $P^u(t, x, B)$, which is a probability measure on R^n for any u, t and x and a Borel function of (u, t, x) for any B. $X(t, w)$ denotes the tth coordinate of $w \in W$ and the system $(X(t), W, F_t, F, P_x^u, x \in R_*^n)$ is a Markov process with transition probability P^u. Define the transition operator P_t^u by

(6.1) $$P_t^u\varphi(x) = E_x^u\left[\varphi(X(t))\,e^{-\int_0^t c^u(X(s))\,ds}\right],$$

i.e. P_t^u is the transition operator with killing rate c^u, which is bounded uniformly in u. We assume that P_t^u is a semi-group which satisfies conditions (A1)–(A5) when φ runs through C.

Let d be a Γ-valued progressively F_t-measurable step function on $[0,\infty)\times W$, i.e.

(6.2) $$d(t,w) = d(k\Delta, w) \quad \text{for} \quad t\in[k\Delta,(k+1)\Delta),$$

where $\Delta = 2^{-N}$. By $\mathfrak{A}_N$ we denote the set of all functions expressed by (6.2). Put $\mathfrak{A} = \bigcup_{N=1}^{\infty}\mathfrak{A}_N$ and call an element of $\mathfrak{A}$ an *admissible control*, more precisely: a *control of switchings*. According to [13], [14] we formulate the following problem: An admissible control d written by (6.2) defines the new measure Q_x^d on (W, F), which satisfies the following conditions:

(6.3) $$\begin{gathered} Q_x^d(A) = P_x^{d(0)}(A), \quad A\in F_\Delta, \\ Q_x^d\big(X(t_i)\in B_i,\ i=1\ldots k/F_\Delta\big) = P_{X(\Delta)}^{d(\Delta)}\big(X(t_i-\Delta)\in B_i,\ i=1\ldots k\big), \\ \Delta < t_i \leqslant 2\Delta, \\ Q_x^d\big(X(t_i)\in B_i,\ i=1\ldots k/F_{l\Delta}\big) = P_{X(l\Delta)}^{d(l\Delta)}\big(X(t_i-l\Delta)\in B_i,\ i=1\ldots k\big), \\ l\Delta < t_i \leqslant (l+1)\Delta. \end{gathered}$$

From (A5) we can see that

(6.4) $$E_x^d\Big[|\varphi(X(t))|\, e^{-\int_0^t c^u(X(s))ds}\Big] \leqslant e^{-ct}\|\varphi\|,$$

where E_x^d means the expectation w.r. to Q_x^d. Let τ be an F_s-stopping time, i.e. a $[0,\infty]$-valued random variable with $(\tau\leqslant t)\in F_{t+}$. By $\mathcal{T}$ we denote the set of all F_s-stopping times. Define $V^{d,\tau}(x)$ by

(6.5) $$V^{d,\tau}(x) = E_x^d\Big[\int_0^\tau f^{d(s)}(X(s))\, e^{-\int_0^s c^{d(\theta)}(X(\theta))d\theta} ds + g(X(\tau))\, e^{-\int_0^\tau c^{d(s)}(X(s))ds}\Big].$$

By (6.4) we mean $g(X(\infty))e^{-\int_0^\infty c^d} = 0$. The problem is to maximize $V^{d,\tau}$.

(6.6) $$V(x) = \sup_{d\in\mathfrak{A}}\sup_{\tau\in\mathcal{T}} V^{d,\tau}(x).$$

We review the randomization of stopping according to Krylov [4], [6], which reduces V to v_φ. When the path stops at t, the gain on the path is given as follows:

(6.7) $$\varrho^d(t) = \int_0^t f^{d(s)}(X(s))\, e^{-\int_0^s c^d} ds + g(X(t))\, e^{-\int_0^t c^d}.$$

We apply the following randomization:

(6.8) $$\text{Prob. (stop at } (t, t+dt)/\text{stop does not occur before}) = r(t)\,e^{-\int_0^t r(s)ds},$$

where $r(t)$ is non-negative F_t-progressively measurable. The condition "$\int_0^\infty r(s)ds = \infty$ w.p. 1" means that the path stops w.p. 1. When we apply the randomized

stopping of (6.8), the gain on the path is given by (6.9):

(6.9) $$\begin{aligned} &\int_0^\infty \varrho^d(t)\, r(t)\, e^{-\int_0^t r(s)ds} dt + \varrho^d(\infty)\, e^{-\int_0^\infty r(s)ds} \\ &\qquad = \int_0^\infty e^{-\int_0^t r+c^d}\big(f^{d(t)}(X(t)) + r(t)g(X(t))\big)dt \quad (\equiv I(d,r)). \end{aligned}$$

By $\mathcal{R}$ we denote the set of all bounded and non-negative F_s-progressively measurable processes. For any $\tau\in\mathcal{T}$ we define $r = r_{n,i}$ by

(6.10) $$r_{n,i}(t) = \begin{cases} n\chi_{(\tau,\tau+i/n)}(t) & \text{for} \quad \tau<\infty, \\ 0 & \text{for} \quad \tau=\infty. \end{cases}$$

Since $(\tau < t < \tau+i/n) = (\tau < t)\cap(t-i/n < \tau)\in F_t$, $r_{n,i}$ is F_t-measurable. Moreover, we can see that, as $N\nearrow\infty$

(6.11) $$\int_0^\infty f^{d(t)}(X(t))\, e^{-\int_0^t r+c^d} dt \to \int_0^\tau e^{-\int_0^s c^d} f^{d(s)}(X(s))ds + e^{-i}\int_0^\infty e^{-\int_0^s c^d} f^{d(s)}(X(s))ds$$

and

(6.12) $$\int_0^\infty g(X(t))\, r(t)\, e^{-\int_0^t r+c^d} dt \to (1-e^{-i})\, g(X(\tau))\, e^{-\int_0^\tau c^d}.$$

Hence

(6.13) $$\lim_i\lim_n I(d, r_{n,i}) = \varrho^d(\tau)$$

and

(6.14) $$I(d,r) \leqslant \int_0^\infty e^{-\int_0^t c^d}\,|f^{d(t)}(X(t))|\,dt + |g|.$$

Since the right side of (6.14) is Q_x^d-integrable by (6.4), the convergence theorem implies

(6.15) $$\lim_i\lim_n E_x^d I(d, r_{n,i}) = V^{d,\tau}(x).$$

Therefore, putting $V^d(x) = \sup_{\tau\in\mathcal{T}} V^{d,\tau}(x)$, we have

(6.16) $$V^d(x) \leqslant \sup_{r\in\mathcal{R}} E_x^d I(d,r).$$

Now we show the converse inequality. For $r\in\mathcal{R}$ we put $r_k = r+1/k$. So r_k decreases to r and $\|r_k\| \leqslant \|r\|+1$. Hence

(6.17) $$\int_0^\infty f^{d(t)}(X(t))\, e^{-\int_0^t r_k+c^d} dt \to \int_0^\infty f^{d(t)}(X(t))\, e^{-\int_0^t r+c^d} dt \quad \text{as} \quad k\to\infty,$$

in $L_1(Q_x^d)$, since $\int_0^\infty e^{-\int^t c^d} |f^{d(t)}(X(t))| dt$ is Q_x^d-integrable. Similarly

$$(6.18)\quad E_x^d\left|\int_0^\infty g(X(t))\, r_k(t)\, e^{-\int^t r_k+c^d} dt - \int_0^\infty g(X(t))\, r(t)\, e^{-\int^t r+c^d} dt\right|$$

$$\le 2(\|r\|+1)E_x^d \int_T^\infty e^{-\int^t c^d}|g(X(t))|\,dt + E_x^d\int_0^T |g(X(t))|\,\Big|r_k(t)e^{-\int^t r_k+c^d} - r(t)e^{-\int^t r+c^d}\Big|\,dt$$

$$\le 2(\|r\|+1)e^{-cT}\|g\| + 2\text{nd term}.$$

Since the 2nd term tends to 0 as $k\to\infty$, we have

$$(6.19)\quad \lim_k E_x^d I(d,r_k) = E_x^d I(d,r).$$

Define the subset $\mathscr{R}_+$ of $\mathscr{R}$ by

$$(6.20)\quad \mathscr{R}_+ = \{r\in\mathscr{R};\ \inf_{tw} r(t,w) > 0\}.$$

From (6.19) we can see that

$$(6.21)\quad \sup_{r\in\mathscr{R}} E_x^d I(d,r) = \sup_{r\in\mathscr{R}_+} E_x^d I(d,r).$$

Suppose $0<m\le \inf r(t,w) \le \sup r(t,w) \le M<\infty$. Then we can uniquely choose τ_k so that

$$(6.22)\quad e^{-\int_0^{\tau_k} r(s)ds} = 1-k2^{-N}, \quad k=0,1,\dots,2^N.$$

τ_k is an F_s-stopping time, because

$$(6.23)\quad (\tau\le t) = \big(e^{-\int_0^t r(s)ds} \le 1-k2^{-N}\big) \in F_t.$$

For simplicity we drop the superscript d if no confusion arises.

$$(6.24)\quad \int_0^\infty e^{-\int^t r+c} f(X(t))\,dt = \sum_{i=0}^{2^N-1}\int_{\tau_i}^{\tau_{i+1}} e^{-\int^t r+c} f(X(t))\,dt$$

$$= \sum_{i=0}^{2^N-1}\int_{\tau_i}^{\tau_{i+1}} \big(e^{-\int^t r} - e^{-\int^{\tau_i} r}\big) e^{-\int^t c} f(X(t))\,dt +$$

$$+ \sum_{i=0}^{2^N-1} e^{-\int^{\tau_i} r}\int_{\tau_i}^{\tau_{i+1}} e^{-\int^t c} f(X(t))\,dt \equiv I_1+I_2.$$

Recalling (6.22), we have

$$(6.25)\quad \big|e^{-\int^t r} - e^{-\int^{\tau_i} r}\big| \le e^{-\int^{\tau_i} r} - e^{-\int^{\tau_{i+1}} r} = 2^{-N}, \quad \tau_i\le t\le\tau_{i+1}.$$

Hence we can see that

$$(6.26)\quad |I_1| \le 2^{-N}\int_0^\infty e^{-\int^t c}|f(X(t))|\,dt \to 0 \quad \text{as} \quad N\to\infty$$

at Q_x^d-a.e. For I_2 we have by (6.22)

$$(6.27)\quad I_2 = 2^{-N}\sum_{i=0}^{2^N-1}(2^N-i)\Big(\int_0^{\tau_{i+1}} e^{-\int^t c} f(X(t))\,dt - \int_0^{\tau_i} e^{-\int^t c}f(X(t))\,dt\Big)$$

$$= 2^{-N}\sum_{i=0}^{2^N}\int_0^{\tau_i} e^{-\int^t c} f(X(t))\,dt.$$

Combining (6.26) and (6.27) with (6.24), we get

$$(6.28)\quad \left|\int_0^\infty e^{-\int^t r+c} f(X(t))\,dt - 2^{-N}\sum_{i=0}^{2^N}\int_0^{\tau_i} e^{-\int^t c} f(X(t))\,dt\right| \le 2^{-N}\int_0^\infty e^{-\int^t c}|f(X(t))|\,dt \to 0$$

as $N\to\infty$.

$$(6.29)\quad \int_0^\infty e^{-\int^t r+c}\, r(t)g(X(t))\,dt = \sum_{i=0}^{2^N-1}\int_{\tau_i}^{\tau_{i+1}} e^{-\int^t r+c}\, r(t) g(X(t))\,dt.$$

Again by (6.22) we have

$$(6.30)\quad \int_{\tau_i}^{\tau_{i+1}} e^{-\int^t r+c}\, r(t) g(X(t))\,dt$$

$$= \int_{\tau_i}^{\tau_{i+1}} e^{-\int^t r} r(t)\big(e^{-\int^t c} g(X(t)) - e^{-\int^{\tau_i} c} g(X(\tau_i))\big)dt + e^{-\int^{\tau_i} c} g(X(\tau_i)) \int_{\tau_i}^{\tau_{i+1}} e^{-\int^t r} r(t)\,dt$$

$$= J_i + 2^{-N} g(X(\tau_i))\, e^{-\int^{\tau_i} c}.$$

For $\varepsilon>0$ we fix T so that "$e^{-mT}<\varepsilon$" and define $j=j(T)$ by $\tau_j\le T<\tau_{j+1}$. Put

$$(6.31)\quad J \equiv \sum_{i=0}^{j-1} J_i + J_j + \sum_{i=j+1}^{2^N-1} J_i.$$

Then we have

$$(6.32)\quad |2\text{nd}| \le 2\|g\|\big(e^{-\int^{\tau_j} r} - e^{-\int^{\tau_{j+1}} r}\big) = 2\|g\|2^{-N}$$

and

$$|3\text{rd}| \le 2\|g\|\int_T^\infty e^{-\int^t r} r(t)\,dt = 2\|g\|\, e^{-\int_0^T r} \le 2\|g\|e^{-mT} < 2\|g\|\varepsilon.$$

For the first term of (6.31) we observe from (6.22) that

$$2^{-N} = e^{-\int^{\tau_{i+1}} r}\big(e^{\int_{\tau_i}^{\tau_{i+1}} r} - 1\big)$$

$$\ge e^{-MT}\int_{\tau_i}^{\tau_{i+1}} r(s)\,ds \ge e^{-MT} m(\tau_{i+1}-\tau_i), \quad i=0,1,\dots,j-1;$$

namely,

$$(6.33)\quad \tau_{i+1}-\tau_i \le e^{MT}m^{-1}2^{-N}, \quad i=0,\dots,j-1.$$

On the other hand, for any fixed path there are only finitely many Δ-jumping times until T, say

$$|X(t)-X(t-)| \geqslant \Delta \quad \text{for} \quad t = \theta_1, \ldots, \theta_p \leqslant T,$$

$$|X(t)-X(t-)| < \Delta \quad \text{for} \quad t \in [0, T]-\{\theta_1, \ldots \theta_p\}.$$

Moreover, there exists a positive constant δ, depending on the path, such that $\sup_{t,s\in[a,\min(T,a+\delta)]} |X(t)-X(s)| < 4\Delta$ whenever $\theta_i \notin [a, \min(T, a+\delta)]$, $i = 1, \ldots, p$. Thus by virtue of (6.33) there exists a large integer N such that

$$\sum_{i=1}^{j-1} |J_i| < \varepsilon \tag{6.34}$$

since g is uniformly continuous and c is bounded. Therefore from (6.30)–(6.34) we can see that

$$\left|\int_0^\infty e^{-\int^t r+c} r(t) g(X(t))\,dt - 2^{-N} \sum_{i=0}^{2^N-1} e^{-\int^{\tau_i} c} g(X(\tau_i))\right| \to 0 \tag{6.35}$$

as $N \to \infty$ at Q_x^d-a.e. Combining (6.35) with (6.28), we get

$$I_N \equiv \left|I(d, r) - 2^{-N} \sum_{i=0}^{2^N} \varrho^d(\tau_i)\right| \to 0 \quad \text{as} \quad N \to \infty \tag{6.36}$$

at Q_x^d-a.e. On the other hand, "$E_x^d \varrho(\tau_i) \leqslant V^d(x)$" and

$$I_N \leqslant 2 \int_0^\infty e^{-\int^t c} |f(X(t))|\,dt + 2\|g\| \tag{6.37}$$

and the right side is Q_x^d-integrable. Hence

$$E_x^d I(d, r) = \lim_N 2^{-N} \sum_{i=0}^{2^N} E_x^d \varrho(\tau_i) \leqslant V^d(x), \qquad r \in \mathscr{R}_+ \tag{6.38}$$

by recalling the definition of $V^d(x)$. Therefore, we obtain the converse inequality of (6.16) by (6.21) and (6.38). This means that

$$V(x) = \sup_{d\in\mathfrak{A}} \sup_{\tau\in\mathscr{T}} V^{d,\tau}(x) = \sup_{d\in\mathfrak{A}} \sup_{r\in\mathscr{R}} E_x^d I(d, r). \tag{6.39}$$

Put $\mathscr{R}_N = \{r \in \mathscr{R};\ r(t) = r(k2^{-N}),\ t \in [k2^{-N}, (k+1)2^{-N})\}$. Using the same calculation as (6.19), we see that $\sup_{r\in\mathscr{R}_N} E_x^d I(d, r)$ increases to $\sup_{r\in\mathscr{R}} E_x^d I(d, r)$. Hence we have by (6.39)

$$V(x) = \lim_{N\nearrow\infty} \sup_{d\in\mathfrak{A}_N, r\in\mathscr{R}_N} E_x^d I(d, r). \tag{6.40}$$

Putting $V_N(x) = \sup_{d\in\mathfrak{A}_N, r\in\mathscr{R}_N} E_x^d I(d, r)$, we shall show

$$V_N(x) = v_\varphi^{(N)}(x), \tag{6.41}$$

where $v_\varphi^{(N)}$ is defined in (5.42). Define $V_t^{d,r}\varphi(x)$ and $V_{N,k}\varphi(x)$ by

$$V_t^{d,r}\varphi(x) = E_x^d \int_0^t e^{-\int^s r+c^d} \big(f^{d(s)}(X(s)) + r(s)\big) g(X(s))\,ds + e^{-\int^t r+c^d} \varphi(X(t))$$

and

$$V_{N,k}\varphi(x) = \sup_{d\in\mathfrak{A}_N, r\in\mathscr{R}_N} V_{k2^{-N}}^{d,r}\varphi(x),$$

respectively. Following the randomized stopping (6.9), if we stop the path before t, the gain becomes

$$\int_0^t \varrho^d(s) r(s) e^{-\int_0^s r(\theta)d\theta} ds + \varrho^d(t) e^{-\int_0^t r(s)ds}$$

$$= \int_0^t e^{-\int^s r+c} \big(f(X(s)) + r(s) g(X(s))\big)\,ds + g(X(t)) e^{-\int^t r+c}.$$

So φ is g in the usual stopping problem.

In order to prove (6.41) we show

$$J^k\varphi(x) = V_{N,k}\varphi(x). \tag{6.42}$$

Recalling the definition of J, we have, putting $\Delta = 2^{-N}$,

$$\begin{aligned} J\varphi(x) &= \sup_{u\lambda} T_\Delta^{u\lambda g}\varphi(x) \\ &= \sup_{u\lambda} E_x^u \int_0^\Delta e^{-\lambda s - \int^s c^u} \big(f^u(X(s)) + \lambda g(X(s))\big)\,ds + e^{-\lambda\Delta - \int^\Delta c^u} \varphi(X(\Delta)) \\ &\leqslant \sup_{d\in\mathfrak{A}_N, r\in\mathscr{R}_N} E_x^{d(0)} \int_0^\Delta e^{-r(0)s - \int^s c^{d(0)}} \big(f^{d(0)}(X(s)) + r(0) g(X(s))\big)\,ds + \\ &\qquad + e^{-r(0)\Delta - \int^\Delta c^{d(0)}} \varphi(X(\Delta)) = V_{N,1}\varphi(x). \end{aligned}$$

Conversely, for any $d \in \mathfrak{A}_N$ and $r \in \mathscr{R}_N$, we get

$$E_x^d \left[\int_0^\Delta e^{-r(0)s - \int^s c^{d(0)}} \big(f^{d(0)}(X(s)) + r(0) g(X(s))\big)\,ds + e^{-r(0)\Delta - \int^\Delta c^{d(0)}} \varphi(X(\Delta))\right]$$

$$\leqslant \sup_{u\lambda} E_x^u \left[\int_0^\Delta e^{-\lambda s - \int^s c^u} \big(f^u(X(s)) + \lambda g(X(s))\big)\,ds + e^{-\lambda\Delta - \int^\Delta c^u} \varphi(X(\Delta))\right] = J\varphi(x).$$

Hence $V_{N,1}\varphi(x) \leqslant J\varphi(x)$. So (6.42) holds for $k = 1$. Suppose that (6.42) holds for k. Thus $V_{N,k}\varphi$ is in C.

$$V^{d,r}_{(k+1)\Delta}\varphi(x) = E^d_x\Big[\int_0^{\Delta} e^{-r(0)s-\int_0^s c^{d(0)}}\big(f^{d(0)}(X(s))+r(0)g(X(s))\big)\,ds +$$
$$+e^{-r(0)\Delta-\int_0^{\Delta} c^{d(0)}} E\Big[\int_{\Delta}^{(k+1)\Delta} e^{-\int_\Delta^s r+c}\big(f^{d(s)}(X(s))+r(s)g(X(s))\big)\,ds +$$
$$+e^{-\int_\Delta^{(k+1)\Delta} r+c}\varphi(X(k+1)\Delta)/F_\Delta\Big]\Big].$$

On the regular conditional probability space the conditional expectation of the right side turns into $V^{\tilde d,\tilde r}_{k\Delta}\varphi(X(\Delta))$ with some $\tilde d\in\mathfrak{A}_N$ and $\tilde r\in\mathscr{R}_N$. So we have

$$V^{d,r}_{(k+1)\Delta}\varphi(x) \leqslant E^d_x\Big[\int_0^{\Delta} e^{-r(0)s-\int_0^s c^{d(0)}}\big(f^{d(0)}(X(s))+r(0)g(X(s))\big)\,ds +$$
$$+e^{-r(0)\Delta-\int_0^{\Delta} c^{d(0)}} V_{N,k}\varphi(X(\Delta))\Big] \leqslant V_{N,1}(V_{N,k}\varphi)(x) = J(J^k\varphi)(x) = J^{k+1}\varphi(x);$$

namely,

$$V_{N,k+1}\varphi(x)\leqslant J^{k+1}\varphi(x). \tag{6.43}$$

Let Γ_l be a sequence of compact sets which increases to Γ as $l\to\infty$. Replace $\Gamma\times\times[0,\infty)$ by $\Gamma_l\times[0,l]$; we define $J_l(N)$ in the same way. Put $F(x,u,\lambda) = T^{u\lambda g}_\Delta\psi(x)$, $\psi\in C$. Since $F(x,u,\lambda)$ is continuous in (x,u,λ) and $J_l\psi(x)\in F(x,\Gamma_l\times[0,l])$, the implicit function theorem gives us the existence of a Borel function m, $\boldsymbol{R}^n\to\Gamma_l\times\times[0,l]$ such that

$$F(x,m(x)) = J_l\psi(x).$$

Hence $F(x,m(x))$ is in C. So we can choose a $\Gamma_l\times[0,l]$-valued Borel functions m_i, $i=1,\ldots,k$, such that

$$T^{m_k(x),g}_\Delta\varphi(x) = J_l\varphi(x),\ T^{m_{k-1}(x),g}_\Delta(J_l\varphi)(x) = J^2\varphi(x),\ldots$$
$$\ldots,\ T^{m_0(x),g}_\Delta(J^k_l\varphi)(x) = J^{k+1}_l\varphi(x).$$

Define $d\in\mathfrak{A}_N$ and $r\in\mathscr{R}_N$ by

$$(d(t),r(t)) = m_j(X(j\Delta)),\quad j\Delta\leqslant t<(j+1)\Delta,\ j=0,\ldots,k.$$

Then

$$V_{N,k+1}\varphi(x)\geqslant E^d_x\Big[\int_0^{(k+1)\Delta} e^{-\int_0^t r+c}\big(f^{d(t)}(X(t))+r(t)g(X(t))\big)\,dt + \tag{6.44}$$
$$+e^{-\int_0^{(k+1)\Delta} r+c}\varphi(X(k+1)\Delta)\Big] = T^{m_0 g}_\Delta\ldots T^{m_k g}_\Delta\varphi(x) = J^{k+1}_l\varphi(x),\qquad\forall l.$$

By routine we can show

$$0_l\text{-}\lim_l J^k_l\varphi = J^k\varphi. \tag{6.45}$$

Using (6.44) and (6.45), we can show the converse inequality of (6.43). This completes the proof of (6.42).

Since $\lim_{k\nearrow\infty} J^k\varphi(x) = v^{(N)}_\varphi(x)$, we see that

$$\lim_{k\nearrow\infty} V_{N,k}\varphi(x) = v^{(N)}_\varphi(x). \tag{6.46}$$

When T tends to ∞, $\int_0^T e^{-\int_0^t r+c}\big(f^{d(t)}(X(t))+r(t)g(X(t))\big)\,dt+e^{-\int_0^T r+c}\varphi(X(T))$ converges to $I(d,r)$ at Q^d_x-a.e. by virtue of (6.4), and is dominated by $\int_0^\infty e^{-\int_0^t c}\big|f^{d(t)}(X(t))\big|\,dt+ +\|g\|$. Hence we have for $d\in\mathfrak{A}_N$ and $r\in\mathscr{R}_N$

$$E^d_x I(d,r) = \lim_{k\nearrow\infty} V^{d,r}_{k2^{-N}}\varphi(x)\leqslant\lim_{k\nearrow\infty}V_{N,k}\varphi(x) = v^{(N)}_\varphi(x). \tag{6.47}$$

Taking the supremum w.r. to d and r, we get

$$V_N(x)\leqslant v^{(N)}_\varphi(x). \tag{6.48}$$

Now we shall show the converse of (6.48). For any $r\in\mathscr{R}_N$ and positive $T = k2^{-N}$, we define r^T by

$$r^T(t) = \begin{cases} r(t) & \text{for}\quad t<T,\\ 0 & \text{for}\quad t\geqslant T.\end{cases}$$

Then

$$\int_0^T e^{-\int_0^t r+c}\big(f^{d(t)}(X(t))+r(t)g(X(t))\big)\,dt+e^{-\int_0^T r+c}\varphi(X(T)) \tag{6.49}$$
$$= I(d,r^T)-e^{-\int_0^T r}\int_T^\infty e^{-\int_0^t c}f^{d(t)}(X(t))\,dt+e^{-\int_0^T r+c}\varphi(X(T)).$$

Recalling (6.4) we see that

$$E^d_x|\text{2nd term}|\leqslant e^{-cT}\|f\|,\qquad E^d_x|\text{3rd term}|\leqslant e^{-cT}\|\varphi\|, \tag{6.50}$$

where $\|f\| = \sup_{ux}|f^u(x)|$. Therefore, from (6.49) and (6.50) we have

$$V^{d,r}_T\varphi(x)\leqslant E^d_x I(d,r^T)+e^{-cT}(\|f\|+\|\varphi\|) \tag{6.51}$$
$$\leqslant V_N(x)+e^{-cT}(\|f\|+\|\varphi\|)\quad\text{for}\quad d\in\mathfrak{A}_N,\ r\in\mathscr{R}_N.$$

Hence

$$V_{N,k}\varphi(x)\leqslant V_N(x)+e^{-cT}(\|f\|+\|\varphi\|).$$

As k tends to ∞, we have

$$\lim_{k\to\infty}V_{N,k}\varphi(x)\leqslant V_N(x), \tag{6.52}$$

namely the converse to (6.48). This completes the proof of (6.41). As N tends to ∞ in (6.41), we get

$$V(x) = v_\varphi(x) = v(x)$$

recalling the Remark in § 5. Thus V of (6.6) is a stochastic construction of the least S_t-excessive majorant of g.

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О МОМЕНТАХ ОСТАНОВКИ ПОЛУУСТОЙЧИВЫХ ДИФФУЗИОННЫХ ПРОЦЕССОВ

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Рассматриваются некоторые классы моментов остановки полуустойчивых диффузионных процессов. Получены точные и асимптотические результаты, характеризующие распределения рассматриваемых моментов остановки.

I. Введение

Одномерный марковский процесс $(X_t, \mathscr{F}_t, P_x)$ со значениями на полупрямой $\boldsymbol{R}^+$ называется *полуустойчивым*, если его переходная функция $P(t, x, B)$ ($B \in \mathscr{B}^+$ — борелевская σ-алгебра на $\boldsymbol{R}^+$) обладает автомодельным свойством, т.е.

$$P(rt, X, B) = P(t, r^{-\alpha}x, r^{-\alpha}B)$$

при всех $r > 0$ и некотором показателе $\alpha > 0$. Полуустойчивые марковские процессы на $\boldsymbol{R}^+$ ввел и описал Ламперти в работе [5], в которой показано, что все невырожденные полуустойчивые диффузионные процессы порождаются оператором

$$L_x f(x) = bx^{1-1/\alpha}f'(x)+dx^{2-1/\alpha}f''(x),$$

где $d > 0$ и b — параметры. Мы будем далее обозначать $\varrho = \frac{\alpha b}{d}+1-\alpha$. Из результатов Ламперти следует, что граница $x = \infty$ при всех значениях параметров является естественной, а граница $x = 0$ является естественной только при $\varrho \geqslant 1$; при $\varrho \geqslant 0$ является захватывающей и при $0 < \varrho < 1$ является регулярной, причем в силу предположения о непрерывности траекторий на границе $x = 0$ может быть либо поглощение либо отражение.

В этой заметке рассматриваются следующие классы моментов остановки:

$$\tau_a = \inf\{t \geqslant 0: X_t \leqslant a(t+y)^\alpha\},$$

$$\sigma_c = \inf\{t \geqslant 0: X_t \geqslant c(t+y)^\alpha \text{ или } X_t = 0\},$$

$$\sigma_c = \inf\{t \geqslant 0: X_t \geqslant c(t+y)^\alpha\},$$