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ON TRAJECTORIES OF GAUSSIAN MARKOV RANDOM FIELDS

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A class of generalized Gaussian random fields $X = X(\varphi)$, $\varphi \in C_0^\infty(T)$, $T \subset \mathbf{R}^d$, is considered. It is characterized by the condition that the scalar product $\langle f, g\rangle$ in the reproducing kernel Hilbert space $\mathscr{H}(X) = \overline{C_0^\infty(T)}_{\langle\cdot,\cdot\rangle}$ is given by a pth order Dirichlet form

$$\langle f, g\rangle = \sum_{|\alpha|, |\beta| \leq p} \int_T a_{\alpha\beta}(t) D^\alpha f \overline{D^\beta g}\, dt$$

such that the norm $\langle f, f\rangle^{1/2}$ is equivalent to the Sobolev norm $\|f\|_p$,

$$\|f\|_p^2 = \sum_{|\alpha| \leq p} \int_T |D^\alpha f|^2 dt.$$

For such X, the existence of the canonical white noise W_x is established and under some additional assumptions stochastic equation for trajectories

$$P^{1/2} X = W_x$$

is derived, where P is a self-adjoint extension of

$$P_0 = \sum_{|\alpha|, |\beta| \leq p} (-1)^{|\beta|} D^\beta (a_{\alpha\beta}(t) D^\alpha)$$

with $D(P_0) = C_0^\infty(T)$ (similar results for Lévy Brownian motion were obtained by Z. Ciesielski in [5], [6]). We prove also that trajectories of such Gaussian random fields are α times continuously differentiable for every α with $|\alpha| < p - d/2$.

0. Introduction

In this paper sample path properties of a class of Gaussian random fields (the so-called *p-th order Markov fields*) are considered with an attempt to derive a stochastic (pseudodifferential) equation for trajectories with a white noise on the right hand side, and in this context to investigate the smoothness properties of trajectories. The work was motivated by the recent papers of Z. Ciesielski [5], [6] on the Lévy Brownian motion $X = X(t)$, $t \in \mathbf{R}^d$, where he proved the existence of the canonical white noise W_x for X and derived a stochastic equation for trajectories

$$\sigma_d D^{d+1/4} X = W_x,$$

σ_d being a constant and D the negative Laplace operator in R^d. The reader will find many ideas of Z. Ciesielski in this paper, and I am greatly indebted to him for the generous help which I received from him while working on the problem.

We shall deal with the class, denoted by $\mathscr{F}_p$, of (generalized) Gaussian random fields (GRF's) $X = X(\varphi)$, $\varphi \in C_0^\infty(T) \equiv \mathscr{D}(T)$, where T is a smooth open domain in R^d. This class is characterized by the condition that the reproducing kernel Hilbert space (RKHS) $\mathscr{H}(X)$ corresponding to a GRF $X \in \mathscr{F}_p$ is obtained by the completion of $\mathscr{D}(T)$ in the norm

$$\langle f,f\rangle^{1/2} = \Big(\sum_{|\alpha|,|\beta|\leqslant p} \int_T a_{\alpha\beta}(t) D^\alpha f \overline{D^\beta f}\, dt\Big)^{1/2}$$

and the norm $\langle f,f\rangle^{1/2}$ is supposed to be equivalent to the Sobolev norm $\|f\|_p \equiv (f,f)_p^{1/2}$, where

$$(f,f)_p = \sum_{|\alpha|\leqslant p} \int_T |D^\alpha f|^2 dt.$$

This assumption (denoted by (A) below) is rather restrictive and not satisfied for example by the Lévy Brownian motion; however, it enables us to apply without difficulty some of the well-known functional analytic machinery which is needed to prove the above-mentioned facts (it might be worth recalling too that condition (A) was used in [17] to prove the Markov property of X).

According to [5], [6] we call the canonical white noise corresponding to a given GRF X the unitary mapping W_x: $L^2(T) \to H(X)$ satisfying

$$X(\varphi) = W_x(P^{-1/2}\varphi), \quad \varphi \in \mathscr{D}(T), \tag{0.1}$$

where $H(X)$ is the closed subspace in $L^2(\Omega, F, P)$ spanned by $\{X(\varphi), \varphi \in \mathscr{D}(T)\}$, and P is a self-adjoint extension of the operator

$$P_0 = \sum_{|\alpha|,|\beta|\leqslant p} (-1)^{|\beta|} D^\beta\big(a_{\alpha\beta}(t) D^\alpha\big). \tag{0.2}$$

Theorem 4.2 (cf. Theorem 3.1, [5]) states that, for every $X \in \mathscr{F}_p$, the canonical white noise W_x exists. It should be noted to avoid confusion that the word canonical in this context simply means that the white noise which is defined by means of a given O.N. basis in $L^2(T)$ does not, in fact, depend on the basis, and so the white noise representation (0.1) is different from the canonical white noise representation of a Gaussian process in the Lévy–Hida sense ([12], [10]).

In Section 5 we obtain a stochastic equation for trajectories of a GRF $X \in \mathscr{F}_p$ of the form

$$P^{1/2}X = W_x, \tag{0.3}$$

where $P^{1/2}$ is the positive (self-adjoint) square root of P. Equation (0.3) should be understood in the distribution sense. However, to prove that the left-hand side of (0.3) is a random distribution (i.e. that $X(P^{1/2}\varphi)$ is continuous in $\mathscr{D}(T)$ a.s.) we had to impose additional restrictions on X (see Theorem 5.1). It is clear that the form of equation (0.3) (more exactly, the operator on the left-hand side) is due to the factorization of P in the form $P = P^{1/2}P^{1/2}$, while another factorization of P: $P = L^+L^-$ ($L^\pm = (L^\mp)^*$) may result in a simpler (say, differential) equation for trajectories of X (see the Example in Section 5).

Finally in Section 6 we prove a conjecture of Z. Ciesielski about the smoothness properties of trajectories of a GRF $X \in \mathscr{F}_p$, namely that all the derivatives $D^\alpha X$ up to order $p-d/2-1$ if d is even and $p-d/2-1/2$ if it is odd exist in the usual sense and are continuous a.s. (Theorem 6.1).

1. Preliminaries

In this section we introduce the definitions and notation and recall some properties of the self-adjoint extension of the operator P_0 which will be used later.

Let T be an open smooth domain in R^d, $d \geqslant 1$, let $a_{\alpha\beta} \in C^\infty(T) \cap C(\overline{T})$, $|\alpha|$, $|\beta| \leqslant p$ be given (complex-valued) functions, and let $\langle f, g\rangle$ be a symmetric positive Dirichlet form of order p on $\mathscr{D}(T)\times\mathscr{D}(T)$, $\mathscr{D}(T) \equiv C_0^\infty(T)$:

$$\langle f, g\rangle = \sum_{|\alpha|,|\beta|\leqslant p} \int_T a_{\alpha\beta}(t) D^\alpha f \overline{D^\beta g}\, dt = \overline{\langle g, f\rangle}, \tag{1.1}$$

$f, g \in \mathscr{D}(T)$. We assume that the following condition is satisfied, even if it is not explicitly mentioned below:

(A) *the norm* $\langle f,f\rangle^{1/2}$ *is equivalent to the Sobolev norm* $\|f\|_p = (f,f)_p^{1/2}$, *where* $(f,f)_p = \sum_{|\alpha|\leqslant p} \int_T |D^\alpha f|^2 dt$.

Clearly (A) implies that the form $\langle f, g\rangle$ is positive definite, and so the operator $P_0 = \sum_{|\alpha|,|\beta|\leqslant p} (-1)^{|\beta|} D^\beta(a_{\alpha\beta}(t)D^\alpha)$ with $D(P_0) = \mathscr{D}(T)$ is symmetric and positive definite, $D(P_0)$ is dense in $L^2(T)$, and thus by Friedrichs' theorem there exists a self-adjoint extension P of P_0, the so-called *Friedrichs' extension*, with the following properties:

P.1. $R(P) = L^2(T)$ and $D(P) = \mathscr{H}(T) \cap D(P_0^*)$, where $\mathscr{H}(T)$ is the Hilbert space completion of $\mathscr{D}(T)$ in the norm $\langle f,f\rangle^{1/2}$;

P.2. P^{-1} exists and is a continuous mapping $L^2(T) \to L^2(T)$;

P.3. If $P^{1/2}$ is the positive (self-adjoint) square root of P, then $D(P^{1/2}) = \mathscr{H}(T)$;

P.4. $P^{1/2}(D(P)) \subset \mathscr{H}(T)$;

P.5. $P^{1/2}$ is isometry $\mathscr{H}(T) \to L^2(T)$.

For P.1, see [13], p. 110 or [8], p. 1240. For P.2 and P.3, see [18], p. 620–621. P.4 is obvious from P.3 and the definition of $P^{1/2}$. P.5 follows from [1] $(P^{1/2}f, P^{1/2}f) = \langle f,f\rangle, f \in \mathscr{H}(T)$ ([18], p. 621), i.e. $P^{1/2}$ is a unitary mapping $\mathscr{H}(T) \to L^2(T)$, and the fact that $R(P) = L^2(T)$ implies $R(P^{1/2}) = L^2(T)$.

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[1] $(f,f) = \int_T |f|^2 dt$, $\|f\| = (f,f)^{1/2}$.

Remark 1.1. As follows from Garding's inequality ([1], p. 78), condition (A) is implied by the uniform strong ellipticity and positivity of the Dirichlet form $\langle f, g\rangle$, i.e. by the following two conditions:

(E) there exists a constant $C > 0$ such that for all $x \in \boldsymbol{R}^d$, $t \in T$

$$\Big|\operatorname{Re} \sum_{|\alpha|,|\beta|=p} a_{\alpha\beta}(t) x^\alpha x^\beta\Big| \geqslant C|x|^{2p};$$

(P) there exists a constant $C > 0$ such that for all $f \in \mathscr{D}(T)$,

$$\langle f, f\rangle \geqslant C(f, f).$$

2. The class $\mathscr{F}_p$ of GRF's

We denote by $\mathscr{D}'(T)$ the dual space of $\mathscr{D}(T)$ and by $\mathscr{D}^*(T)$ the space of all continuous antilinear functionals on $\mathscr{D}(T)$. Hence if $F(\varphi) \in \mathscr{D}'(T)$ $(\mathscr{D}^*(T))$ then $F^*(\varphi) \equiv F(\overline{\varphi}) \in \mathscr{D}^*(T)$ ($\mathscr{D}'(T)$, respectively). We embedd $\mathscr{H}(T)$ in $\mathscr{D}^*(T)$ by setting $f^*(\varphi) = f(\overline{\varphi}) = (f, \varphi)$ for $f \in \mathscr{H}(T)$. Now, as $f^*(\varphi)$ is a continuous mapping $\mathscr{H}(T) \to C^1$ for every $\varphi \in \mathscr{D}(T)$, by the well-known theorem of Aronszajn ([2], p. 343) $\mathscr{H}(T)$ is an RKHS and there exists a kernel $R(\varphi, \psi)$ on $\mathscr{D}(T)\times\mathscr{D}(T)$ such that

R.1. $R(\varphi, \cdot) \in \mathscr{H}(T)$ for each $\varphi \in \mathscr{D}(T)$;

R.2. $\{R(\varphi, \cdot), \varphi \in \mathscr{D}(T)\}$ span $\mathscr{H}(T)$;

R.3 (the *reproducing property*). $\langle f, R(\varphi, \cdot)\rangle = f^*(\varphi)$ for every $f \in \mathscr{H}(T)$ and $\varphi \in \mathscr{D}(T)$.

As $R(\varphi, \psi)$ is positive definite, there exists a (generalized) GRF $X = X(\varphi)$, $\varphi \in \mathscr{D}(T)$ on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$E[X(\varphi)\overline{X(\psi)}] = R(\varphi, \psi), \quad \varphi, \psi \in \mathscr{D}(T).$$

Clearly $\mathscr{H}(T)$ is an RKHS corresponding to X, and we shall occasionally write $\mathscr{H}(X)$ instead of $\mathscr{H}(T)$. We denote by $\mathscr{F}_p$ the set of all GRF's $X = X(\varphi)$, $\varphi \in \mathscr{D}(T)$ for which the RKHS $\mathscr{H}(X)$ is obtained by completion of $\mathscr{D}(T)$ in the scalar product $\langle f, g\rangle$ given by (1.1) and satisfying condition (A).

Remark 2.1. It is known ([17], Theorem 5.2) that if $p-d/2 > 0$ then $X \in \mathscr{F}_p$ is a Markov (and continuous a.s.) random field. A slight modification of the argument in [17] enables us to prove the Markov property for all (generalized) GRF $X \in \mathscr{F}_p$ (see also [14], [11]). The following statement is technical but it will often be used later:

PROPOSITION 2.1. $R(\varphi, \psi) = (P^{-1}\varphi, \psi)$, $\varphi, \psi \in \mathscr{D}(T)$.

Proof. Integration by parts gives

$$\langle f, R(\psi, \cdot)\rangle = \sum_{|\alpha|,|\beta|\leqslant p} \int_T a_{\alpha\beta}(t) D^\alpha f(t) \overline{D^\beta R(\psi, t)}\, dt = (Pf, R(\psi, \cdot)), \quad f, \psi \in \mathscr{D}(T).$$

Hence, by the reproducing property,

$$(2.1) \qquad (Pf, R(\psi, \cdot)) = (f, \psi)$$

for all $f, \psi \in \mathscr{D}(T)$. Observe that this equality holds for all $f \in D(P) \subset \mathscr{H}(T)$ as well. In fact,

$$(2.2) \qquad (f, \psi) = (P^{1/2}f, P^{1/2}R(\psi, \cdot))$$

as $R(\psi, \cdot) \in \mathscr{H}(T) = D(P^{1/2})$. Now if $\{f_n\} \subset \mathscr{D}(T)$ is a sequence convergent in $\mathscr{H}(T)$ to $f \in \mathscr{H}(T)$, then $f_n \to f$ in $L^2(T)$ and $P^{1/2}f_n \to P^{1/2}f$ in $L^2(T)$ by P.5. Hence (2.2) is true for all $f \in \mathscr{H}(T)$. If $f \in D(P)$, then $P^{1/2}f \in \mathscr{H}(T) = D(P^{1/2})$ by P.4 and so (2.1) holds for all $f \in D(P)$. But $R(P) = L^2(T) \supset \mathscr{D}(T)$ by P.1, so that, for every $\varphi \in \mathscr{D}(T)$, $f = P^{-1}\varphi$ belongs to $D(P)$, and substituting $f = P^{-1}\varphi$ into (2.1) we complete the proof.

3. The dual space of $\mathscr{H}(X)$

We define $\mathscr{H}_-(X)$ to be the completion of $\mathscr{D}(T)$ in the norm $\langle f, f\rangle_-^{1/2} = (R(f, f))^{1/2}$ and $H(X)$ to be the completion of $\{X(\varphi), \varphi \in \mathscr{D}(T)\}$ in $L^2(\Omega, \mathscr{F}, P)$. We also write $\mathscr{H}_+(X)$ and $\langle f, f\rangle_+$ for $\mathscr{H}(X)$ and $\langle f, f\rangle$, respectively. It is easy to see that the spaces $H(X)$, $\mathscr{H}_-(X)$ and $\mathscr{H}_+(X)$ are isometric while the isometry J: $H(X) \to \mathscr{H}_+(X)$ is given by $J[X(\varphi)] = R(\varphi, \cdot)$, $\varphi \in \mathscr{D}(T)$, and the isometry R: $\mathscr{H}_-(X) \to \mathscr{H}_+(X)$ is given by $R[\varphi] = R(\varphi, \cdot)$. It follows from Proposition 2.1 that $R = P^{-1}$ on $\mathscr{D}(T)$.

PROPOSITION 3.1. *$\mathscr{H}_+(X)$ and $\mathscr{H}_-(X)$ are dual as Banach spaces and the canonical pairing between them on $\mathscr{D}(T)\times\mathscr{D}(T)$ is given bythe sca lar product in $L^2(T)$.*

Proof. As $\mathscr{D}(T)$ is dense both in $\mathscr{H}_+(X)$ and $\mathscr{H}_-(X)$, it is enough to check that

$$(3.1) \qquad (f, g)^2 \leqslant \langle f, f\rangle_+ \langle g, g\rangle_-, \quad f, g \in \mathscr{D}(T),$$

and

$$(3.2) \qquad \langle f, f\rangle_\pm = \sup_{\substack{g \in \mathscr{D}(T)\\ \langle g, g\rangle_\mp \leqslant 1}} |(f, g)|^2, \quad f \in \mathscr{D}(T).$$

But (3.1) follows easily from the reproducing property:

$$|(f, g)|^2 = |\langle f, R(g, \cdot)\rangle_+|^2 \leqslant \langle f, f\rangle_+ \langle R(g, \cdot), R(g, \cdot)\rangle_+$$
$$= \langle f, f\rangle_+ R(g, g) = \langle f, f\rangle_+ \langle g, g\rangle_-.$$

The first part of (3.2) is also obvious:

$$\sup_{I_+} |(f, g)|^2 = \sup_{I_+} |\langle g, R(f, \cdot)\rangle_+|^2 = \langle R(f, \cdot), R(f, \cdot)\rangle_+$$
$$= R(f, f) = \langle f, f\rangle_-,$$

where $I_\pm = \{g \in \mathscr{D}(T): \langle g, g\rangle_\pm \leqslant 1\}$. To prove the second part, recall (2.1) and Proposition 2.1:

$$\sup_{I_-} |(f, g)|^2 = \sup_{I_-} |(Pf, R(g, \cdot))|^2 = \sup_{I_-} |R(Pf, g)|^2$$
$$= \sup_{I_-} |\langle Pf, g\rangle_-|^2 = \langle Pf, Pf\rangle_- = R(Pf, Pf)$$
$$= (f, Pf) = \langle f, f\rangle_+.$$

Remark 3.1. We shall retain the same notation (f, g) for the canonical pairing between $\mathscr{H}_+(X)$ and $\mathscr{H}_-(X)$, i.e. (f, g) will denote the unique bicontinuous sesquilinear form on $\mathscr{H}_+(X)\times\mathscr{H}_-(X)$ such that

$$(f, g) = \int_T f(t)\overline{g(t)}\,dt \quad \text{on} \quad \mathscr{D}(T)\times\mathscr{D}(T).$$

Remark 3.2. Let $H_p(T)$ be the completion of $\mathscr{D}(T)$ in the norm $||f||_p$, and let $H_{-p}(T)$ be the corresponding dual space with $||f||_{-p} = \sup\limits_{\substack{g\in\mathscr{D}(T)\\ ||g||_p\leqslant 1}} |(f, g)|$ (see e.g. [19], p. 142–145).

Now, as the norms $||f||_p$ and $\langle f,f\rangle_+^{1/2}$ are equivalent, the negative norms $||f||_{-p}$ and $\langle f,f\rangle_-^{1/2}$ are equivalent too, and so we can (and will) identify algebraically the spaces $H_p(T)$ and $\mathscr{H}_+(T)$ as well as $H_{-p}(T)$ and $\mathscr{H}_-(T)$, respectively.

4. The canonical white noise

In this section we generalize the concept of the canonical white noise W_x associated with a GRF X, which was introduced in [5], [6] for the Lévy Brownian motion. Let $\{Y_j, j = 1, 2, \ldots\}$ and $\{f_j, j = 1, 2, \ldots\}$ be the O.N. basis in $H(X)$ and $L^2(T)$, respectively. Define

$$Y(\varphi) = \sum_j Y_j\,(\varphi, P^{-1/2}f_j), \qquad \varphi\in\mathscr{D}(T).$$

The series are convergent a.s. as

$$\sum_{j\leqslant N} |(\varphi, P^{-1/2}f_j)|^2 = \sum_{j\leqslant N} |(P^{-1/2}\varphi, f_j)|^2 \leqslant ||P^{-1/2}\varphi||^2 < \infty.$$

It is easy to see that $Y = Y(\varphi)$, $\varphi\in\mathscr{D}(T)$ is a (generalized) GRF with mean zero and the same covariance as that of X:

$$E[Y(\varphi)\overline{Y(\psi)}] = \sum_j (\varphi, P^{-1/2}f_j)\,\overline{(\psi, P^{-1/2}f_j)} = \sum_j (P^{-1/2}\varphi, f_j)\overline{(P^{-1/2}\psi, f_j)}$$
$$= (P^{-1/2}\varphi, P^{-1/2}\psi) = (\varphi, P^{-1}\psi).$$

Clearly $H(Y)\subset H(X)$. In fact, they coincide.

PROPOSITION 4.1. $H(Y) = H(X)$.

Proof. It suffices to show that the set $\mathscr{Z} = \{z = (z_1, z_2, \ldots), z_j = (P^{-1/2}\varphi, f_j), \varphi\in\mathscr{D}(T)\}$ is dense in $l^2 = \left\{x = (x_1, x_2 \ldots), x_j\in C^1, \sum_j |x_j|^2 < \infty\right\}$. But $\mathscr{D}(T)$ is dense in $L^2(T) \Rightarrow P^{-1/2}\mathscr{D}(T)$ is dense in $\mathscr{H}(T)$ (as $P^{-1/2}$ is an isometry $L^2(T)\to\mathscr{H}(T)$) $\Rightarrow P^{-1/2}\mathscr{D}(T)$ is dense in $L^2(T)\Rightarrow\mathscr{Z}$ is dense in l^2.

THEOREM 4.1 (cf. Theorem 4.1 [6]). *Let $\{f_j, j = 1, 2, \ldots\}$ be a given* O.N. *basis in $L^2(T)$. Then, given the* GRF *$X\in\mathscr{F}_p$, there exists a unique* O.N. *basis $\{X_j, j = 1, 2, \ldots\}$ in $H(X)$ such that*

$$X(\varphi) = \sum_j X_j(\varphi, P^{-1/2}f_j), \qquad \varphi\in\mathscr{D}(T). \tag{4.1}$$

The proof, which utilizes Proposition 4.1 and the fact that $P^{-1/2}$ is an isometry $L^2(T)\to\mathscr{H}(T)$, is quite the same as that of Theorem 4.1 [6].

According to [5], [6] we call the canonical white noise corresponding to a given GRF $X = X(\varphi)$, $\varphi\in\mathscr{D}(T)$ the unitary mapping W_x: $L^2(T)\to H(X)$ defined by the formula

$$W_x(f) = \sum_j X_j(f, f_j),$$

where $\{X_j, j = 1, 2, \ldots\}$ and $\{f_j, j = 1, 2, \ldots\}$ are given as in Theorem 4.1.

THEOREM 4.2. *For a given* GRF *$X = X(\varphi)$, $\varphi\in\mathscr{D}(T)$, $X\in\mathscr{F}_p$ the canonica white noise W_x exists, is independent of the choice of the* O.N. *basis $\{f_j, j = 1, 2, \ldots\}$ in $L^2(T)$ and*

$$X(\varphi) = W_x(P^{-1/2}\varphi). \tag{4.2}$$

Proof. The only thing to prove is that W_x does not depend on the choice of the basis in $L^2(T)$. Let $\{f'_j, j = 1, 2, \ldots\}$ be another O.N. basis in $L^2(T)$. According to Theorem 4.1 there exists an O.N. basis $\{X'_j, j = 1, 2, \ldots\}$ in $H(X)$ such that

$$\sum_j X_j(P^{-1/2}\varphi, f_j) = \sum_j X'_j(P^{-1/2}\varphi, f'_j) = X(\varphi)$$

for every $\varphi\in\mathscr{D}(T)$. We want to prove that

$$W_x(f) = \sum_j X_j(f, f_j) = \sum_j X'_j(f, f'_j) \equiv W'_x(f), \qquad f\in L^2(T).$$

But as W_x and W'_x are unitary maps $L^2(T)\to H(X)$, this follows from the fact that $P^{-1/2}\mathscr{D}(T)$ is dense in $L^2(T)$.

5. Stochastic equation for trajectories

In [5], a stochastic (pseudodifferential) equation for trajectories of the Lévy Brownian motion $X = X(t)$, $t\in \boldsymbol{R}^d$ was obtained, of the following form:

$$\sigma_d D^{(d+1)/4}X = W,$$

where W is a white noise, D the negative Laplace operator, σ_d a constant, and $D^{(d+1)/4}X(\varphi) \overset{\text{df}}{=} X(D^{(d+1)/4}\varphi)$, $\varphi\in\mathscr{D}(\boldsymbol{R}^d)$. We are going to derive a similar equation for (generalized) GRF $X = X(\varphi)$, $\varphi\in\mathscr{D}(T)$, with $D^{(d+1)/4}$ replaced by $P^{1/2}$. Unfortunately we could not obtain it under condition (A), under which all the previous statements were proved, and some additional assumptions seem to be necessary.

The spaces $\mathscr{H}_\pm(X)$ and $H(X)$ are isometric and the isometry $J^{-1}R$: $\mathscr{H}_-(X)\to H(X)$ is given on $\mathscr{D}(T)$ by $J^{-1}R[\varphi] = X(\varphi)$ (see Section 3). We denote $\overline{X}(f) \overset{\text{df}}{=} J^{-1}R[f]$ for all $f\in\mathscr{H}_-(X)$. Clearly each element of $H(X)$ can be written in the form $\overline{X}(f)$ for a certain $f\in\mathscr{H}_-(X)$.

PROPOSITION 5.1. $\overline{X}(P^{1/2}\varphi) = W_x(\varphi)$, $\varphi \in \mathscr{D}(T)$.

Proof. From Theorem 4.2 we have $X(f) = \overline{X}(f) = W_x(P^{-1/2}f)$ for every $f \in \mathscr{D}(T)$. As $P^{1/2}\varphi \in L^2(T) \subset \mathscr{H}_-(T)$ for $\varphi \in \mathscr{D}(T)$, there exists a sequence $\{f_n\} \subset \mathscr{D}(T)$ such that $f_n \to P^{1/2}\varphi$ in $L^2(T)$ and hence in $\mathscr{H}_-(T)$, i.e. $\overline{X}(f_n) \to \overline{X}(P^{1/2}\varphi)$. On the other hand, $P^{-1/2}f_n \to P^{-1/2}P^{1/2}\varphi = \varphi$ in $L^2(T)$ as $P^{-1/2}$ is a continuous map $L^2(T) \to L^2(T)$; hence $W_x(P^{-1/2}f_n) \to W_x(\varphi)$, which proves the proposition.

As follows from a theorem of Minlos ([9], p. 413), the trajectories of $X \in \mathscr{F}_p$ are random distributions a.s., i.e. the mapping $\varphi \to X(\varphi)(\omega)$ is continuous in $\mathscr{D}(T)$ on an ω-set of probability 1. Now, $P^{1/2}X$ is not well-defined for $X(\cdot) \in \mathscr{D}'(T)$ as $P^{1/2}$, in general, does not map $\mathscr{D}(T)$ into $\mathscr{D}(T)$, but it may be well-defined on some special subspaces of $\mathscr{D}'(T)$. This discussion leads us to the consideration of the following three cases where the operation $P^{1/2}$ on trajectories of a (generalized) GRF X is well-defined and continuous a.s. with respect to the convergence in $\mathscr{D}(T)$.

B.1. $X(\cdot) \in \mathscr{H}_-(T)$ a.s;

B.2. $P^{1/2}$ maps $\mathscr{D}(T)$ into $\mathscr{D}(T)$ continuously;

B.3. $T = \boldsymbol{R}^d$ and $P^{1/2}$ maps $\mathscr{D}(\boldsymbol{R}^d)$ into $\mathscr{S}(\boldsymbol{R}^d)$ continuously, where $\mathscr{S}(\boldsymbol{R}^d)$ is the Schwartz space of all rapidly decreasing C^∞-functions on $\boldsymbol{R}^d$.

We define $P^{1/2}X(\varphi) = (X, \overline{P^{1/2}\varphi})$, $\varphi \in \mathscr{D}(T)$ if B.1 holds, recalling that $P^{1/2}$ maps $\mathscr{D}(T)$ into $\mathscr{H}_+(T)$ continuously (this fact is an easy consequence of P.4 and P.5) and (f, g) is well-defined and bicontinuous on $\mathscr{H}_-(T)\times\mathscr{H}_+(T)$ (Section 3). We also set $P^{1/2}X(\varphi) = X(P^{1/2}\varphi)$ if B.2 or B.3 is true, as in the latter case, by another theorem of Minlos ([9], p. 395), $X(\cdot) \in \mathscr{S}'(\boldsymbol{R}^d)$ a.s.

THEOREM 5.1. *Let any of the conditions* B.1–B.3 *be satisfied. Then almost every trajectory of a* GRF $X \in \mathscr{F}_p$ *satisfies the following stochastic equation*:

$$P^{1/2}X = W_x. \tag{5.1}$$

Proof. From Proposition 5.1, it is enough to check that in any case (B.1–B.3) $P^{1/2}X(\varphi) = \overline{X}(P^{1/2}\varphi)$ a.s. for every $\varphi \in \mathscr{D}(T)$. By the definition of both sides this holds with $P^{1/2}\varphi$ replaced by $f \in \mathscr{D}(T)$ and a simple continuity argument (as in the proof of Proposition 5.1) concludes the proof.

The rest of this section is devoted to a more detailed discussion of the conditions B.1–B.3. Starting with B.2, we can only say that it is satisfied if $P^{1/2}$ restricted to $\mathscr{D}(T)$ $(P^{1/2}|\mathscr{D}(T))$ is a differential operator with C^∞-coefficients. In this case equation (5.1) is a stochastic differential equation and the simplest one in many ways. However, we do not know any fairly general conditions on the operator P_0 which guarantee that $P^{1/2}|\mathscr{D}(T)$ is a differential operator. In any case, if $P_0 = L^2$, where L is a differential operator, symmetric and positive with $D(L) = \mathscr{D}(T)$, then $P^{1/2}|\mathscr{D}(T) \neq L$ in general. In this context it should be noted that our Theorem 5.1 only very slightly overlaps a result of Dudley ([7], Theorem 4.4) on Gaussian fields satisfying certain differential equations with white noise on the right-hand side. As for B.1, the sufficient condition for it is given in the well-known Minlos–Sazonov theorem ([9], p. 422–423, see also [16]); namely, the natural embedding $\mathscr{H}_+(T) \to \mathscr{H}_-(T)$ must be Hilbert–Schmidt (H–S).

PROPOSITION 5.2. *Let $T \subset \boldsymbol{R}^d$ be a bounded open domain, and $p > d/4$. Then the natural embedding $\mathscr{H}_+(T) \to \mathscr{H}_-(T)$ is* H–S.

Finally, for B.3 we have

PROPOSITION 5.3. *Let P_0 be a (differential) operator with constant coefficients. Then $P^{1/2}$ maps $\mathscr{S}(\boldsymbol{R}^d)$ (and $\mathscr{D}(\boldsymbol{R}^d)$) into $\mathscr{S}(\boldsymbol{R}^d)$ continuously.*

Both statements (Propositions 5.2 and 5.3) may be known, but we could not find them in the literature and we give the proofs in the Appendix below (they are proved under assumption (A), of course).

We end up this section with an example of the so-called Ornstein–Uhlenbeck process $X = X(t)$, $t \in \boldsymbol{R}^1$, which illustrates our Theorem 5.1 as well as the remark (see Introduction) that the form of the stochastic equation for trajectories of a given GRF X depends on the factorization of the operator P.

EXAMPLE. Let $X = X(t)$, $-\infty < t < +\infty$ be a real-valued Gaussian process with mean zero and the covariance $R(t, s) = e^{-|t-s|}/2$. Then $\mathscr{H}_+(X) = H_1(\boldsymbol{R}^1)$ $= \{u = u(t),\ t \in \boldsymbol{R}^1:\ u$ absolutely continuous and $\langle u, u\rangle = \int_{-\infty}^{\infty} (|u|^2+|u'|^2)\,dt < \infty\}$. The operator P is given by $Pu = u-u''$ on $D(P) = H_2(\boldsymbol{R}^1) = \{u = u(t), t \in \boldsymbol{R}^1$: u, u' absolutely continuous and $\int_{-\infty}^{\infty} (|u|^2+|u'|^2+|u''|^2)\,dt < \infty\}$; $D(P^{1/2}) = H_1(\boldsymbol{R}^1)$ and $P^{1/2}u(t) = 1/\sqrt{2\pi}\int_{-\infty}^{\infty} e^{itx}(1+|x|^2)^{1/2}\hat{u}(x)\,dx$. By Theorem 5.1 and Proposition 5.3, the trajectories of X satisfy the pseudodifferential equation $P^{1/2}X = W$, while, on the other hand, they solve the differential equations $L^{\pm}X = W^{\pm}$, where $L^{\pm}u$ $= u\pm u'$, $D(L^{\pm}) = H_1(\boldsymbol{R}^1)$ and $W^{\pm}$ are white noises on $\boldsymbol{R}^1$. Note that L^+L^- $= L^-L^+ = P$.

6. Regularity of trajectories

In this section we prove a conjecture of Z. Ciesielski about the differentiability of trajectories of a GRF $X = X(\varphi)$, $\varphi \in \mathscr{D}(T)$, $T \subset \boldsymbol{R}^d$ of the class $\mathscr{F}_p$; namely,

THEOREM 6.1. *If $p > d/2$, then almost every trajectory of a* GRF $X \in \mathscr{F}_p$ *is $|\alpha|$ times continuously differentiable for every α such that $|\alpha| < p-d/2$ (i.e. $D^\alpha X(\cdot) \in C(T)$ a.s.).*

Remark 6.1. The continuity of trajectories of a GRF $X \in \mathscr{F}_p$ if $p > d/2$ was proved in [17].

Proof of Theorem 6.1. The idea of the proof goes back to [4], p. 633. From Theorem 4.1 we have

$$X(\varphi) = \sum_j X_j(\varphi, F_j), \tag{6.1}$$

where $\{X_j, j = 1, 2, \ldots\}$ is an O.N. basis in $H(X)$ and $\{F_j, j = 1, 2, \ldots\}$ is an O.N. basis in $\mathscr{H}(X)$ (we can let it be such that $F_j \in \mathscr{D}(T)$ for $j = 1, 2, \ldots$). If $|\alpha| < p-d/2$, define

$$X^\alpha(a) = \sum_j X_j \overline{D_a^\alpha F_j(a)}, \quad a \in T. \tag{6.2}$$

We prove the theorem if we show that

(a) the series (6.2) are convergent uniformly on every compact subset of T with probability 1;

(b) $(X^\alpha, \varphi) = (-1)^{|\alpha|}(X, D^\alpha\varphi)$ for all $\varphi \in \mathscr{D}(T)$ a.s.

Now, (b) easily follows from (a) and

$$(X^\alpha, \varphi) = \lim_{N\to\infty} \sum_{j=1}^N X_j(\overline{D^\alpha F_j}, \varphi)$$
$$= \lim_{N\to\infty} \sum_{j=1}^N X_j(\overline{F}_j, D^\alpha\varphi)(-1)^{|\alpha|} = (-1)^{|\alpha|}(X, D^\alpha\varphi).$$

Thus, it remains to establish (a). Define

$$X_N^\alpha(a) = \sum_{j=1}^N X_j \overline{D^\alpha F_j(a)},$$

$$I_N(a) = E|X_N^\alpha(a)|^2 = \sum_{j=1}^N |D^\alpha F_j(a)|^2;$$

and

$$I_N(a, b) = E|X_N^\alpha(a) - X_N^\alpha(b)|^2 = \sum_{j=1}^N |D^\alpha F_j(a) - D^\alpha F_j(b)|^2.$$

The uniform convergence of the series (6.2) will result ([3], cf. [4], Theorem 2) from the following

PROPOSITION 6.1. *There exists a constant $C > 0$ such that uniformly in $a, b \in T$ and $N = 1, 2, \ldots$* (²)

$$I_N(a) \leqslant C, \tag{6.3}$$

$$I_N(a, b) \leqslant C|a-b|^{1/4}. \tag{6.4}$$

Proof. We shall only prove (6.4) as (6.3) is proved analogically. Since $F_j \in \mathscr{D}(T)$, $j = 1, 2, \ldots$, we have

$$I_N(a, b) = \sum_{j=1}^N \big(D^\alpha F_j(a) - D^\alpha F_j(b)\big)\big(\overline{D^\alpha F_j(a)} - \overline{D^\alpha F_j(b)}\big)$$
$$= \sum_{j=1}^N D^{\alpha,\alpha}\big(F_j(a)\overline{F_j(a)} - F_j(a)\overline{F_j(b)} - F_j(b)\overline{F_j(a)} + F_j(b)\overline{F_j(b)}\big),$$

where $D^{\alpha,\alpha} = \partial^{2|\alpha|}/\partial a_1^{\alpha_1} \ldots \partial a_d^{\alpha_d} \partial b_1^{\alpha_1} \ldots \partial b_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$.

(²) Note that T may be unbounded.

By the reproducing property,

$$F_j(a)\overline{F_j(b)} = \langle F_j, R(a,\cdot)\rangle \overline{\langle F_j, R(b,\cdot)\rangle};$$

therefore

$$I_N(a, b) = D^{\alpha,\alpha}\big(\langle R(a,\cdot), P_N R(a,\cdot)\rangle - \langle R(b,\cdot), P_N R(a,\cdot)\rangle +$$
$$+ \langle R(b,\cdot), P_N R(b,\cdot)\rangle - \langle R(a,\cdot), P_N R(b,\cdot)\rangle\big),$$

where $R(a, b) = E[X(a)\overline{X(b)}]$ exists if $p-d/2 > 0$ and is jointly continuous ([17]), and P_N is the orthoprojector from $\mathscr{H}(T)$ onto $\bigvee\{F_j, j = 1, \ldots, N\}$.

Denote $\Lambda(a, b) = \langle R(a,\cdot), P_N R(b,\cdot)\rangle$. Since the norms $\langle f, f\rangle^{1/2}$ and $\|f\|_p$ are equivalent, there exists a bounded symmetric operator B with bounded inverse B^{-1} such that (see also [17])

$$(f, g)_p = \langle f, Bg\rangle$$

and

$$R(a,\cdot) = B\varrho_T(a,\cdot),$$

where $\varrho_T(a, b)$ is the reproducing kernel of $H_p(T)$. Hence

$$\Lambda(a, b) = \big(\varrho_T(a,\cdot), P_N\varrho_T(b,\cdot)\big)_p.$$

Next, $H_p(T)$ can be naturally embedded as a subspace into $H_p(\boldsymbol{R}^d)$. Therefore

$$\varrho_T(a,\cdot) = P_T\varrho(a,\cdot),$$

where P_T is the orthoprojector from $H_p(\boldsymbol{R}^d)$ onto $H_p(T)$ and $\varrho(a, b)$, $a, b \in \boldsymbol{R}^d$ is the reproducing kernel of $H_p(\boldsymbol{R}^d)$, $\varrho(a, b) = \varrho(a-b)$, and

$$\varrho(t) = 1/(2\pi)^d \int_{R^d} e^{itx}\Big(\sum_{|\alpha|\leqslant p} x^{2\alpha}\Big)^{-1} dx.$$

All these manipulations result in a final expression for $\Lambda(a, b)$ with which we shall deal below; namely:

$$\Lambda(a, b) = \big(\varrho(a,\cdot), A\varrho(b,\cdot)\big)_p, \tag{6.5}$$

where $A = P_N B P_T$ is a bounded operator $H_p(\boldsymbol{R}^d) \to H_p(\boldsymbol{R}^d)$, and $\|A\|$ is bounded uniformly in $N = 1, 2, \ldots$ We shall prove that $D^{\alpha,\alpha}\Lambda(a, b)$ exists in the usual sense for $|\alpha| < p-d/2$ and is Hölder continuous with exponent $1/4$, i.e., in fact, Proposition 6.1 and Theorem 6.1. This will result from the following steps:

PROPOSITION 6.2. $D^\alpha\varrho(a,\cdot) \in H_p(\boldsymbol{R}^d)$, $|\alpha| < p-d/2$.

Proof. Compute

$$\|D^\alpha\varrho(a,\cdot)\|_p^2 = \text{const}\int_{R^d} x^{2\alpha}\Big(\sum_{|\beta|\leqslant p} x^{2\beta}\Big)^{-1} dx < \infty$$

if $|\alpha| < p-d/2$ ([19], p. 244).

PROPOSITION 6.3. $D^{\alpha,\beta}\Lambda(a, b) \equiv \partial^{|\alpha|+|\beta|}\Lambda(a, b)/\partial a_1^{\alpha_1} \ldots \partial a_d^{\alpha_d} \partial b_1^{\beta_1} \ldots \partial b_d^{\beta_d}$ *exists in the usual sense for* $|\alpha|, |\beta| < p-d/2$ *and*

$$D^{\alpha,\beta}\Lambda(a, b) = \big(D^\alpha\varrho(a,\cdot), AD^\beta\varrho(b,\cdot)\big)_p, \quad a, b \in \boldsymbol{R}^d. \tag{6.6}$$

Proof (by induction). Let (6.6) hold for some α, β: $|\alpha| < p-d/2-1$, $|\beta| < p-d/2$. We shall see that $D^{\tilde{\alpha},\beta}\Lambda(a,b)$ exists and (6.6) holds for $\tilde{\alpha} = \alpha+e_i$, $e_i = (\underbrace{0,\ldots,0}_{i-1},1,0,\ldots,0)$. Let

$$\Delta_h D^{\alpha,\beta}\Lambda(a,b) \equiv h^{-1}(D^{\alpha,\beta}\Lambda(a+he_i,b)-D^{\alpha,\beta}\Lambda(a,b))$$
$$= \big((h^{-1}(D^{\alpha}\varrho(a+he_i,\cdot)-D^{\alpha}\varrho(a,\cdot)), AD^{\beta}\varrho(b,\cdot)\big)_p.$$

Then

$$\big|\Delta_h D^{\alpha,\beta}\Lambda(a,b)-(D^{\tilde{\alpha}}\varrho(a,\cdot), AD^{\beta}\varrho(b,\cdot))_p\big|$$
$$\leqslant \mathrm{const}\big\|h^{-1}(D^{\alpha}\varrho(a+he_i,\cdot)-D^{\alpha}\varrho(a,\cdot))-D^{\tilde{\alpha}}\varrho(a,\cdot)\big\|_p$$
$$\leqslant \mathrm{const}\Big(\int_{R^d}|h^{-1}(e^{ihxe_i}-1)-ix_i|^2x^{2\alpha}\Big(\sum_{|\beta|\leqslant p}x^{2\beta}\Big)^{-1}dx\Big)^{1/2} \to 0$$

as $h \to 0$, since the integrand tends to zero as $h \to 0$ at every $x \in R^d$ and is dominated by the function $2x_i^2x^{2\alpha}\big(\sum_{|\beta|\leqslant p}x^{2\beta}\big)^{-1}$ which is in $L^1(R^d)$.

The proof of the following inequality was reported to the author by Z. Ciesielski (private communication).

PROPOSITION 6.4. *Let $|\alpha| < p-d/2$. Then there exists a constant $C = C(p,d)$ such that*

$$|D^{2\alpha}\varrho(a)-D^{2\alpha}\varrho(0)| \leqslant C\min(1, |a|^{1/2}).$$

Proof: We have

$$D^{2\alpha}\varrho(a)-D^{2\alpha}\varrho(0) = \frac{(-1)^{|\alpha|}}{(2\pi)^d}\int_{R^d}x^{2\alpha}(e^{iax}-1)\Big(\sum_{|\beta|\leqslant p}x^{2\beta}\Big)^{-1}dx.$$

Let $|a| \leqslant 1$, and let λ be an arbitrary number, $0 < \lambda < 1$. Making use of the inequalities $|e^{iax}-1| \leqslant |ax|$ and

$$\sum_{|\beta|\leqslant p}x^{2\beta} \geqslant 1+\sum_{i=1}^{d}x_i^{2p} = 1+d\cdot\frac{1}{d}\sum_{i=1}^{d}x_i^{2p}$$
$$\geqslant 1+d\Big(\frac{1}{d}\sum_{i=1}^{d}x_i^2\Big)^p = 1+\frac{1}{d^{p-1}}|x|^{2p}$$
$$\geqslant \frac{1}{d^{p-1}}(1+|x|^{2p}),$$

we get

$$I \equiv |D^{2\alpha}\varrho(a)-D^{2\alpha}\varrho(0)|$$
$$\leqslant \frac{d^{p-1}}{(2\pi)^d}\int_{|ax|\leqslant|a|^{\lambda}}\frac{|x|^{2|\alpha|}|ax|}{1+|x|^{2p}}dx+\frac{2d^{p-1}}{2\pi^d}\int_{|ax|\geqslant|a|^{\lambda}}\frac{|x|^{2|\alpha|}dx}{1+|x|^{2p}}$$
$$\equiv I_1+I_2.$$

Here

$$I_1 \leqslant |a|^{\lambda}\mathrm{const}\int_{R^d}\frac{|x|^{2|\alpha|}}{1+|x|^{2p}}dx = C|a|^{\lambda};$$

and

$$I_2 \leqslant \mathrm{const}\int_{|x|\geqslant|a|^{\lambda-1}}\frac{|x|^{2|\alpha|}}{1+|x|^{2p}}dx$$
$$= \mathrm{const}\int_{|a|^{\lambda-1}}^{\infty}\frac{r^{2|\alpha|+d-1}}{1+r^{2p}}dr$$
$$\leqslant C|a|^{(1-\lambda)(2p-d-2|\alpha|)}.$$

Therefore $I \leqslant C(|a|^{\lambda}+|a|^{(1-\lambda)(2p-d-2|\alpha|)})$. Now, the following equation for λ has a unique solution λ_0:

$$\lambda = (1-\lambda)(2p-d-2|\alpha|),$$
$$\lambda_0 = \frac{2p-d-2|\alpha|}{2p-d-2|\alpha|+1}, \quad \lambda_0 < 1.$$

Thus, $|D^{2\alpha}\varrho(a)-D^{2\alpha}\varrho(0)| \leqslant C|a|^{\lambda_0}$, $|a| \leqslant 1$. See that $\lambda_0 \geqslant 1/2$. In fact, $\lambda_0 = \varphi(u_0)$, where $\varphi(u) = u(1+u)^{-1}$ and $u_0 = 2p-d-2|\alpha| > 0$, i.e. $u_0 \geqslant 1$ as u_0 is an integer. But $\varphi'(u) = (1+u)^{-2} > 0$, which implies that $\lambda_0 = \varphi(u_0) \geqslant \varphi(1) = 1/2$. Clearly $|a|^{\lambda_0} \leqslant |a|^{1/2}$ for $|a| \leqslant 1$. Consequently,

$$|D^{2\alpha}\varrho(a)-D^{2\alpha}\varrho(0)| \leqslant C|a|^{1/2}, \quad |a| \leqslant 1. \tag{6.7}$$

It is easy to see that the left-hand side of (6.7) is bounded uniformly in $a \in R^d$, which together with (6.7) implies Proposition 6.4.

Now to end the proof of Proposition 6.1 and, in fact, of Theorem 6.1 it suffices to show that there exists a constant $C > 0$ such that for all $a', a'', b \in R^d$, α: $|\alpha| < p-d/2$ and operators A: $H_p(R^d) \to H_p(R^d)$ with (say) $\|A\| \leqslant 1$,

$$|D^{\alpha,\alpha}\Lambda(a',b)-D^{\alpha,\alpha}\Lambda(a'',b)| \leqslant C|a'-a''|^{1/4}.$$

But this easily follows from (6.6), (6.7) and the reproducing property:

$$|D^{\alpha,\alpha}\Lambda(a',b)-D^{\alpha,\alpha}\Lambda(a'',b)|^2$$
$$\leqslant C\|D^{\alpha}\varrho(a',\cdot)-D^{\alpha}\varrho(a'',\cdot)\|^2$$
$$\leqslant C\big|D^{\alpha,\alpha}(\varrho(a',\cdot),\varrho(a',\cdot))_p-D^{\alpha,\alpha}(\varrho(a',\cdot),\varrho(a'',\cdot))_p\big|+$$
$$+C\big|D^{\alpha,\alpha}(\varrho(a'',\cdot),\varrho(a'',\cdot))_p-D^{\alpha,\alpha}(\varrho(a'',\cdot),\varrho(a',\cdot))_p\big|$$
$$= 2C|D^{2\alpha}\varrho(a'-a'')-D^{2\alpha}\varrho(0)| \leqslant \mathrm{const}|a'-a''|^{1/2}.$$

Remark 6.1. It is known ([17]) that a GRF $X \in \mathscr{F}_p$, $p-d/2 > 0$, is the so-called Markov random field of order p, i.e. if $\Gamma \subset T$ is a smooth $(d-1)$-dimensional surface, then the boundary data $H(\Gamma) = \bigcap_{\varepsilon>0}H(\Gamma_\varepsilon)$, where Γ_ε is an ε-neighbourhood of Γ, is the subspace spanned by $p-1$ 'weak' normal derivatives of X on Γ (see [17] for details). Now, Theorem 6.1 implies that among those $p-1$ 'weak' normal derivatives the first $p-d/2-1/2$ if d is odd and $p-d/2-1$ if it is even are 'strong', i.e. are derivatives of the trajectory in the usual sense.

7. Appendix

We present here the proofs of Propositions 5.2 and 5.3. Let us start with 5.2: it will follow clearly from Propositions 7.1 and 7.2 below.

PROPOSITION 7.1. *Let H_+ be a Hilbert space completion of $\mathscr{D}(T)$ in the norm $[f,f]_+^{1/2}$, which is equivalent to the norm $\langle f,f\rangle_+^{1/2}$, and let H_- be the dual of H_+. Then the natural embedding $\mathscr{H}_+(T)\to\mathscr{H}_-(T)$ is* H–S *iff such is the embedding $H_+\to H_-$.*

Proof. As the norms $\langle f,f\rangle_+^{1/2}$ and $[f,f]_+^{1/2}$ are equivalent, there exists a bounded symmetric operator B: $\mathscr{H}_+(T)\to\mathscr{H}_+(T)$ with a bounded inverse B^{-1} such that

$$[f,g]_+ = \langle Bf, g\rangle_+, \quad f,g\in\mathscr{H}_+(T). \tag{7.1}$$

The embedding $\mathscr{H}_+(T)\to\mathscr{H}_-(T)$ is H–S iff it is 2-summable ([15]), i.e. iff there exists a constant $C>0$ such that

$$\sum_i \langle f_i, f_i\rangle_- \leqslant C \sup_{\substack{g\in\mathscr{H}_+(T)\\ \langle g,g\rangle_+\leqslant 1}} \sum_i |\langle f_i, g\rangle_+|^2$$

for any sequence $\{f_i\}\subset\mathscr{H}_+(T)$. Let $H_+\to H_-$ be H–S. Then ($[f,f]_-^{1/2}$ denotes the 'negative' norm: $[f,f]_- = \sup\limits_{\substack{g\in\mathscr{D}(T)\\ [g,g]_+\leqslant 1}} |[f,g]|^2$):

$$\sum_i \langle f_i, f_i\rangle_- \leqslant C\sum_i [f_i,f_i]_- \leqslant C \sup_{\substack{g\in H_+\\ [g,g]_+\leqslant 1}} \sum_i |[f_i,g]_+|^2$$

$$= C \sup_{\substack{g'\in\mathscr{H}_+(T)\\ \langle B^{-1}g',g'\rangle_+\leqslant 1}} \sum_i |\langle f_i, g'\rangle_+|^2 \leqslant C \sup_{\substack{g\in\mathscr{H}_+(T)\\ \langle g,g\rangle_+\leqslant 1}} \sum_i |\langle f_i, g\rangle_+|^2$$

for some constant $C>0$ and any sequence $\{f_i\}\subset\mathscr{H}_+(T)$, i.e. $\mathscr{H}_+(T)\to\mathscr{H}_-(T)$ is H–S. The converse statement is proved in the same way.

PROPOSITION 7.2. *Let $T\subset \boldsymbol{R}^d$ be an open bounded domain, and $p>d/4$. Then the embedding $H_p(T)\to H_{-p}(T)$ is* H–S.

Proof. Unfortunately we could not find the proof of this fact in the literature but we shall reduce the problem to the embedding $H_{2p}(T)\to H_0(T) = L^2(T)$, which is known to be H–S under the assumptions of this proposition (see e.g. [19], p. 385 or [13], p. 336). Let $T\subset\Pi = \{x\in\boldsymbol{R}^d\colon 0<x_i<2\pi\}$ and let $\widetilde{\|f\|}_p \equiv \left(\int (1-\Delta)^p f(x)\,dx\right)^{1/2}$ be another Sobolev norm which is equivalent to the norm $\|f\|_p$. We denote by C_π^∞ the set of all C^∞-functions on Π which can be extended to C^∞-functions on $\boldsymbol{R}^d$ periodically with period Π, and by $\tilde{H}_p(T)$ ($\tilde{H}_p(\Pi)$, $\tilde{H}_{p,\pi}$) the Hilbert space completions of $\mathscr{D}(T)$ ($\mathscr{D}(\Pi)$, C_π^∞, respectively) in the norm $\widetilde{\|f\|}_p$. By Proposition 7.1, it suffices to prove that the embedding $\tilde{H}_p(T)\to\tilde{H}_{-p}(T)$ is H–S, where $\tilde{H}_{-p}(T)$ is the dual of $\tilde{H}_p(T)$. If $f\in\mathscr{D}(T)$, then $f\in\tilde{H}_p(\Pi)$ if extended so as

to be zero on $\Pi\setminus T$, $\widetilde{\|f\|}_p^2 = (2\pi)^{-d}\sum\limits_\beta |(f, e^{i\beta x})|^2(1+|\beta|^2)^p$, and

$$\widetilde{\|f\|}_{-p}^2 = \sup_{\substack{g\in\mathscr{D}(T)\\ \widetilde{\|g\|}_p\leqslant 1}} |(f,g)|^2 \leqslant (2\pi)^{-d} \sup_{(g_\beta)\colon \sum_\beta |g_\beta|^2\leqslant 1} \Big|\sum_\beta (f, e^{i\beta x})(1+|\beta|^2)^{-p/2} g_\beta\Big|^2 \tag{7.2}$$

$$= (2\pi)^{-d}\sum_\beta |(f, e^{i\beta x})|^2(1+|\beta|^2)^{-p} \equiv \widetilde{\|f\|}_{-p,\pi}^2.$$

Let $\{f_i\}\subset\mathscr{D}(T)$ be an O.N. basis in $\tilde{H}_p(T)$. We want to prove that

$$\sum_\beta \widetilde{\|f_i\|}_{-p}^2 < \infty. \tag{7.3}$$

Now see that

$$F_i(x) \equiv (2\pi)^{-d/2}\sum_\beta e^{i\beta x}(f_i, e^{i\beta x})(1+|\beta|^2)^{-p/2}, \qquad x\in\Pi,\ i = 1,2,\ldots$$

is an O.N. sequence in $\tilde{H}_{2p,\pi}$ (in fact, $\widetilde{(F_i, F_j)}_{2p} = \widetilde{(f_i,f_j)}_p = \delta(i-j)$), and $(F_i, F_i) = \widetilde{\|f_i\|}_{-p,\pi}^2$. As the embedding $H_{2p,\pi}\to L^2(\Pi)$ is H–S (a simple proof of this fact can be found in [13], p. 336), $\sum\limits_i (F_i, F_i) < \infty$, which implies (7.3) by (7.2).

Remark 7.1. Unfortunately the H–S property of the embedding $H_{2p}(T)\to L^2(T)$ does not hold for T not bounded. For example, the embedding $H_p(\boldsymbol{R}^1)\to L^2(\boldsymbol{R}^1)$ is not H–S for any p; take $\{f_n, n = 1,2,\ldots\}$ such that $\hat{f}_n(t) = e^{itn}(1+|t|^2)^{-p/2}\times$ $\times 1_{(0,2\pi)}(t)$, where $\hat{f}$ denotes the Fourier transform of f. Then $\widetilde{(f_n,f_m)}_p = \int_{-\infty}^{\infty} \hat{f}_n\overline{\hat{f}_m}(1+$ $+|t|^2)^p\,dt = 2\pi\delta(n-m)$ while $(f_n, f_n) = \mathrm{const}$ and so $\sum (f_n,f_n) = \infty$. Next we prove Proposition 5.3:

PROPOSITION 5.3. *Let $T = \boldsymbol{R}^d$ and let P_0 be an operator (given by* (0.2)) *with constant coefficients. Then $P^{1/2}$ is a continuous mapping from $\mathscr{S}(\boldsymbol{R}^d)$ (and $\mathscr{D}(\boldsymbol{R}^d)$) to $\mathscr{S}(\boldsymbol{R}^d)$.*

Proof. By definition,

$$P_0 f = \sum_{|\alpha|,|\beta|\leqslant p} (-1)^{|\beta|} a_{\alpha\beta} D^{\alpha+\beta} f, \qquad D(P_0) = \mathscr{D}(\boldsymbol{R}^d);$$

or, in terms of the Fourier transform,

$$P_0 f(t) = (2\pi)^{-2/d}\int_{\boldsymbol{R}^d} e^{itx}\hat{f}(x) P(x)\,dx, \qquad f\in\mathscr{D}(\boldsymbol{R}^d),$$

where $P(x) = \sum\limits_{|\alpha|,|\beta|\leqslant p} (-1)^{|\beta|} a_{\alpha\beta}(ix)^{\alpha+\beta}$. Now use assumption (A) to prove that

$$C_2(1+|x|^2)^p \leqslant P(x) \leqslant C_1(1+|x|^2)^p \tag{7.4}$$

for some constants C_1, $C_2>0$ and all $x\in\boldsymbol{R}^d$. In fact, according to (A),

$$C_2\big((1-\Delta)^p f, f\big) \leqslant (P_0 f, f) \leqslant C_1\big((1-\Delta)^p f, f\big), \qquad f\in\mathscr{D}(\boldsymbol{R}^d) \tag{7.5}$$

as the norms $\|f\|_p = (f,f)^{1/2}$ and $\widetilde{\|f\|}_p = \big((1-\Delta)^p f, f\big)^{1/2}$ are equivalent ([19], p. 220).

Rewrite (7.5) in terms of the Fourier transform:

$$(7.6)\qquad C_2 \int_{\mathbf{R}^d} |\hat{f}(x)|^2(1+|x|^2)^p dx \leqslant \int_{\mathbf{R}^d} |\hat{f}(x)|^2 P(x) dx \leqslant C_1 \int_{\mathbf{R}^d} |\hat{f}(x)|^2(1+|x|^2)^p dx$$

for all $f \in \mathscr{D}(\mathbf{R}^d)$. But $\widehat{\mathscr{D}(\mathbf{R}^d)}$ is dense in $\mathscr{S}(\mathbf{R}^d)$, hence (7.6) implies (7.4).

PROPOSITION 7.3. *Let P be Friedrichs' self-adjoint extension of P_0. Then $D(P) = H_{2p}(\mathbf{R}^d)$ and*

$$(7.7)\qquad Pf(t) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{itx} \hat{f}(x) P(x) dx, \qquad f \in D(P).$$

Proof. By a theorem of von Neumann ([13], p. 108) we need to check the following facts:

(1) $\tilde{P} = \overline{P}_0$;

(2) $\tilde{P}$ is symmetric;

(3) $R(\tilde{P}) = L^2(\mathbf{R}^d)$;

where $\tilde{P}$ is given by the right-hand side of (7.7) with $D(\tilde{P}) = H_{2p}(\mathbf{R}^d)$, and $\overline{P}_0$ denotes the closure of P_0.

Now, ad (1), if $\{f_n\} \subset D(P_0) = \mathscr{D}(\mathbf{R}^d)$ is convergent in $L^2(\mathbf{R}^d)$ to $f \in L^2(\mathbf{R}^d)$ and $P_0 f_n \to g$ (in $L^2(\mathbf{R}^d)$), then

$$\int_{\mathbf{R}^d} |\hat{f}_n - \hat{f}_m|^2 (1+|x|^2)^{2p} dx \leqslant C \int_{\mathbf{R}^d} |\hat{f}_n - \hat{f}_m|^2 P^2(x) dx$$

$$= C\|P_0(f_n - f_m)\|^2 \to 0, \quad n, m \to \infty,$$

i.e. $\{f_n\}$ is a Cauchy sequence in $H_{2p}(\mathbf{R}^d)$. By (7.4), $\tilde{P}$ is easily seen to be a continuous map $H_{2p}(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$; hence $f \in D(\tilde{P})$ and $g = \tilde{P}f$, i.e. $\tilde{P} = \overline{P}_0$.

Ad (2), apply Parseval's identity to the Fourier transform in $L^2(\mathbf{R}^d)$;

Ad (3); clearly f given by $\hat{f}(x) = \hat{g}(x) P^{-1}(x)$ gives the solution of $\tilde{P}f = g$ for every $g \in L^2(\mathbf{R}^d)$, and $f \in H_{2p}(\mathbf{R}^d)$.

PROPOSITION 7.4. $D(P^{1/2}) = H_p(\mathbf{R}^d)$ *and*

$$(7.8)\qquad P^{1/2} f(t) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{itx} \hat{f}(x) P^{1/2}(x) dx.$$

Proof. Denote by Tf the right-hand side of (7.8) with $D(T) = H_p(\mathbf{R}^d)$. The same argument as above proves that T is self-adjoint (and clearly positive). Elementary computations show that $P = T^2$ on $H_{2p}(\mathbf{R}^d)$, i.e. T is a self-adjoint positive square root of P. But such a root is unique ([8], p. 1247).

Now to end the proof of Proposition 5.3, since the Fourier transform is a continuous mapping $\mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$, it is enough to check by the previous Proposition that such is the mapping $f(x) \to f(x) P^{1/2}(x)$. But this is obvious since $P(x)$ is a strictly positive polynomial.

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