PROBABILITY THEORY
BANACH CENTER PUBLICATIONS, VOLUME 5
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1979

MARTINGALE CRITERIA FOR STOCHASTIC STABILITY*

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0. Introduction

Our purpose is to discuss stability properties of the following simple adjustment scheme which seems to arise in various applications. Consider two non-negative stochastic processes $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ which are adapted to some increasing family of σ-fields $(\mathcal{F}_n)_{n\geq 0}$. Let X react to positive values of the "signal" Y with a trend downwards, and let Y react to sufficiently large values of X by becoming positive. We assume that the interaction is strong enough, which in particular will mean

$$E[X_n - X_{n+1} | \mathcal{F}_n] \geq Y_n \quad \text{on} \quad \{Y_n > 0\}, \tag{0.1}$$

and we show that this implies *positive recurrence* in the sense that

$$\liminf_n \frac{1}{n} \sum_{k=1}^{n} I_{\{Y_n \leq \alpha\}} > 0 \text{ a.s.} \tag{0.2}$$

for any $\alpha > 0$. If in addition the trend of X is "switched off at equilibrium", i.e.,

$$E[X_n - X_{n+1} | \mathcal{F}_n] = 0 \quad \text{on} \quad \{Y_n = 0\}, \tag{0.3}$$

one obtains "*quick convergence to equilibrium*" in the sense that $\sum_{n=0}^{\infty} Y_n < \infty$ a.s. In Section 3 we illustrate the technique by an example, where a process $(Z_n)_{n\geq 0}$ is "stabilized" at some fixed level.

In the theory of Markov processes such martingale criteria for positive recurrence and convergence are essentially well known; cf., for example, Bucy [1], Wonham [9] and Hildenbrand–Radner [4]. The usual setting is $X_n = f(\xi_n)$ and $Y_n = \varepsilon I_{E-A}(\xi_n)$, where X arises by observing some function f along the paths of some Markov process (ξ_n), and where A is some subset of the state space E. (0.1) then means that f is a (weak) *Liapunov function* for A, and (0.2) translates into positive recurrence of A. (0.3) means that f is superharmonic on E and harmonic on A, and it implies

* Research partially supported by the National Science Fundation.

that the process finally stays in A. The present note came out of a discussion with W. Hildenbrand who suggested to drop the Markov property in the context of [4]. J. L. Snell then pointed out to me that martingale criteria for recurrence (but not for positive recurrence) of general stochastic processes appear already in Lamperti [5].

The proofs only involve repeated use of the discrete Doob decomposition and the law of large numbers for martingales. A continuous time version of (0.2) for right-continuous semimartingales, which is based on the representation of semimartingales as signed measures, will appear in [3].

1. A Liapunov criterion for positive recurrence

Let $X = (X_n)_{n\geqslant 0}$ and $Y = (Y_n)_{n\geqslant 0}$ be two stochastic processes with values in $[0, \infty)$, both defined over some basic probability space $(\Omega, \mathscr{F}, P)$ and adapted to an increasing family $(\mathscr{F}_n)_{n\geqslant 0}$ of σ-fields in $\mathscr{F}$. We assume that X has bounded conditional variance in the sense that

(1.1) $$E[(X_{n+1}-X_n)^2\,|\mathscr{F}_n] \leqslant c$$

for some constant $c > 0$.

(1.2) DEFINITION. Let us say that X is a *Liapunov process* for Y, or that (X, Y) is a *Liapunov system*, if

(1.3) $$E[X_n - X_{n+1}|\mathscr{F}_n] \geqslant Y_n \quad \text{on} \quad \{Y_n > 0\},$$

and if

(1.4) $$X^* \equiv \sup_n X_{n+1} I_{\{Y_n\leqslant\alpha\}} < \infty \quad P\text{-a.s.}$$

for some constant $\alpha > 0$.

(1.5) *Remark*. Suppose that X has bounded increments. Then (1.4) is satisfied if X is bounded near $\{Y = \alpha\}$ in the sense that

(1.6) $$\sup_n X_n I_{\{Y_n\leqslant\alpha\}} < \infty \quad P\text{-a.s.}$$

or

(1.7) $$\sup_n X_n I_{\{Y_{n+1}\leqslant\alpha\}} < \infty \quad P\text{-a.s.}$$

(1.3) and (1.6), resp. (1.7) specify the interaction between X and Y. (1.3) means that X reacts to the "signal" Y by tending downwards as soon as $Y > 0$ (the signal is "on"), and that it does so at a pace which depends on the magnitude of Y. On the other hand, Y reacts to X by assuming a value $> \alpha$ if X is above some "critical level", either immediately as in (1.6), or with a time lag as in (1.7).

We now want to show that a Liapunov system does not drift away. More precisely, let us define for any $\beta > 0$ the set

$$A_\beta \equiv \{(\omega, n)|\ Y_n(\omega) < \beta\}$$

in $\Omega\times\{0, 1, \ldots\}$, and let us show that the system spends a positive fraction of the time in A_β:

(1.8) THEOREM. *If X is a Liapunov process for Y, then each set A_β is positive recurrent in the sense that*

(1.9) $$\liminf_n \frac{1}{n}\sum_{k=0}^{n} I_{A_\beta}(\,.\,, k) \geqslant \frac{1}{1+C_\beta(.)} \quad P\text{-a.s.}$$

with $C_\beta(.) \equiv (\alpha\wedge\beta)^{-1}X^*(.)$.

(1.10) EXAMPLE. Let $(\xi_n)_{n\geqslant 0}$ be a stochastic process on some measurable state space $(E, \mathscr{E})$, defined over $(\Omega, \mathscr{F}, P)$ and adapted to $(\mathscr{F}_n)_{n\geqslant 0}$. Let $A\in\mathscr{E}$ be a measurable subset of the state space, and let f be a non-negative function on E which is bounded on A. Now suppose that f is a *Liapunov function* for A in the sense that the process $f(\xi_n)$ has bounded increments and satisfies

$$E[f(\xi_n)-f(\xi_{n+1})|\mathscr{F}_n] \geqslant \varepsilon \quad \text{on} \quad \{\xi_n \notin A\}$$

for some $\varepsilon > 0$. Applying (1.8) with $X_n = f(\xi_n)$ and $Y_n = \varepsilon I_{E-A}(\xi_n)$ we obtain positive recurrence of A in the sense that

$$\liminf_n \frac{1}{n}\sum_{k=0}^{n} I_A(\xi_k) \geqslant \frac{1}{1+C(.)} \quad P\text{-a.s.}$$

with $C(.) \equiv \varepsilon^{-1}\big(c+\sup_{x\in A} f(x)\big)$.

From now on we assume that (X, Y) is a Liapunov system, and we fix $\beta > 0$. Without loss of generality we assume $\beta\leqslant\alpha$ and write $A = A_\beta$.

(1.11) *Remark on notation.* Let S be a stopping time, i.e., a function on Ω with values in $\{0, 1, \ldots, \infty\}$ such that $\{S\leqslant n\}\in\mathscr{F}_n$ for each $n\geqslant 1$. Then $\mathscr{F}_S$ will denote the σ-field of all events $A\in\mathscr{F}$ such that $A\cap\{S\leqslant n\}\in\mathscr{F}_n$ for each $n\geqslant 0$. For any set $B\subseteq\Omega\times\{0, 1, \ldots\}$ we write

$$T_B\circ\theta_S \equiv \inf\{n\geqslant S(.)|\ (\,.\,, n)\in B\},$$

so that $S+T_B\circ\theta_S$ is the first entrance time into B from time S on. Now take the set $A = A_\beta$ and its complement A^c. We set $S_0 = T_0 = 0$, and for $n\geqslant 1$ we define

$$T_n \equiv T_A\circ\theta_{S_{n-1}}, \qquad R_n \equiv T_{A^c}\circ\theta_{S_{n-1}+T_n},$$

where

$$S_n \equiv \sum_{k=1}^{n}(T_k+R_k),$$

the time of the nth return to A^c, is easily seen to be a stopping time. $T_1+\ldots+T_n$ is the total time spent outside of A up to time S_n.

Proof of the Theorem. (1.14) below implies that the average time spent in A is a.s. equal to 1 on $\bigcup_n\{S_n = \infty\}$. On $\{S_n\leqslant m < S_{n+1}\}$ we have

$$\frac{1}{m}\sum_{k=1}^{n} I_{A^c}(\,.\,, k) \leqslant \frac{T_1+\ldots+T_{n+1}}{S_n} \leqslant \frac{T_1+\ldots+T_{n+1}}{T_1+\ldots+T_n+n}.$$

Noting (1.20) it is thus enough to show

$$\limsup_n \frac{T_1+\ldots+T_n}{T_1+\ldots+T_n+n} \leqslant 1-\frac{1}{1+C_\beta}\ P\text{-a.s.} \quad \text{on} \quad \bigcap_n \{S_n<\infty\}.$$

But this follows from (1.18) below since $\frac{x}{n} \leqslant C+\gamma$ implies $\frac{x}{x+n} \leqslant \frac{C}{1+C}+\gamma$.

We are now going to establish the two lemmas which were used in the preceding argument.

(1.12) LEMMA. *For $p = 1,2$ and $m \geqslant 1$ we have*

(1.13) $$E[T_{m+1}^p|\mathscr{F}_{S_m}] \leqslant C^p\ P\text{-a.s.} \quad on \quad \{S_m<\infty\},$$

with

$$C^1(.) \equiv C_\beta(.), \quad C^2(.) \equiv \beta^{-2}[X^{*2}(.)+cC^1(.)].$$

The same is true for $m = 0$ if we replace X^* by X_0 in the definition of C^p.

(1.14) *Remark.* In particular, we have $T_{m+1}<\infty$ P-a.s. on $\{S_m<\infty\}$ for each $m \geqslant 0$, i.e., the system returns to A after each excursion to A^c. The set A is thus *recurrent* in the sense that $P[(.,n) \in A$ infinitely often$] = 1$. We need the estimates in (1.13) in order to show, via (1.18) below, that A is actually *positive* recurrent in the sense of (1.8).

Proof of (1.12). Fix $m \geqslant 0$ and define $T \equiv T_{m+1}$, $\mathscr{G}_n \equiv \mathscr{F}_{S_{m+n}}$ and

$$Z_n \equiv XS_{m+n}I_{\{S_m<\infty,\, T>n\}} \quad (n \geqslant 0).$$

The process $Z = (Z_n)_{n\geqslant 0}$ is non-negative and adapted to $(\mathscr{G}_n)_{n\geqslant 0}$. Consider its Doob decomposition

$$Z_n = M_n - A_n \quad (n \geqslant 0)$$

into a martingale $(M_n)_{n\geqslant 0}$ and a predictable process $(A_n)_{n\geqslant 0}$, where $(A_n)_{n\geqslant 0}$ is defined through $A_0 = 0$ and

$$A_{n+1}-A_n = E[Z_n - Z_{n+1}|\mathscr{G}_n].$$

Due to (1.3) we have

(1.15) $$A_{n+1}-A_n \geqslant \beta I_{\{T>n\}}.$$

This shows that (A_n) is in fact an increasing process, and that the martingale (M_n) is non-negative since $M_n = Z_n + A_n \geqslant A_n \geqslant 0$. Moreover, (1.15) implies

(1.16) $$\beta T = \beta \sum_{k\geqslant 0} I_{\{T>k\}} \leqslant A_T \leqslant M_T,$$

where we set $A_\infty = \lim_n A_n$ and $M_\infty = \lim_n M_n$. But

$$E[M_T|\mathscr{G}_0] \leqslant M_0 = X_{S_m},$$

and so we obtain

(1.17) $$E[T|\mathscr{G}_0] \leqslant \beta^{-1}X_{S_m}$$

which is finite for $m = 0$ and bounded by $C^1(.)$ for $m \geqslant 1$ due to (1.4). This settles

the case $p = 1$. Now use (1.16) to conclude

$$\beta^2 E[T^2|\mathscr{G}_0] \leqslant E[M_T^2|\mathscr{G}_0] \leqslant \liminf_N E[M_{T\wedge N}^2|\mathscr{G}_0],$$

where

$$\begin{aligned} E[M_{T\wedge N}^2|\mathscr{G}_0] &= M_0^2 + E\Big[\sum_{k=0}^{N-1}(M_{k+1}^2-M_k^2)I_{\{T>k\}}|\mathscr{G}_0\Big] \\ &= M_0^2 + \sum_{k=0}^{N-1} E[E[M_{k+1}^2-M_k^2|\mathscr{G}_k]I_{\{T>k\}}|\mathscr{G}_0] \\ &\leqslant M_0^2 + cE[T\wedge N|\mathscr{G}_0], \end{aligned}$$

since

$$\begin{aligned} E[M_{k+1}^2-M_k^2|\mathscr{G}_k] &= E[(M_{k+1}-M_k)^2|\mathscr{G}_k] \\ &\leqslant E[(X_{k+1}-X_k)^2|\mathscr{G}_k] \leqslant c \end{aligned}$$

due to (1.1). This implies

$$E[T^2|\mathscr{G}_0] \leqslant \beta^{-2}(X_{S_m}^2 + cE[T|\mathscr{G}_0]),$$

where the right side is finite for $m = 0$ and bounded by $C^2(.)$ for $m \geqslant 1$.

(1.18) LEMMA. $\limsup_n \frac{T_1+\ldots+T_n}{n} \leqslant C_\beta$ *P-a.s. on* $\bigcap_n \{S_k<\infty\}$.

Proof. Consider the increasing process

$$B_n \equiv \sum_{k=0}^{n} T_n \quad (n \geqslant 0),$$

the associated predictable process $(B_n')_{n\geqslant 0}$ defined through $B_0' \equiv 0$ and

$$B_{n+1}'-B_n' \equiv E[B_{n+1}-B_n|\mathscr{F}_{S_n}] = E[T_{n+1}|\mathscr{F}_{S_n}] \quad (n \geqslant 0),$$

and the associated variance process $(V_n)_{n\geqslant 0}$ with $V_0 \equiv 0$ and

$$V_n \equiv \sum_{k=1}^{n} E[T_k^2|\mathscr{F}_{S_{k-1}}] \quad (n \geqslant 1).$$

By a law of large numbers due to Neveu, resp. Dubins and Freedman, we have

$$\lim_n \frac{B_n - B_n'}{V_n} = 0\ P\text{-a.s.} \quad \text{on} \quad \{V_\infty = \infty\};$$

cf., for example, [6], T. 65, p. 66. But for $n \geqslant 1$ (1.12) implies

$$1 \leqslant V_{n+1}-V_n = E[T_{n+1}^2|\mathscr{F}_{S_n}] \leqslant C^2 \quad \text{on} \quad \{S_n<\infty\}$$

so that $n \leqslant V_{n+1} \leqslant V_1 + nC^2$ P-a.s. on $\bigcap_m \{S_m<\infty\}$. Since $V_1<\infty$ a.s. by (1.12), we may thus conclude

(1.19) $$\lim_n \frac{B_n-B_n'}{n} = 0\ P\text{-a.s.} \quad \text{on} \quad \bigcap_n \{S_n<\infty\}.$$

Due to (1.12) we have $\limsup_n \frac{B'_n}{n} \leqslant C_\beta$, and this together with (1.19) yields (1.18).
Let us also note that (1.19) implies

$$\lim_n \frac{T_{n+1} - E[T_{n+1}|\mathscr{F}_{S_n}]}{n} = 0;$$

hence

(1.20) $$\lim \frac{T_{n+1}}{n} = 0 \ P\text{-a.s.} \quad \text{on} \quad \bigcap_n \{S_n < \infty\}$$

due to (1.12).

2. Convergence towards equilibrium

So far we did not impose any restriction on the trend of X at times where $Y = 0$. Let us now add the assumption that the trend is "switched off at equilibrium":

(2.1) $$E[X_n - X_{n+1}|\mathscr{F}_n] = 0 \quad \text{on} \quad \{Y_n = 0\}$$

for each $n \geqslant 0$.

(2.2) PROPOSITION. *If (X, Y) is a Liapunov system which satisfies* (2.1) *then we have convergence towards equilibrium in the sense that*

(2.3) $$X_n \to X_\infty \leqslant X^* \ P\text{-a.s.},$$

and

(2.4) $$Y_n \to 0 \ P\text{-a.s.}$$

The convergence in (2.4) *is "quick" in the sense that*

(2.5) $$E\Big[\sum_{n=0}^{\infty} Y_n|\mathscr{F}_0\Big] = X_0 < \infty.$$

Proof. Due to (2.1) the Liapunov process $X = (X_n)_{n\geqslant 0}$ is now a non-negative supermartingale, hence a.s. convergent to some finite limit X_∞. (1.8) and (1.4) imply $X_\infty \leqslant X^*$; in fact we have $X_n \leqslant X^*$ for all $n \geqslant n_0(\omega)$. Now consider the Doob decomposition $X_n = M_n - A_n$ of X into a martingale $M = (M_n)$ and a predictable increasing process $A = (A_n)$. Due to (1.3) and (2.1) we have

$$A_{n+1} - A_n = E[X_n - X_{n+1}|\mathscr{F}_n] \geqslant Y_n \ P\text{-a.s.}$$

This implies $\sum_{n\geqslant 0} Y_n \leqslant \lim_n A_n = A_\infty$, and the right side satisfies $E[A_\infty|\mathscr{F}_0] \leqslant X_0 < \infty$ since $E[A_n|\mathscr{F}_0] \leqslant E[M_n|\mathscr{F}_0] = M_0 = X_0$.

(2.6) EXAMPLE. In the situation of example (1.10) the condition (2.1) takes the form

$$E[f(\xi_n) - f(\xi_{n+1})|\mathscr{F}_n] = 0 \quad \text{on} \quad \{\xi_n \in A\}.$$

This means that the Liapunov function f is now superharmonic on E and harmonic on A (in the generalized sense of Doob [2]), and (2.5) says that the total time spent outside of A has finite expectation.

3. Adjusting one process to another

Let us now look at a different setting. Consider two real-valued stochastic processes $X = (X_n)_{n\geqslant 0}$ and $\tilde{X} = (\tilde{X}_n)_{n\geqslant 0}$, both defined over $(\Omega, \mathscr{F}, P)$ and adapted to $(\mathscr{F}_n)_{n\geqslant 0}$. Suppose that X is steered towards $\tilde{X}$ in the sense that

(3.1) $$E[X_n - X_{n+1}|\mathscr{F}_n] \underset{(<)}{>} E[\tilde{X}_n - \tilde{X}_{n+1}|\mathscr{F}_n] \quad \text{on} \quad \{X_n \underset{(<)}{>} \tilde{X}_n\},$$

i.e., X tends downwards (resp. upwards) more than $\tilde{X}$ as long as X is above (resp. below) $\tilde{X}$. This means that the process $Z = (Z_n)_{n\geqslant 0}$ with

$$Z_n \equiv X_n - \tilde{X}_n$$

is steered towards 0 in the sense that

(3.2) $$Z_n E[Z_n - Z_{n+1}|\mathscr{F}_n] > \quad \text{on} \quad \{Z_n \neq 0\}.$$

Let us now formulate conditions which guarantee that the adjustment (3.2) leads to a stabilizations of Z at 0. We assume that increments are bounded so that

(3.2) $$|Z_n - Z_{n+1}| \leqslant c \quad (n \geqslant 0)$$

for some constant c. In addition to the sign rule (3.2) for the direction of the trend, we assume that its absolute value satisfies

(3.3) $$|E[Z_n - Z_{n+1}|\mathscr{F}_n]| \geqslant h(|Z_n|)$$

for some monotone function h on $[0, \infty)$ with $h(0) = 0$ and $h > 0$ on $(0, \infty)$ (we have $h \leqslant c$ due to (3.2)). Now (1.8) implies that the procedure leads at least to positive recurrence in the sense that

(3.4) COROLLARY. $\liminf_n \frac{1}{n} \sum_{k=0}^{n-1} I_{\{|Z_k| \leqslant c\}} \geqslant \frac{1}{1+a} > 0$ *with* $a = 2c/h(c)$.

Proof. On $\{Z_n > c\}$ we have

$$E[|Z_n| - |Z_{n+1}||\mathscr{F}_n] = E[Z_n - Z_{n+1}|\mathscr{F}_n] \geqslant h(Z_n) \geqslant h(c),$$

and in the same way we have

(3.5) $$E[|Z_n| - |Z_{n+1}||\mathscr{F}_n] \geqslant h(c)$$

on $\{Z_n < -c\}$. This means that $(|Z_n|)_{n=0}$ is a Liapunov process for the process $Y_n = h(c) I_{\{|Z_n| > c\}}$ $(n \geqslant 0)$. Now apply (1.8).

Let us now try to get convergence of Z to 0. We assume $Z_n \in L^2$ $(n \geqslant 0)$ and

(3.6) $$a_n(.) \equiv 2Z_n E[Z_n - Z_{n+1}|\mathscr{F}_n] - E[(Z_n - Z_{n+1})^2|\mathscr{F}_n] \geqslant -c_n(.)$$

with $c_n(.) \geqslant 0$ and $\sum_{n=0} c_n(.) \in L^1$. Note that we have $a_n \geqslant 0$ for large enough Z_n, so that (3.6) essentially means that the "variance is more and more tuned down" near 0.

(3.7) PROPOSITION. (3.6) *implies* $Z_n \to 0$ *P*-a.s.

Proof. Since $a_n = E[Z_n^2 - Z_{n+1}^2 | \mathscr{F}_n]$, the Doob decomposition of $(Z_n^2)_{n \geqslant 0}$ has the form

$$Z_n^2 = M_n - \sum_{k=1}^{n} a_k = S_n + \sum_{k=1}^{n} c_k,$$

where

$$S_n \equiv M_n - \sum_{k=1}^{n} (a_k + c_k) \geqslant -\sum_{k=1}^{n} c_k$$

is a supermartingale bounded from below in L^1, hence a.s. convergent to some finite limit. This implies the convergence of Z_n^2, resp. Z_n to some finite limit and, in particular, $Z_n - Z_{n+1} \to 0$ P-a.s. Now the lemma of Hunt yields

$$E[Z_n - Z_{n+1} | \mathscr{F}_n] \to 0$$

due to (3.2); cf., for example, [7], p. 143. Thus (3.3) implies $h(|Z_n|) \to 0$, hence $|Z_n| \to 0$ P-a.s.

(3.8) *Remark.* The argument for almost sure convergence of (Z_n^2) is included only for the sake of completeness, since I learned from D. Siegmund that it is contained in Theorem 1 of [8]. Condition (3.6) means in fact that (Z_n^2) is an "almost supermartingale" in the sense of [8].

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Presented to the Semester
Probability Theory
February 11–June 11, 1976

PROBABILITY THEORY
BANACH CENTER PUBLICATIONS, VOLUME 5
PWN—POLISH SCIENTIFIC PUBLISHERS
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НЕКОТОРЫЕ ЗАМЕЧАНИЯ ПО ПОВОДУ ЗАКОНА БОЛЬШИХ ЧИСЕЛ В R^d

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Пусть

(1) $$X_1, X_2, \ldots, X_n, \ldots; \quad X_i = (X_{i1}, \ldots, X_{id})$$

последовательность независимых, одинаково распределенных случайных векторов принимающих значения из евклидового пространства $\boldsymbol{R}^d$, $d \geqslant 1$.

Обозначим через $a = (a_1, \ldots, a_d)$ вектор математических ожиданий и B — ковариационную матрицу вектора X_1.

Если $x \in \boldsymbol{R}^d$, то положим

$$|x| = \sqrt{x_1^2 + \ldots + x_d^2}\,.$$

Пусть $S_n = X_1 + \ldots + X_n$, ε — любое положительное число, $I_n(\varepsilon)$ индикатор события $\{|S_n - na| > n\varepsilon\}$. Тогда

$$\nu_\varepsilon = \sum_{n=1}^{\infty} I_n(\varepsilon)$$

есть „считающая величина", т.е. число осуществлений события $\{|S_n - na| > n\varepsilon\}$.

Легко понять, что конечность почти всюду „считающей величины" ν_ε (для любого $\varepsilon > 0$) означает выполнение усиленного закона больших чисел для случайных векторов последовательности (1).

Имеет место следующая теорема, являющаяся многомерным аналогом одного результата П. Эрдеша [4].

Теорема 1. *Для того, чтобы при любом фиксированном* $\varepsilon > 0$ $E\nu_\varepsilon < \infty$ *необходимо и достаточно*

$$EX_1 = a, \quad E|X_1|^2 < \infty.$$

Очевидно, что в силу известной леммы Бореля–Контелли из теоремы 1 следует применимость усиленного закона больших чисел для случайных векторов последовательности (1).