$$X^*(n) = (I+Q(n))X(n),$$
$$X(n+1) = X^*(n)+an^{-\alpha}\{R(n+1, X^*(n))+G(n+1, X^*(n), \omega)\}.$$

(This is the dynamic Robbins–Monro procedure for tracking $\theta(n)$, the unique root of $R(n, x)$.)

THEOREM. *For $n \to \infty$ and every $x \in E_k$, the distribution of $n^{\alpha/2}(X(n)-\theta(n))$ tends to the normal distribution with mean value $a^{-1}B^{-1}q_\infty$ and the covariance matrix*

$$S = a\int_0^\infty e^{Bv}S_0 e^{B^Tv}dv.$$

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ON A GENERALIZATION OF A THEOREM OF W. SUDDERTH AND SOME APPLICATIONS

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Introduction and basic definitions

1. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and $(\mathscr{F}_n)_{n\geqslant 0}$ an increasing family of sub-σ-algebras of $\mathscr{F}$. We shall consider a sequence $X = (X_n)_{n\geqslant 0}$ of (real-valued) random variables which always is assumed to be adapted to the family $(\mathscr{F}_n)_{n\geqslant 0}$. A nonnegative (possibly, infinite) random variable T is called a *stopping time* (of the family $(\mathscr{F}_n)_{n\geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$) if for all n the event $\{T = n\}$ belongs to $\mathscr{F}_n$. By $\overline{\mathfrak{M}}$ we shall denote the set of all stopping times and by $\mathfrak{M}$ the set of all a.s. finite stopping times.

Let $T \in \overline{\mathfrak{M}}$. Define the random variable X_T by

$$X_T(\omega) = \begin{cases} X_n(\omega) & \text{if} \quad T(\omega) = n, \\ \limsup\limits_n X_n(\omega) & \text{if} \quad T(\omega) = \infty. \end{cases}$$

Let us introduce the class $\overline{\mathfrak{M}}(X)$ of all stopping times T satisfying the condition that the integral $\mathbf{E}X_T$ exists, i.e. $\mathbf{E}X_T^+ < \infty$ or $\mathbf{E}X_T^- < \infty$.(1) Finally, we set $\mathfrak{M}(X) = \overline{\mathfrak{M}}(X) \cap \mathfrak{M}$.

2. In the problem of optimal stopping (cf. Shiryaev [6] or Chow, Robbins, and Siegmund [4]) one considers the value(2)

$$V = \sup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$$

which is interpreted as the maximal gain that can be obtained by stopping the reward sequence $(X_n)_{n\geqslant 0}$ in an optimal way. Analogously, for any stopping time $S \in \mathfrak{M}(X)$ the value

$$V_S = \sup_{\substack{T\in\overline{\mathfrak{M}}(X)\\ T\geqslant S}} \mathbf{E}X_T$$

(1) For any real number x, we set $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$.
(2) Of course, $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$.

is interpreted as the maximal gain which can be obtained by stopping the reward sequence $(X_n)_{n\geqslant 0}$ after S. We now introduce the value

$$V_\infty^{\bar{\mathfrak{M}}} = \inf_{S\in\mathfrak{M}(X)} V_S.$$

This value can be seen as the maximal gain that can still be obtained by stopping the reward sequence $(X_n)_{n\geqslant 0}$ after an arbitrarily long period of time.

3. In the present note we investigate the *value at time infinity* V_∞. We give an explicit expression for evaluating it. Using this characterization of the value at infinity V_∞, we come to necessary and sufficient conditions for almost sure convergence of a sequence of random variables $(X_n)_{n\geqslant 0}$. Connections with other work are pointed out. Our results are stated under most general assumptions. The main result of the paper is Theorem 8.

The proofs will be omitted. They will be published elsewhere.

4. Before formulating the main result of the paper we still introduce some definitions and notations.

Obviously, the set $\bar{\mathfrak{M}}$ is partially ordered by the relation $\geqslant$: $T\geqslant S$ if $T(\omega)\geqslant S(\omega)$ for all $\omega\in\Omega$. Moreover, the ordered set is directed since with two stopping times S and T the random variable $\max(S, T)$ also is a stopping time. In general, the ordered set $\mathfrak{M}(X)$ need not be directed, nevertheless it is natural to introduce the notation

$$(1)\qquad \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \inf_{S\in\mathfrak{M}(X)} \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqslant S}} \mathbf{E}X_T.$$

In our notation we thus have

$$V_\infty = \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T.$$

Analogously, we define

$$(2)\qquad \liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \sup_{S\in\mathfrak{M}(X)} \inf_{\substack{T\in\mathfrak{M}(X)\\ T\geqslant S}} \mathbf{E}X_T.$$

It is obvious that $-\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ is equal to the value at infinity of the sequence $(-X_n)_{n\geqslant 0}$.

We say that the limit of the generalized sequence $\mathbf{E}X_T$ for $T\in\mathfrak{M}(X)$ exists if

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T.$$

In this case we write $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ for this value.

5. Let us introduce the class $\mathbf{C}^+$ of all random sequences $X = (X_n)_{n\geqslant 0}$ satisfying the following two conditions:

(1) $\mathbf{E}\limsup_n X_n$ exists,

(2) $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T < +\infty$.

By definition, the class $\mathbf{C}^-$ consists of all random sequences $(X_n)_{n\geqslant 0}$ such that $(-X_n)_{n\geqslant 0}$ belongs to $\mathbf{C}^+$. Furthermore, we set $\mathbf{C} = \mathbf{C}^+\cap\mathbf{C}^-$.

6. Condition 5(1) means that the stopping time identically equal to $+\infty$ belongs to $\bar{\mathfrak{M}}(X)$.

Suppose now that (X_n) or $(-X_n)$ is satisfying condition 5(1). Using Lévy's martingale convergence theorem, it can easily be seen that for any $S\in\mathfrak{M}$ there exists $T\in\mathfrak{M}(X)$ such that $T\geqslant S$. Thus the ordered set $\mathfrak{M}(X)$ is directed and, moreover, the sets $\{T\in\mathfrak{M}(X)\colon T\geqslant S\}$ in definitions (1) and (2) are nonempty.

Inequalities and equalities for the value at infinity

We begin with the following result which was proved by W. Sudderth [7] in the case $\mathfrak{M}(X) = \mathfrak{M}$, using Lévy's martingale convergence theorem.

7. Theorem (1) *If* $\mathbf{E}\limsup_n X_n$ *exists then*

$$\mathbf{E}\limsup_n X_n \leqslant \limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T.$$

(2) *If* $\mathbf{E}\liminf_n X_n$ *exists then*

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqslant \mathbf{E}\liminf_n X_n.$$

Of course, (2) follows applying (1) to $(-X_n)_{n\geqslant 0}$. Statement (1) is well known in optimal stopping (cf. Shiryaev [6] or Chow, Robbins, and Siegmund [4]): Replacing $\mathfrak{M}(X)$ by $\bar{\mathfrak{M}}(X)$, we do not change the value $\sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqslant S}} \mathbf{E}X_T$. But by assumption $\infty\in\bar{\mathfrak{M}}(X)$ and thus

$$\mathbf{E}\limsup_n X_n \leqslant \sup_{\substack{T\in\mathfrak{M}(X)\\ T\geqslant S}} \mathbf{E}X_T \quad \text{for all } S\in\mathfrak{M}.$$

Now we formulate the main result of the present note.

8. Theorem. (1) *Suppose that* $(X_n)_{n\geqslant 0}$ *belongs to* $\mathbf{C}^+$. *Then*

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\limsup_n X_n.$$

(2) *If* $(X_n)_{n\geqslant 0}$ *belongs to* $\mathbf{C}^-$ *then*

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\liminf_n X_n.$$

Theorem 8 was earlier stated and proved by W. Sudderth [7] but only under the assumption for statement (1) (resp. statement (2)) that the random sequence $(X_n)_{n\geqslant 0}$ is bounded above (resp. below) by an integrable random variable. In this case, the proof of Theorem 8 is rather simple and, in fact, is an easy consequence of Theorem 7 and Fatou's lemma. Recently, R. Chen [3] generalized this result assuming only that the family $(X_T^+)_{T\in\mathfrak{M}}$ (resp. $(X_T^-)_{T\in\mathfrak{M}}$) is uniformly integrable. His proof is only a slight modification of that of W. Sudderth [7]. Our proof of Theorem 8 is not quite trivial and requires some basic ideas of the theory of optimal stopping.

9. *Remark.* It should be noticed that statement (1) of Theorem 8, in general, is not true if $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = +\infty$. W. Sudderth [7] gave an example of a uniformly integrable sequence $(X_n)_{n\geqslant 0}$ such that $\lim_n X_n = 0$ a.s. but $\limsup_{T\in\mathfrak{M}} \mathbf{E}X_n \geqslant 1$. This example does not contradict Theorem 8 since one easily verifies that, in fact, $\limsup_{T\in\mathfrak{M}} \mathbf{E}X_T = +\infty$. Of course, the equality in statement (1) (resp. statement (2)) of Theorem 8 also holds if $\mathbf{E}\limsup_n X_n = +\infty$ (resp. $\mathbf{E}\liminf_n X_n = -\infty$) (cf. Theorem 7).

We now consider one special case in which Theorem 8 is true without the condition that $(X_n)_{n\geqslant 0}$ belongs to $\mathbf{C}^+$ (resp. $\mathbf{C}^-$). Let $\mathscr{F}_\infty$ be the smallest σ-algebra containing $\mathscr{F}_n$ for all n. For any $T\in\mathfrak{M}$, define the σ-algebra $\mathscr{F}_T$ as the collection of all $A\in\mathscr{F}_\infty$ such that $A\cap\{T=n\}$ belongs to $\mathscr{F}_n$ for every $n\geqslant 0$.

10. THEOREM. *Suppose that for some $T\in\mathfrak{M}$ we have $\mathscr{F}_T = \mathscr{F}_\infty$.*

(1) *If* $\mathbf{E}\limsup_n X_n$ *exists then*

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\limsup_n X_n.$$

(2) *If* $\mathbf{E}\liminf_n X_n$ *exists then*

$$\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\liminf_n X_n.$$

11. *Remarks.* (1) In particular, the condition of Theorem 10 is fulfilled if $\mathscr{F}_n = \mathscr{F}$ for all $n\geqslant 0$. In this case $\mathfrak{M}$ consists of all nonnegative random variables defined on $(\Omega, \mathscr{F}, \mathbf{P})$.

(2) It is interesting to notice that the value $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ does not change if we replace $(\mathscr{F}_n)_{n\geqslant 0}$ by a new family $(\mathscr{G}_n)_{n\geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$. The only conditions are that $(X_n)_{n\geqslant 0}$ belongs to $\mathbf{C}^+$ for both $(\mathscr{F}_n)_{n\geqslant 0}$ and $(\mathscr{G}_n)_{n\geqslant 0}$ and, of course, $\mathscr{F}_n^X \subseteq \mathscr{G}_n$ for all $n\geqslant 0$ where $\mathscr{F}_n^X$ is the smallest σ-algebra relative to which X_m is measurable for every $m\leqslant n$. This follows immediately from Theorem 8. The analogous remark is valid for $\liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ and $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ if it exists.

(3) Now we consider the case where $\mathscr{G}_n = \mathscr{F}$ for all $n\geqslant 0$. Using Theorem 10, we observe that $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ with respect to $(\mathscr{G}_n)_{n\geqslant 0}$ cannot be equal to $+\infty$ if $(X_n, \mathscr{F}_n)$ belongs to $\mathbf{C}^+$. Thus, if $(X_n, \mathscr{F}_n)$ belongs to $\mathbf{C}^+$ the value $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ does not change by passing over to $(\mathscr{G}_n)_{n\geqslant 0}$. In particular, this is true for $(\mathscr{F}_n)_{n\geqslant 0} = (\mathscr{F}_n^X)_{n\geqslant 0}$.

(4) Without the condition that $(X_n, \mathscr{F}_n)$ belongs to $\mathbf{C}^+$ the latter remark does not hold. Indeed, consider the example given by W. Sudderth [7] (cf. Remark 9). We then have $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = +\infty$ with respect to $(\mathscr{F}_n^X)_{n\geqslant 0}$. However, by Theorem 10 we obtain

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\limsup_n X_n = 0$$

with respect to $(\mathscr{G}_n)_{n\geqslant 0}$.

(5) Suppose now that $(X_n, \mathscr{G}_n)$ is a Markov sequence and $(\mathscr{F}_n)_{n\geqslant 0}$ is such that $\mathscr{F}_n^X \subseteq \mathscr{F}_n \subseteq \mathscr{G}_n$ for all $n\geqslant 0$. Then it can be proved that $(X_n, \mathscr{G}_n)$ belongs to $\mathbf{C}^+$ if $(X_n, \mathscr{F}_n)$ does. Therefore, if $(X_n, \mathscr{F}_n)$ belongs to $\mathbf{C}^+$ then the extension of $(\mathscr{F}_n)_{n\geqslant 0}$ to $(\mathscr{G}_n)_{n\geqslant 0}$ does not change $\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$.

Necessary and sufficient conditions for almost sure convergence

Now we investigate the connection between equalities for the value at infinity and almost sure convergence of random sequences. To begin with, we formulate a sufficient condition.

11. THEOREM. *Let $(X_n)_{n\geqslant 0}$ be a random sequence such that* $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist. Suppose, moreover, that* $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists. Then*

$$\mathbf{E}\limsup_n X_n = \lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\liminf_n X_n.$$

If one (and therefore all) of the values $\mathbf{E}\limsup_n X_n$, $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$, *and* $\mathbf{E}\liminf_n X_n$ *is finite then* $\lim_n X_n$ *exists a.s. and is integrable.*

This result follows from Theorem 7.

We now apply Theorem 11 to generalized regular supermartingales.

12. DEFINITION. A random sequence $(X_n)_{n\geqslant 0}$ is called a *generalized regular supermartingale* if for all $S, T\in\mathfrak{M}(X)$ such that $S\leqslant T$ we have

$$\mathbf{E}X_T \leqslant \mathbf{E}X_S.$$

The following lemma shows that in the above definition expectations can be replaced by conditional expectations.

13. LEMMA. *A random sequence $(X_n)_{n\geqslant 0}$ is a generalized regular supermartingale if and only if*

$$\mathbf{E}(X_T|\mathscr{F}_S) \leqslant X_S \quad a.s.$$

for all $S, T\in\mathfrak{M}(X)$ with $S\leqslant T$ and such that $-\infty < \mathbf{E}X_T$ and $\mathbf{E}X_S < +\infty$.

14. THEOREM. *Let $(X_n)_{n\geqslant 0}$ be a generalized regular supermartingale. Suppose that* $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist. Then*

$$\mathbf{E}\liminf_n X_n = \mathbf{E}\limsup_n X_n$$

and this value is equal to $\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$. *If one (and therefore all) of the values* $\mathbf{E}\liminf_n X_n$,

$\mathbf{E}\limsup_n X_n$, *and* $\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *is finite, then*

$$\lim_n X_n \text{ exists a.s.}$$

and is integrable.

We next state the connection between generalized regular supermartingales and uniform integrability properties.

15. THEOREM. *Let $(X_n)_{n\geqslant 0}$ be a generalized regular supermartingale. The following conditions are equivalent:*

(1) $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist and one (and therefore all) of the values* $\mathbf{E}\liminf_n X_n$, $\mathbf{E}\limsup_n X_n$, *and* $\inf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *is not equal to* $-\infty$.

(2) *The family* $(X_T^-)_{T\in\mathfrak{M}(X)}$ *is uniformly integrable.*

It seems that the implication (1) → (2) was not known previously.

16. COROLLARY. *Let $(X_n)_{n\geqslant 0}$ be a generalized supermartingale (i.e. $\mathbf{E}X_n$ exists and $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \leqslant X_n$ for all $n \geqslant 0$). The following conditions are equivalent:*

(1) $(X_n^-)_{n\geqslant 0}$ *is uniformly integrable.*

(2) $(X_T^-)_{T\in\mathfrak{M}}$ *is uniformly integrable.*

(3) $(X_n)_{n\geqslant 0}$ *is regular and* $\inf_{T\in\mathfrak{M}} \mathbf{E}X_T > -\infty$.

(4) $(X_n)_{n\geqslant 0}$ *is regular and* $\mathbf{E}\liminf_n X_n$ *exists and is not equal to* $-\infty$.

(5) $(X_n)_{n\geqslant 0}$ *is regular,* $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist, and* $\mathbf{E}\limsup_n X_n > -\infty$.

If one (and therefore all) of these conditions is satisfied then $\lim_n X_n$ *exists a.s. and* $\mathbf{E}\lim_n X_n > -\infty$. *If, moreover, there exists $T\in\mathfrak{M}$ such that* $\mathbf{E}X_T^+ < \infty$ *then* $\lim_n X_n$ *is integrable.*

We next come to the general situation and give necessary and sufficient conditions for almost sure convergence.

17. THEOREM. *Let $(X_n)_{n\geqslant 0}$ belong to* $\mathbf{C}$. *The following conditions are equivalent:*

(1) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists.*

(2) $\lim_n X_n$ *exists a.s.*

If one of these conditions is satisfied then $\lim_n X_n$ *is integrable and*

$$\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\lim_n X_n.$$

This theorem looks like Lebesgue's theorem on changing the order of limit and integral. The proof immediately follows from Theorem 8.

From Theorem 10 one derives the following result.

18. THEOREM. *Suppose that for some $T\in\mathfrak{M}$ we have $\mathcal{F}_T = \mathcal{F}_\infty$ and let* $\mathbf{E}\liminf_n X_n$ *and* $\mathbf{E}\limsup_n X_n$ *exist. The following conditions are equivalent:*

(1) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists and is finite.*

(2) $\lim_n X_n$ *exists a.s. and is integrable.*

If one of the conditions is satisfied then

$$\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \mathbf{E}\lim_n X_n.$$

In general, the existence of $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ is equivalent to the equality $\mathbf{E}\liminf_n X_n = \mathbf{E}\limsup_n X_n$ and in this case the values occurring are equal.

Theorem 18 can be applied to the case where $\mathcal{F}_n = \mathcal{F}$ for all $n \geqslant 0$, i.e. if the parameter T for the generalized sequence $\mathbf{E}X_T$ is ranging over all nonnegative finite random variables such that the integral of X_T makes sense.

Connection to amarts

Next we consider the set $\mathfrak{M}_b$ of all bounded stopping times T, i.e. $T \leqslant N$ for some integer $N \geqslant 0$. Let $\mathfrak{M}_b(X)$ be the collection of all $T\in\mathfrak{M}_b$ such that $\mathbf{E}X_T$ exists. Analogously to the case of $\mathfrak{M}(X)$ we define

$$\limsup_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T = \inf_{S\in\mathfrak{M}_b(X)} \sup_{\substack{T\in\mathfrak{M}_b(X)\\ T\geqslant S}} \mathbf{E}X_T$$

and

$$\liminf_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T = \sup_{S\in\mathfrak{M}_b(X)} \inf_{\substack{T\in\mathfrak{M}_b(X)\\ T\geqslant S}} \mathbf{E}X_T.$$

We say that the limit of $\mathbf{E}X_T$, where T ranges over $\mathfrak{M}_b(X)$, exists if

$$\liminf_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T = \limsup_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T$$

and write $\lim_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T$ for this value.

The following theorem was proved by R. Chen [3], although his proof is not the simplest.

19. THEOREM. (1) *If $(X_n^-)_{n\geqslant 0}$ is uniformly integrable, then*

$$\limsup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T \leqslant \limsup_{T\in\mathfrak{M}_b} \mathbf{E}X_T.$$

(2) *If $(X_n^+)_{n\geqslant 0}$ is uniformly integrable then*

$$\liminf_{T\in\mathfrak{M}_b} \mathbf{E}X_T \leqslant \liminf_{T\in\mathfrak{M}(X)} \mathbf{E}X_T.$$

20. DEFINITION (cf. Edgar and Sucheston [5]). A random sequence $(X_n)_{n\geq 0}$ is called a *(generalized) amart* if $\lim_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T$ exists and is finite.

As a direct consequence of Theorem 7 and Theorem 19 we obtain

21. THEOREM. *Let $(X_n)_{n\geq 0}$ be a uniformly integrable amart. We then have:*

(1) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ *exists and is finite.*

(2) $\lim_n X_n$ *exists a.s. and is integrable.*

(3) $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T = \lim_{T\in\mathfrak{M}_b} \mathbf{E}X_T = \mathbf{E}\lim_n X_n$.

Statement (3) does not remain valid without the assumption of the uniform integrability.

22. EXAMPLE. Let $(Y_n)_{n\geq 0}$ be a sequence of independent random variables such that $\mathbf{P}(Y_n = 1) = \mathbf{P}(Y_n = 0) = \frac{1}{2}$. Define $X_n = 2^n \cdot Y_1 \ldots Y_n$ and $\mathscr{F}_n = \mathscr{F}_n^X$. Then $(X_n)_{n\geq 0}$ is a nonnegative martingale, and hence an amart, which is not uniformly integrable. Obviously, $\lim_n X_n = 0$ a.s. and $\lim_{T\in\mathfrak{M}_b} \mathbf{E}X_T = 1$. From Fatou's lemma we obtain $\mathbf{E}X_T \leq 1$ for all $T\in\mathfrak{M}$. By Theorem 8, $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is equal to zero.

We proceed with a lemma of R. Chen [3].

23. LEMMA. (1) *Let $(X_T^+)_{T\in\mathfrak{M}_b}$ be uniformly integrable. Then*

$$\limsup_{T\in\mathfrak{M}_b} \mathbf{E}X_T \leq \mathbf{E}\limsup_n X_n.$$

(2) *If $(X_T^-)_{T\in\mathfrak{M}_b}$ is uniformly integrable, then*

$$\mathbf{E}\liminf_n X_n \leq \liminf_{T\in\mathfrak{M}_b} \mathbf{E}X_T.$$

This lemma, Theorem 8, and Theorem 21 yield the following theorem which is due to R. Chen [3].

24. THEOREM. *Let $(X_T)_{T\in\mathfrak{M}_b}$ be uniformly integrable. The following conditions then are equivalent:*

(1) $(X_n)_{n\geq 0}$ *is an amart.*

(2) $\lim_{T\in\mathfrak{M}} \mathbf{E}X_T$ *exists and is finite.*

(3) $\lim_n X_n$ *exists a.s. and is integrable.*

The next result, known as the *amart convergence theorem*, is due to Austin, Edgar, and Ionescu Tulcea [1] and R. V. Chacon [2].

25. THEOREM. *Let $(X_n)_{n\geq 0}$ be an amart such that $\sup_n \mathbf{E}|X_n| < \infty$. Then $\lim_n X_n$ exists a.s. and is integrable.*

It is clear that the theorem also is true if we only assume $\limsup_n \mathbf{E}|X_n| < +\infty$.

The converse of this theorem is false as the following example shows.

26. EXAMPLE. Let $(X_n)_{n\geq 0}$ be as in Example 22 and define

$$Z_{2n+1} = X_n \quad \text{and} \quad Z_{2n} = 0$$

for all $n \geq 0$. Then $\lim_n Z_n = 0$ a.s. but $\liminf_{T\in\mathfrak{M}_b} \mathbf{E}Z_T = 0$ and $\limsup_{T\in\mathfrak{M}_b} \mathbf{E}Z_T = 1$.

We now present the following interesting result.

27. THEOREM. *Let $(X_n)_{n\geq 0}$ be an amart such that $\limsup_n \mathbf{E}|X_n| < \infty$. Then $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite.*

It should be noticed that under the assumptions of Theorem 27 $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ and $\lim_{T\in\mathfrak{M}_b} \mathbf{E}X_T$ are not equal in general (cf. Example 26). Moreover, the converse statement to Theorem 27 does not hold, i.e. if $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite then $(X_n)_{n\geq 0}$ need not be an amart (cf. Example 26). Unfortunately, the proof of this theorem is based on the amart convergence theorem. A direct proof is not known to us. However, a direct proof of Theorem 27 would be of interest because Theorem 27 and Theorem 8 imply the amart convergence theorem.

The proof of Theorem 27 is based on Theorem 8 and Theorem 25 using the following lemma which is interesting in its own right.

28. LEMMA. (1) *Suppose $\limsup_n \mathbf{E}X_n^- < \infty$ and $\limsup_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T < \infty$. Then $(X_n)_{n\geq 0}$ belongs to $\mathbf{C}^+$.*

(2) *Suppose $\limsup_n \mathbf{E}X_n^+ < \infty$ and $\liminf_{T\in\mathfrak{M}_b(X)} \mathbf{E}X_T > -\infty$. Then $(X_n)_{n\geq 0}$ belongs to $\mathbf{C}^-$.*

Summarizing the results of the paper, we conclude that the class of random sequences $(X_n)_{n\geq 0}$ satisfying the property that $\lim_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$ exists and is finite is, possibly, more interesting than the class of the so-called amarts. In particular, under the assumption $\limsup_n \mathbf{E}|X_n| < \infty$ the class of amarts is smaller and therefore, in our opinion, the notion of an amart too restrictive. Finally, it should be noticed the consequence of Theorem 27 that $\lim_{T\in\mathfrak{M}} \mathbf{E}X_T$ exists and is finite for every supermartingale $(X_n)_{n\geq 0}$ satisfying $\sup_n \mathbf{E}X_n^- < \infty$. Before, this result was not known.

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ON ABSOLUTE CONTINUITY AND SINGULARITY OF PROBABILITY MEASURES

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Introduction

1. Let $(\Omega, \mathcal{F})$ be a measurable space and $\boldsymbol{Q}$, $\boldsymbol{P}$ two probability measures on it.

The probability measure $\boldsymbol{Q}$ is called *absolutely continuous* with respect to $\boldsymbol{P}$ ($\boldsymbol{Q} \ll \boldsymbol{P}$) if for every $A \in \mathcal{F}$ such that $\boldsymbol{P}(A) = 0$ we have $\boldsymbol{Q}(A) = 0$. The probability measures $\boldsymbol{Q}$ and $\boldsymbol{P}$ are called *equivalent* ($\boldsymbol{Q} \sim \boldsymbol{P}$) if both conditions $\boldsymbol{Q} \ll \boldsymbol{P}$ and $\boldsymbol{P} \ll \boldsymbol{Q}$ are satisfied. Finally, we say that $\boldsymbol{Q}$ and $\boldsymbol{P}$ are *singular* ($\boldsymbol{Q} \perp \boldsymbol{P}$) if there exists a set $N \in \mathcal{F}$ such that $\boldsymbol{Q}(N) = 0$ and $\boldsymbol{P}(N) = 1$.

2. We now assume that we are given an increasing family $(\mathcal{F}_n)_{n \geqslant 0}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the condition that $\mathcal{F}$ is the smallest σ-algebra containing $\mathcal{F}_n$ for all $n \geqslant 0$. Denote the restrictions of $\boldsymbol{Q}$ and $\boldsymbol{P}$ on the σ-algebra $\mathcal{F}_n$ by $\boldsymbol{Q}_n$ and $\boldsymbol{P}_n$, respectively. The problem which will be studied here is the following. Suppose $\boldsymbol{Q}_n \ll \boldsymbol{P}_n$ for every $n \geqslant 0$. We want to find conditions for absolute continuity and singularity of $\boldsymbol{Q}$ and $\boldsymbol{P}$.

3. Since Kakutani's famous work [3] on the equivalence of infinite product measures many authors have been investigated equivalence and singularity of certain probability measures. One of the fundamental results is the equivalence-singularity dichotomy for Gaussian measures on function spaces of J. Feldman [1] and J. Hajek [2]. Many efforts were done to give conditions for absolute continuity of special processes (for example, diffusion processes) and to find the explicite expression of the Radon–Nikodym derivative (cf. Lipcer and Shiryaev [4]). Problems of this kind play a fundamental role in many areas of probability theory and, above all, in statistics.

In the present paper we shall prove a general theorem giving necessary and sufficient conditions for absolute continuity $\boldsymbol{Q} \ll \boldsymbol{P}$ and singularity $\boldsymbol{Q} \perp \boldsymbol{P}$ in terms