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ON ABSOLUTE CONTINUITY AND SINGULARITY OF PROBABILITY MEASURES

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Introduction

1. Let $(\Omega, \mathscr{F})$ be a measurable space and $\boldsymbol{Q}$, $\boldsymbol{P}$ two probability measures on it.

The probability measure $\boldsymbol{Q}$ is called *absolutely continuous* with respect to $\boldsymbol{P}$ ($\boldsymbol{Q} \ll \boldsymbol{P}$) if for every $A \in \mathscr{F}$ such that $\boldsymbol{P}(A) = 0$ we have $\boldsymbol{Q}(A) = 0$. The probability measures $\boldsymbol{Q}$ and $\boldsymbol{P}$ are called *equivalent* ($\boldsymbol{Q} \sim \boldsymbol{P}$) if both conditions $\boldsymbol{Q} \ll \boldsymbol{P}$ and $\boldsymbol{P} \ll \boldsymbol{Q}$ are satisfied. Finally, we say that $\boldsymbol{Q}$ and $\boldsymbol{P}$ are *singular* ($\boldsymbol{Q} \perp \boldsymbol{P}$) if there exists a set $N \in \mathscr{F}$ such that $\boldsymbol{Q}(N) = 0$ and $\boldsymbol{P}(N) = 1$.

2. We now assume that we are given an increasing family $(\mathscr{F}_n)_{n \geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$ satisfying the condition that $\mathscr{F}$ is the smallest σ-algebra containing $\mathscr{F}_n$ for all $n \geqslant 0$. Denote the restrictions of $\boldsymbol{Q}$ and $\boldsymbol{P}$ on the σ-algebra $\mathscr{F}_n$ by $\boldsymbol{Q}_n$ and $\boldsymbol{P}_n$, respectively. The problem which will be studied here is the following. Suppose $\boldsymbol{Q}_n \ll \boldsymbol{P}_n$ for every $n \geqslant 0$. We want to find conditions for absolute continuity and singularity of $\boldsymbol{Q}$ and $\boldsymbol{P}$.

3. Since Kakutani's famous work [3] on the equivalence of infinite product measures many authors have been investigated equivalence and singularity of certain probability measures. One of the fundamental results is the equivalence-singularity dichotomy for Gaussian measures on function spaces of J. Feldman [1] and J. Hajek [2]. Many efforts were done to give conditions for absolute continuity of special processes (for example, diffusion processes) and to find the explicite expression of the Radon–Nikodym derivative (cf. Lipcer and Shiryaev [4]). Problems of this kind play a fundamental role in many areas of probability theory and, above all, in statistics.

In the present paper we shall prove a general theorem giving necessary and sufficient conditions for absolute continuity $\boldsymbol{Q} \ll \boldsymbol{P}$ and singularity $\boldsymbol{Q} \perp \boldsymbol{P}$ in terms of almost sure convergence with respect to the probability measure Q of the densities dQ_n/dP_n. In contrary to most of other papers, in general we only assume $Q_n \ll P_n$ (instead of $Q_n \sim P_n$) for all $n \geqslant 0$. In many situations the characterization of absolute continuity and singularity proposed here is advantageous and leads to easy proofs of known and unknown results.

From our general theorem we can derive necessary and sufficient conditions for absolute continuity and singularity in "predictable terms" in the sense of the modern theory of stochastic processes. The conditions are very similar to that of Kolmogorov's three series theorem. The development of these problems needs some essential facts from martingale theory. Here we do not treat these questions. The authors intend to publish a paper on effective criteria for absolute continuity and singularity of the type described above.

In this note we only present some general examples illustrating the basic theorem. First we consider the independent case proving Kakutani's theorem. As a second example we give another proof of Feldman's [1] and Hajek's [2] dichotomy for Gaussian measures. Then we investigate the Markov case. For non-homogeneous discrete time Markov processes $(X_n)_{n\geqslant 0}$ with values in arbitrary state spaces we shall prove that either $Q \ll P$ or $Q \perp P$ (of course, if $Q_n \ll P_n$ for all $n \geqslant 0$) if $(X_n)_{n\geqslant 0}$ is a 0-1 sequence with respect to Q.

For homogeneous Markov chains we can say even more. For example, let $(X_n)_{n\geqslant 0}$ be stationary and ergodic with respect to Q. Then the alternative $Q \ll P$ or $Q \perp P$ holds. The assumption of stationarity and ergodicity can still be weakened. In particular, if the state space is countable, then it is sufficient to know that except for, possibly, transient states there is one class of positive recurrent states with respect to Q.

The main theorem

Let Z_n be the Radon–Nikodym derivative dQ_n/dP_n, i.e., the density of Q_n with respect to P_n. By Z_∞ we denote the (possibly, infinite) random variable $\limsup_n Z_n$. It is well known (and easy to verify) that $(Z_n, \mathscr{F}_n, P)$ is a nonnegative martingale. Thus the limit of $(Z_n)_{n\geqslant 0}$ exists P-a.s. and of course equals to Z_∞ P-a.s. By Fatou's lemma we have[1] $E_P Z_\infty \leqslant 1$. In particular, $P(0 \leqslant Z_\infty < \infty) = 1$. Recall the well-known facts that the conditions $Q \ll P$, $E_P Z_\infty = 1$, and $(Z_n, \mathscr{F}_n, P)$ uniformly integrable are equivalent. In the following theorem we give the explicit form of the Lebesgue decomposition of Q with respect to P. As an immediate consequence we obtain necessary and sufficient conditions for $Q \ll P$ and $Q \perp P$ in terms of almost sure convergence.

4. THEOREM. *Suppose $Q_n \ll P_n$ for all $n \geqslant 0$. We then have the Lebesgue decomposition*

$$Q(B) = \int_B Z_\infty \, dP + R(B)$$

(1) By E_Q and E_P we denote the expectation with respect to Q and P, respectively.

for all $B \in \mathscr{F}$, where $P \perp R$. Moreover, the following conditions are satisfied:

(1) $P(Z_\infty < \infty) = 1$.

(2) $R(Z_\infty < \infty) = 0$.

Proof. By Fatou's lemma,

$$\int_B Z_\infty \, dP \leqslant Q(B)$$

for all $B \in \mathscr{F}_n$ and $n \geqslant 0$, and hence for all $B \in \mathscr{F}$. We now set

$$R(B) = Q(B) - \int_B Z_\infty \, dP$$

for all $B \in \mathscr{F}$. Then R is a non-negative and finite measure on $\mathscr{F}$. For proving the theorem it now suffices to verify that conditions (1) and (2) hold. Condition (1) was mentioned before the formulation of the theorem. To prove (2) we first notice that there exist at most countably many constants c such that $P(Z_\infty = c) > 0$. Therefore it suffices to verify that

$$R(Z_\infty < c) = 0$$

for all finite $c \geqslant 0$ such that $P(Z_\infty = c) = 0$. We have

$$\{Z_\infty < c\} \subseteq \liminf_n \{Z_n < c\}$$

and P-a.s.

$$\{Z_\infty < c\} = \lim_n \{Z_n < c\}.$$

Using Fatou's lemma, from this follows

$$R(Z_\infty < c) \leqslant \liminf_n Q(Z_n < c) - \limsup_n \int_{\{Z_n < c\}} Z_n \, dP.$$

Hence

$$R(Z_\infty < c) \leqslant \liminf_n \Big[Q(Z_n < c) - \int_{\{Z_n < c\}} Z_n \, dP\Big] = 0$$

and the proof is finished.

5. THEOREM. *Suppose $Q_n \ll P_n$ for all $n \geqslant 0$.*

(1) *$Q \ll P$ if and only if $Q(Z_\infty < \infty) = 1$.*

(2) *$Q \perp P$ if and only if $Q(Z_\infty = \infty) = 1$.*

Proof. Since $P(Z_\infty = \infty) = 0$, from $Q \ll P$ follows $Q(Z_\infty = \infty) = 0$ and therefore $Q(Z_\infty < \infty) = 1$. Conversely, assume $Q(Z_\infty < \infty) = 1$. Then from Theorem 4 we obtain

$$Q(B) = Q(B \cap \{Z_\infty < \infty\}) = \int_B Z_\infty \, dP.$$

Thus $Q \ll P$.

We now verify (2). Let $Q \perp P$. By Theorem 4, $Q(B) = R(B)$. Using 4(2), we obtain $Q(Z_\infty < \infty) = 0$ and hence $Q(Z_\infty = \infty) = 1$. Conversely, if $Q(Z_\infty = \infty) = 1$, then Theorem 4 implies $Q(B) = R(B)$ for all $B \in \mathscr{F}$. Since $R \perp P$ we get $Q \perp P$.

6. Corollary. *Let $Q_n \sim P_n$ for all $n \geqslant 0$.*

(1) *$Q \sim P$ if and only if $Q(Z_\infty < \infty) = P(Z_\infty > 0) = 1$.*

(2) *$Q \perp P$ if and only if $Q(Z_\infty = \infty) = 1$ or $P(Z_\infty = 0) = 1$.*

Proof. Theorem 5 can be applied to the densities

$$Z'_n = \frac{dP_n}{dQ_n} = \frac{1}{Z_n} \quad \text{for } n \geqslant 0.$$

7. Corollary. *Suppose $Q_n \ll P_n$ for all $n \geqslant 0$. Then $\lim_n Z_n$ exists and is finite on the set $\{\sup Z_n < \infty\}$ Q-a.s.*

Indeed, $\lim_n Z_n$ exists and is finite P-a.s. But in view of Theorem 4, for every $B \in \mathscr{F}$ with $B \subseteq \{Z_\infty < \infty\} = \{\sup_n Z_n < \infty\}$, we have

$$Q(B) = \int_B Z_\infty \, dP.$$

Setting $B = \{\liminf_n Z_n < \limsup_n Z_n < \infty\}$, we obtain $Q(B) = 0$.

Absolute continuity on $\mathscr{F}_T$

A non-negative (possibly, infinite) integer-valued random variable T is called a *stopping time* (of the family $(\mathscr{F}_n)_{n \geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$) if the event $\{T = n\}$ belongs to $\mathscr{F}_n$ for all $n \geqslant 0$.

Let T be a stopping time. By $\mathscr{F}_T$ we denote the σ-algebra of events A from $\mathscr{F}$ such that $A \cap \{T = n\}$ belongs to $\mathscr{F}_n$ for all $n \geqslant 0$. Let Q_T and P_T be the restrictions of Q and P on $\mathscr{F}_T$, respectively.

We now give a necessary and sufficient criterion for absolute continuity of Q_T with respect to P_T.

8. Theorem. *Suppose $Q_n \ll P_n$ for all $n \geqslant 0$. Let T be a stopping time.*

(1) *$Q_T \ll P_T$ if and only if $\{T = \infty\} \subseteq \{Z_\infty < \infty\}$ Q-a.s.*

(2) *Then the Radon–Nikodym derivative dQ_T/dP_T is Z_T, where*

$$Z_T(\omega) = \begin{cases} Z_n(\omega) & \text{if} \quad T(\omega) = n, \\ Z_\infty(\omega) & \text{if} \quad T(\omega) = \infty. \end{cases}$$

Proof. Let $A \in \mathscr{F}_T$. We then have

$$Q(A) = \sum_{n=0}^{\infty} Q(A \cap \{T = n\}) + Q(A \cap \{T = \infty\}).$$

Notice that $A \cap \{T = n\}$ belongs to $\mathscr{F}_n$ and $Q(A \cap \{T = \infty\}) = Q(A \cap \{T = \infty\} \cap \cap \{Z_\infty < \infty\})$ by assumption. Consequently,

$$Q(A) = \sum_{n=0}^{\infty} \int_{A \cap \{T=n\}} Z_n \, dP + \int_{A \cap \{T=\infty\} \cap \{Z_\infty < \infty\}} Z_\infty \, dP$$

and since $P(Z_\infty < \infty) = 1$, we obtain

$$Q(A) = \int_A Z_T \, dP.$$

Thus $Q_T \ll P_T$ and $dQ_T/dP_T = Z_T$.

Conversely, if $Q_T \ll P_T$, then

$$Q(A \cap \{Z_\infty = \infty\}) = 0$$

for all $A \in \mathscr{F}_T$ by Theorem 4. Setting $A = \{T = \infty\}$, we obtain

$$\{T = \infty\} \subseteq \{Z_\infty < \infty\} \qquad Q\text{-a.s.}$$

9. Corollary. *Suppose that $Q(T < \infty) = 1$. Then $Q_T \ll P_T$.*

Let $(X_n)_{n \geqslant 0}$ be a random sequence defined on $(\Omega, \mathscr{F})$ and taking values in arbitrary measurable spaces $(E_n, \mathscr{E}_n)$. The following notation will be used. Let $\mathscr{F}^X_n$ be the smallest σ-algebra with respect to which X_m is measurable for all $m \leqslant n$. In this case we always assume $\mathscr{F}_n = \mathscr{F}^X_n$ for all $n \geqslant 0$ and $\mathscr{F} = \mathscr{F}^X_\infty$.

10. Example. Let $(X_n)_{n \geqslant 0}$ be a sequence of independent random variables defined on $(\Omega, \mathscr{F}, P)$ such that $P(X_n = 1) = P(X_n = 0) = \frac{1}{2}$. Suppose that there is a $\omega_0 \in \Omega$ with $X_n(\omega_0) = 1$ for all $n \geqslant 0$. Define the probability measure Q as the δ-distribution of ω_0. Clearly, $Q_n \ll P_n$ and $Z_n = dQ_n/dP_n = 2^{n+1} \cdot X_0 \cdot \ldots \cdot X_n$. Furthermore, $Z_\infty = 0$ P-a.s. But we have $Z_\infty = \infty$ Q-a.s. By Theorem 4 we get $Q \perp P$. We now define

$$T(\omega) = \min\{n \geqslant 0\colon Z_n(\omega) = 0\},$$

where $T(\omega) = \infty$ if there is no such n. Then T is a stopping time and we have $Q(T = \infty) = 1$. By Theorem 8, Q_T is not absolutely continuous with respect to P_T. Moreover, since $P(T < \infty) = 1$, we observe that $Q_T \perp P_T$.

Independent random sequences

Let $(X_n)_{n \geqslant 0}$ be a random sequence defined on $(\Omega, \mathscr{F})$ and such that X_n is taking values in $(E_n, \mathscr{E}_n)$ for all $n \geqslant 0$. Assume that $(X_n)_{n \geqslant 0}$ is independent with respect to both Q and P.

11. Theorem. *If $Q_n \ll P_n$ for all $n \geqslant 0$, then $Q \ll P$ or $Q \perp P$.*

Proof. Let Q_{X_n} and P_{X_n} be the distributions of X_n in $(E_n, \mathscr{E}_n)$ with respect to Q and P. Obviously, $Q_{X_n} \ll P_{X_n}$ for all $n \geqslant 0$. We then have

$$Z_n = q_0(X_0) \cdot \ldots \cdot q_n(X_n) \qquad Q\text{-a.s. and } P\text{-a.s.}$$

for all $n \geqslant 0$, where $q_n = dQ_{X_n}/dP_{X_n}$. From Kolmogorov's 0-1 law for independent sequences we get $Q(Z_\infty < \infty) = 1$ or 0. By Theorem 5, $Q \ll P$ or $Q \perp P$.

If we assume $Q_n \sim P_n$ for all $n \geqslant 0$, then by symmetry we obtain Kakutani's [3] equivalence-singularity dichotomy.

Gaussian random functions

We now give a proof of the Feldman–Hajek dichotomy for Gaussian measures that is essentially based on Theorem 5. In our opinion this proof is advantageous, because it is not difficult and shows down clearly which basic properties of the normal distribution are needed. The fundamental property for the validity of the dichotomy is the following well-known fact: *Let* $(\xi_n)_{n\geq 0}$ *be an arbitrary Gaussian sequence. Then either* $\sum_{n=0}^{\infty}\xi_n^2<+\infty$ *a.s. or* $\sum_{n=0}^{\infty}\xi_n^2=+\infty$ *a.s. in dependence on the convergence or divergence of* $\sum_{n=0}^{\infty}E\xi_n^2$. *In the sequel we shall use this property without further comment.*

Before we consider the Gaussian case we state an important lemma that will be derived from our general Theorem 5.

LEMMA. *Let*(2) $\alpha_n=Z_nZ_{n-1}^{-1}$ *for* $n\geq 1$ *and* $\alpha_0=1$. *Suppose, moreover, that* $Q_n\sim P_n$ *for all* $n\geq 0$. *If for some* $p\geq 1$

$$\sum_{n=0}^{\infty}\ln E_P(\alpha_{n+1}^{1/p}|\mathcal{F}_n)=-\infty\quad P\text{-a.s.}$$

then we have $Q\perp P$.

Proof. It can easily be seen that

$$0<E_P(\alpha_{n+1}^{1/p}|\mathcal{F}_n)\leq 1\qquad Q\text{-a.s.}$$

and since $Q_n\sim P_n$ this inequality also holds P-a.s. Define the random variables

$$Y_{n+1}=\alpha_{n+1}^{1/p}[E_P(\alpha_{n+1}^{1/p}|\mathcal{F}_n)]^{-1},\qquad Y_0=0,$$

and

$$W_n=\prod_{k=1}^{n}Y_k.$$

One immediately verifies that $(W_n,\mathcal{F}_n,P)$ forms a nonnegative martingale and, consequently, the finite limit $W_\infty=\lim_n W_n$ exists P-a.s. Notice that

$$Z_n^{1/p}=W_n\prod_{k=0}^{n-1}E_P(\alpha_{k+1}^{1/p}|\mathcal{F}_k)\qquad P\text{-a.s.}$$

In view of the assumption of the lemma we have $\prod_{k=0}^{\infty}E_P(\alpha_{k+1}^{1/p}|\mathcal{F}_k)=0$ P-a.s. From this immediately follows

$$P(Z_\infty=0)=1.$$

By Corollary 6, we obtain $Q\perp P$, proving the lemma.

(2) We make the convention $a^{-1}=0$ if $a=0$.

Let now I be an arbitrary parameter set and $(X_i)_{i\in I}$ a family of random variables defined on $(\Omega,\mathcal{F})$. Let $\mathcal{F}$ be generated by this family. Assume that $(X_i)_{i\in I}$ is Gaussian, i.e. for all $i_1,\dots,i_n\in I$ the random vector $(X_{i_1},\dots,X_{i_n})$ has (possibly, degenerated) normal distributions, with respect to both probability measures Q and P.

The Feldman–Hajek dichotomy asserts that either $Q\sim P$ or $Q\perp P$. For proving this dichotomy first we notice that, obviously, it suffices to consider the case $I=\{0,1,2,\dots\}$. Furthermore, we can and do assume $Q_n\sim P_n$ for all $n\geq 0$. Otherwise we have $Q_n\perp P_n$ for some n and thus $Q\perp P$. Let

$$a_n=E_Q(X_n|\mathcal{F}_{n-1}),\qquad \lambda_n^2=E_Q[(X_n-a_n)^2|\mathcal{F}_{n-1}]$$

for all $n\geq 0$ where $\mathcal{F}_{-1}=\{\emptyset,\Omega\}$. The random variables b_n and σ_n^2 have the same meaning with respect to P. Note that in view of the theorem of normal correlation (cf., for example, [4], Theorem 13.1) a_n and b_n are linear functions of $X_0,\dots,X_{n-1}$ and λ_n as well as σ_n are nonnegative constants. Define $D=\{n\geq 0\colon \lambda_n>0\}$ and $\varrho_n=\sigma_n/\lambda_n$ for all $n\in D$. (Clearly, $n\in D$ if and only if $\sigma_n>0$.) For the Radon–Nikodym derivative one easily deduces the following formula:

$$(*)\qquad Z_n=\frac{dQ_n}{dP_n}=\prod_{\substack{k=0\\k\in D}}^{n}\varrho_k\exp\left\{-\frac{1}{2}\frac{(X_k-a_k)^2}{\lambda_k^2}+\frac{1}{2}\frac{(X_k-b_k)^2}{\sigma_k^2}\right\}.$$

In the next theorem we give necessary and sufficient conditions for absolute continuity and singularity of Q and P in case of $I=\{0,1,2,\dots\}$. The conditions are formulated in terms of a_n, b_n, λ_n, and σ_n. In particular, the dichotomy either $Q\sim P$ or $Q\perp P$ is an immediate consequence of this theorem.

12. THEOREM. *Let* $Q_n\sim P_n$ *for all* $n\geq 0$.

(1) $Q\sim P$ *if and only if* $\sum_{n\in D}\left((\ln\varrho_n)^2+E_Q\frac{(a_n-b_n)^2}{\lambda_n^2+\sigma_n^2}\right)<+\infty$.

(2) $Q\perp P$ *if and only if* $\sum_{n\in D}\left((\ln\varrho_n)^2+E_Q\frac{(a_n-b_n)^2}{\lambda_n^2+\sigma_n^2}\right)=+\infty$.

Proof. The proof consists of two parts:

(i) $\sum_{n\in D}\left((\ln\varrho_n)^2+E_Q\frac{(a_n-b_n)^2}{\lambda_n^2+\sigma_n^2}\right)<+\infty\Rightarrow Q\ll P$.

(ii) $\sum_{n\in D}\left((\ln\varrho_n)^2+E_Q\frac{(a_n-b_n)^2}{\lambda_n^2+\sigma_n^2}\right)=+\infty\Rightarrow Q\perp P$.

In fact, by symmetry the inclusions then hold if we replace E_Q by E_P, too. The assertions (1) and (2) of the theorem then can easily be verified.

(i) From $(*)$ we observe

$$\ln\alpha_n=\ln\varrho_n-\frac{1}{2}\frac{(X_n-a_n)^2}{\lambda_n^2}+\frac{1}{2}\frac{(X_n-b_n)^2}{\lambda_n^2}$$

for all $n \in D$ and $\ln \alpha_n = 0$ otherwise. Note that the sequence $(X_n - a_n)/\lambda_n$ is independent with mean zero and dispersion 1 with respect to $\boldsymbol{Q}$. From this we obtain

$$\boldsymbol{E}_Q \ln \alpha_n = \ln \varrho_n - \frac{\varrho_n^2 - 1}{2\varrho_n^2} + \frac{1}{2} \boldsymbol{E}_Q \frac{(a_n - b_n)^2}{\sigma_n^2}$$

for all $n \in D$. Notice that the convergence of $\sum_{n \in D} (\ln \varrho_n)^2$ implies the absolute convergence of $\sum_{n \in D} \left(\ln \varrho_n - \frac{1}{2} \frac{\varrho_n^2 - 1}{\varrho_n^2} \right)$. Furthermore, in view of $\lim \varrho_n = 1$ the series of

$$\boldsymbol{E}_Q \frac{(a_n - b_n)^2}{\sigma_n^2} \quad \text{and} \quad \boldsymbol{E}_Q \frac{(a_n - b_n)^2}{\lambda_n^2 + \sigma_n^2}$$

converge simultaneously. Consequently,

$$\sum_{n=0}^{\infty} \boldsymbol{E}_Q \ln \alpha_n < +\infty .$$

From this the assertion follows.(3)

(ii) Let us assume that $\boldsymbol{Q}$ and $\boldsymbol{P}$ are not singular. By the lemma proven above, we obtain for $p = 2$

$$\sum_{n=0}^{\infty} \ln \boldsymbol{E}_P (\alpha_{n+1}^{1/2} | \mathscr{F}_n) > -\infty$$

with positive probability with respect to $\boldsymbol{P}$. An easy computation shows that for $n \in D$

$$\ln \boldsymbol{E}_P (\alpha_{n+1}^{1/2} | \mathscr{F}_n) = \frac{1}{2} \ln \frac{2\varrho_n}{\varrho_n^2 + 1} - \frac{(a_n - b_n)^2}{4(\lambda_n^2 + \sigma_n^2)} .$$

Because both terms are nonpositive we get

$$\sum_{n \in D}^{\infty} \ln \frac{2\varrho_n}{\varrho_n^2 + 1} > -\infty \quad \text{and} \quad \boldsymbol{P}\left(\sum_{n \in D} \frac{(a_n - b_n)^2}{\lambda_n^2 + \sigma_n^2} < +\infty \right) > 0 .$$

But the convergence of the first series is equivalent to the convergence of $\sum_{n \in D} (\ln \varrho_n)^2$. We now use that $\left((a_n - b_n)/\sqrt{\lambda_n^2 + \sigma_n^2}\right)_{n \geq 0}$ is Gaussian with respect to $\boldsymbol{Q}$ and $\boldsymbol{P}$. Therefore

$$\sum_{n \in D} \frac{(a_n - b_n)^2}{\lambda_n^2 + \sigma_n^2} < +\infty \quad \boldsymbol{P}\text{-a.s.}$$

Since $\boldsymbol{Q}$ and $\boldsymbol{P}$ are not singular we get that

$$\boldsymbol{Q}\left(\sum_{n \in D} \frac{(a_n - b_n)^2}{\lambda_n^2 + \sigma_n^2} < +\infty \right) > 0 .$$

(3) This fact is well known: We have $\sum_{k=0}^{n} \boldsymbol{E}_Q \ln \alpha_k = \boldsymbol{E}_Q \ln Z_n = \boldsymbol{E}_P Z_n \ln Z_n$. Therefore, $\sup_n \boldsymbol{E}_P Z_n \ln Z_n < +\infty$ and the martingale $(Z_n, \mathscr{F}_n, P)$ is uniformly integrable. This yields $\boldsymbol{Q} \ll \boldsymbol{P}$.

Consequently,

$$\sum_{n \in D} \boldsymbol{E}_Q \frac{(a_n - b_n)^2}{\lambda_n^2 + \sigma_n^2} < +\infty .$$

Thus (ii) is established. This completes the proof of the theorem.

Non-homogeneous Markov chains

Let $(X_n)_{n \geq 0}$ be a random sequence defined on $(\Omega, \mathscr{F}, \boldsymbol{P})$ and taking values in measurable spaces $(E_n, \mathscr{E}_n)$. We call $(X_n)_{n \geq 0}$ a *Markov chain* if there is a transition probability $P^{(n)}$ of $(E_n, \mathscr{E}_n)$ in $(E_{n+1}, \mathscr{E}_{n+1})$ (i.e., a function $(x, A) \rightsquigarrow P^{(n)}(x, A)$ for $x \in E_n$ and $A \in \mathscr{E}_{n+1}$ which is $\mathscr{E}_n$-measurable in x and a probability measure in A) such that

$$\boldsymbol{P}(X_{n+1} \in A | \mathscr{F}_n^X) = P^{(n)}(X_n, A) \quad \boldsymbol{P}\text{-a.s.}$$

for every $A \in \mathscr{E}_{n+1}$ and $n \geq 0$. A Markov chain $(X_n)_{n \geq 0}$ is called *homogeneous* if $(E_n, \mathscr{E}_n)$ and $P^{(n)}$ do not depend on n and, moreover, there is a translation operator Θ satisfying the condition

$$X_n(\Theta \omega) = X_{n+1}(\omega) \quad \text{for} \quad n \geq 0 \text{ and } \omega \in \Omega . \tag{1}$$

We say that $(X_n)_{n \geq 0}$ is a 0-1 sequence if the asymptotical σ-algebra $\bigcap_n \sigma(X_k, k \geq n)$ only consists of events of probability 0 or 1. Notice that $(X_n)_{n \geq 0}$ is a 0-1 sequence with respect to $\boldsymbol{P}$ if $(X_n)_{n \geq 0}$ is a 0-1 sequence with respect to a probability measure $\tilde{\boldsymbol{P}}$ and $\boldsymbol{P} \ll \tilde{\boldsymbol{P}}$.

The following theorem is due to Lodkin [5]. We give a simpler proof of it.

13. THEOREM. *Let $(X_n)_{n \geq 0}$ be a Markov chain with respect to both $\boldsymbol{Q}$ and $\boldsymbol{P}$ and suppose that $(X_n)_{n \geq 0}$ is a 0-1 sequence with respect to $\boldsymbol{Q}$. If $\boldsymbol{Q}_n \ll \boldsymbol{P}_n$ for all $n \geq 0$, then either $\boldsymbol{Q} \ll \boldsymbol{P}$ or $\boldsymbol{Q} \perp \boldsymbol{P}$.*

Proof. Let $R^{(n)}$ be any transition probability of $(E_n, \mathscr{E}_n)$ in $(E_{n+1}, \mathscr{E}_{n+1})$ such that $Q^{(n)}(x, \cdot) \ll R^{(n)}(x, \cdot)$ and $P^{(n)}(x, \cdot) \ll R^{(n)}(x, \cdot)$ for every $x \in E_n$. We can take, for example, $R^{(n)} = \frac{1}{2}(Q^{(n)} + P^{(n)})$. Define

$$p_n(x, y) = \frac{dP^{(n)}(x, \cdot)}{dR^{(n)}(x, \cdot)}(y)$$

and

$$q_n(x, y) = \frac{dQ^{(n)}(x, \cdot)}{dR^{(n)}(x, \cdot)}(y)$$

for all $x \in E_n$ and $y \in E_{n+1}$. Let f be the Radon–Nikodym derivative of the initial distributions of $(X_n)_{n \geq 0}$ with respect to $\boldsymbol{Q}$ and $\boldsymbol{P}$. It can easily be verified that

$$Z_n = \frac{d\boldsymbol{Q}_n}{d\boldsymbol{P}_n} = f(X_0) \cdot \frac{q_0(X_0, X_1) \cdot \ldots \cdot q_{n-1}(X_{n-1}, X_n)}{p_0(X_0, X_1) \cdot \ldots \cdot p_{n-1}(X_{n-1}, X_n)} \tag{2}$$

Q-a.s. and P-a.s. for every $n \geq 1$. (For this we need no measurability assumptions of p_n and q_n in x). Obviously, the event $\{Z_\infty = \infty\}$ belongs to the asymptotical σ-algebra. Since $(X_n)_{n\geq 0}$ is a 0-1 sequence with respect to Q it follows $Q(Z_\infty = \infty) = 0$ or 1 and by Theorem 5 either $Q \ll P$ or $Q \perp P$.

Homogeneous Markov chains

In this section we present some interesting and, as far as we know, new results on the dichotomy $Q \ll P$ or $Q \perp P$ in the homogeneous Markov case.

Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with respect to both Q and P, taking values in an arbitrary state space $(E, \mathscr{E})$.

14. THEOREM. *Let $(X_n)_{n\geq 0}$ be stationary and ergodic with respect to Q. If $Q_n \ll P_n$ for all $n \geq 0$, then either $Q \ll P$ or $Q \perp P$.*

Proof. From (2) we see that for the Radon–Nikodym derivative $Z_n = dQ_n/dP_n$

$$\ln Z_n = \ln f(X_0) + \sum_{k=0}^{n-1} \ln \frac{q(X_k, X_{k+1})}{p(X_k, X_{k+1})}$$

Q-a.s. and P-a.s. for all $n \geq 0$. This yields that the event $\{Z_\infty = \infty\} = \{\limsup_n \ln Z_n = \infty\}$ is invariant with respect to the translation operator Θ. Since Q is ergodic we obtain $Q(Z_\infty = \infty) = 0$ or 1. By Theorem 5, $Q \ll P$ or $Q \perp P$.

It should be noticed that in the stationary case every 0-1 sequence is ergodic. The converse is not true and thus our assumption of ergodicity is weaker.

It is worth noting the following simple fact. Let $\tilde{P}$ and P be two probability measures such that either $\tilde{P} \ll P$ or $\tilde{P} \perp P$. Let Q be a third probability measure satisfying $Q \ll \tilde{P}$. Then the alternative $Q \ll P$ or $Q \perp P$ holds. This allows us to apply Theorem 14 to the non-stationary case.

We now weaken the assumption of Theorem 14 that Q is stationary and ergodic.

15. THEOREM. *Suppose that there exists a probability measure $\tilde{Q}$ on $\mathscr{F}$ such that the following two conditions are satisfied:*

(1) *$\tilde{Q}$ is stationary and ergodic.*

(2) *$Q \ll \tilde{Q}$.*

If $Q_n \ll P_n$ for all $n \geq 0$, then either $Q \ll P$ or $Q \perp P$.

Proof. We know that the event $\{Z_\infty = \infty\}$ is invariant and therefore $\tilde{Q}(Z_\infty = \infty) = 0$ or 1. Consequently, $Q(Z_\infty = \infty) = 0$ or 1, proving the theorem because of Theorem 5.

16. COROLLARY. *Let $(X_n)_{n\geq 0}$ be Markov with respect to $\tilde{Q}$, having the same transition probability as Q, such that $Q_0 \ll \tilde{Q}_0$. Suppose, moreover, that $\tilde{Q}$ is stationary and ergodic.*

If $Q_n \ll P_n$ for all $n \geq 0$, then either $Q \ll P$ or $Q \perp P$.

Indeed, one easily verifies $Q \ll \tilde{Q}$.

We close the paper with some applications to homogeneous Markov chains in a discrete state space.

17. THEOREM. *Let E be countable. Let E_0 be a non-empty positive recurrent class and assume $Q(X_0 \in E_0) = 1$.*

If $Q_n \ll P_n$ for all $n \geq 0$, then the alternative $Q \ll P$ or $Q \perp P$ holds.

Proof. Let $X(\omega) = (X_n(\omega))_{n\geq 0}$ for all $\omega \in \Omega$ and denote the image probability measures of Q and P on the space of all sequences of elements from E by Q^X and P^X, respectively. We then have $Q \ll P$ if and only if $Q^X \ll P^X$ and $Q \perp P$ if and only if $Q^X \perp P^X$. Therefore we can assume without loss of generality that $\Omega = E^{\{0,1,2,\ldots\}}$ and $X_n(\omega) = \omega(n)$ for all $\omega \in \Omega$ and $n \geq 0$.

Let μ be the unique stationary initial distribution for the transition probability $Q(x, A)$ which is carried by E_0. By $\tilde{Q}$ we denote the unique probability measure on $\mathscr{F}$ satisfying the property that $(X_n)_{n\geq 0}$ is Markov with respect to it, admitting the transition probability $Q(x, A)$ and having initial distribution μ. The Markov chain $(X_n)_{n\geq 0}$ then is stationary and ergodic with respect to $\tilde{Q}$. Notice that $\mu(\{x\}) > 0$ for all $x \in E_0$. Consequently, $Q_0 \ll \tilde{Q}_0$ and the assertion of the theorem follows from Corollary 16.

Without the assumption $Q(X_0 \in E_0) = 1$ for some positive recurrent class E_0 the statement of Theorem 17 does not remain valid. We give a simple example for illustrating this situation.

18. EXAMPLE. Let $E = \{1, 2, \ldots, 2N\}$ and define the transition probabilities

$$q_{ij} = \begin{cases} \mu_1(\{j\}) & \text{if } i \leq N \text{ and } j \leq N, \\ \mu_2(\{j\}) & \text{if } i > N \text{ and } j > N, \\ 0 & \text{otherwise}, \end{cases}$$

$$p_{ij} = \begin{cases} \mu_1(\{j\}) & \text{if } i \leq N \text{ and } j \leq N, \\ \mu_3(\{j\}) & \text{if } i > N \text{ and } j > N, \\ 0 & \text{otherwise}, \end{cases}$$

where μ_1, μ_2, and μ_3 are probability measures on E such that μ_1 is concentrated on $\{1, \ldots, N\}$, μ_2 and μ_3 are concentrated on $\{N+1, \ldots, 2N\}$, $\mu_3(\{j+N\}) = \mu_1(\{j\})$, and $\mu_2 \ll \mu_3$. Of course, we assume $\mu_2 \neq \mu_3$. For any initial distribution μ, let Q_μ and P_μ be the probability measures on the space of all sequences of elements from E constructed from q_{ij} and p_{ij} and μ by the theorem of Ionescu Tulcea. Obviously, $Q_{\mu_1} = P_{\mu_1}$. One easily verifies $Q_{\mu_2} \perp P_{\mu_3}$. Let now $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ and $\nu = \frac{1}{2}(\mu_1 + \mu_3)$. We then have neither $Q_\mu \ll P_\nu$ nor $Q_\mu \perp P_\nu$.

Finally, we consider one special case for which we always have the dichotomy $Q \ll P$ or $Q \perp P$.

19. THEOREM. *Let E be countable and suppose that $E = E_0 \cup E_t$, where E_0 is a non-empty positive recurrent class and E_t the set of transient states with respect to Q.*

If $Q_n \ll P_n$ for all $n \geq 0$, then the dichotomy "either $Q \ll P$ or $Q \perp P$" holds.

Proof. Let T be the first entry time in the set E_0. Of course, $Q(T < \infty) = 1$. Let $\Omega_T = \{T < \infty\}$ and define a random sequence $(Y_n)_{n \geq 0}$ on Ω_T by

$$Y_n(\omega) = X_{T+n}(\omega).$$

By Q^Y we denote the restriction of Q on $\mathscr{F}^Y$ (which is a σ-algebra of subsets of Ω_T). Set

$$P^Y(A) = \frac{P(A)}{P(T < \infty)}$$

for all $A \in \mathscr{F}^Y$. (Notice that in view of Corollary 9 we have $P(T < \infty) > 0$.) By the strong Markov property $(Y_n)_{n \geq 0}$ is Markov with respect to both Q^Y and P^Y, admitting the transition probabilities $Q(x, A)$ and $P(x, A)$, respectively. Now we observe $Q^Y(Y_0 \in E_0) = 1$. In view of Corollary 9 we find $Q_n^Y \ll P_n^Y$ for all $n \geq 0$. According to Theorem 17, $Q^Y \ll P^Y$ or $Q^Y \perp P^Y$. Using Theorem 5, we get $Q(Z_\infty^Y = \infty) = 0$ or 1, where $Z_n^Y = dQ_n^Y/dP_n^Y$ and $Z_\infty^Y = \limsup_n Z_n^Y$. From formula (2) we derive

$$Z_{T+n} = Z_T(Z_0^Y)^{-1} Z_n^Y \quad \text{on } \{T < \infty\} \quad P\text{-a.s. and } Q\text{-a.s.}$$

Consequently, $Q(Z_\infty = \infty) = 0$ or 1. Applying Theorem 5, the assertion now follows.

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ON MARKOVIAN DECISION PROCESSES WITH UNBOUNDED REWARDS

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1. Introduction

The concept of Markovian decision processes was first introduced by R. Bellman in 1957.

With the important contributions of R. A. Howard, D. Blackwell and others the mathematical foundations and the applications of this part of dynamic programming developed rapidly. It is very interesting to note how special applied topics make it necessary to extend the standard decision model.

If we consider e.g. queueing systems, it is natural to choose, for modelling, a countable state space — the number of customers waiting to be served — and an arbitrary action space. Furthermore, the queueing process carries rewards, where the negative rewards sometimes have a component that increases without bound with the state of the system.

When studying stochastic systems of this kind, it is not possible to apply directly the model of Blackwell [1], which assumes a uniformly bounded reward. Thus, we have two options: one is to transform the queueing model into a model of the uniformly bounded case, cf. [12], the other is to weaken Blackwell's assumption. The papers given by Harrison [4], [5], Lippman [8], [9], van Nunen [14], [15] and Wessels [16] about models whose state or action space is countable and which are sufficient for treating queueing systems lead in this direction.

This paper aims at giving a further generalization of Blackwell's model necessary for inventory systems.

An essential property of some inventory models is that their state and action space have the same structure.

Therefore, we consider Markovian decision processes with both state and action space being uncountable and rewards unbounded.

In Section 2 we shall outline fundamental definitions and results of the standard model (cf. [1]), which is to be modified in Section 3 and applied to an inventory system in Section 4. An elaborate discussion of problems and results of this paper may be found in [2], [3], [7] and [10].