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WEAK CONVERGENCE OF LINEAR RANK STATISTICS

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Introduction

We say a sequence $\{\xi_n\}$ of random variables *converges weakly* to the random variable ξ, and we write $\xi_n \Rightarrow \xi$ if $\mathscr{F}_n(x) \to \mathscr{F}(x)$ at continuity points x of $\mathscr{F}(x)$, where $\mathscr{F}_n(x)$ and $\mathscr{F}(x)$ are the distribution functions of ξ_n and ξ, respectively.

In this lecture the author summarized a part of his results about the weak convergence of linear rank statistics, which can be found in his papers [1], [2], [3], [4], and [5]. This paper contains these results.

In the first section the necessary definitions are given. In the second section one can find the mean limit theorem for the characteristic function of linear rank statistics, which is of fundamental importance for the whole theory of linear rank statistics. Then there follow some limit theorems as consequences of this fundamental theorem. The third section contains applications of the results of the second. Especially as a generalization of the well-known limit theorem of the Wilcoxon statistic, we give necessary and sufficient conditions for simple linear rank statistics to have a normal limit distribution. Then we give methods of constructing linear rank statistics with a given asymptotic with the aid of the Riemann integrability criterion, of mechanical quadrature, of Szegö's result concerning the eigenvalues of Toeplitz matrices, and of pseudo-random numbers.

The results obtained are suitable for constructing tests to decide whether two random variables have a common continuous distribution function or not, provided that at least one of the sample sizes is large.

1

Let $\{(x_1, \ldots, x_\nu)\} = \boldsymbol{R}_\nu$ be the real vector space of dimension $\nu = m+n$, where $m > 0$, $n \geqslant 0$ are integers. Denote by ω_0 the set of the vectors of $\boldsymbol{R}_\nu$ for which at least two components are equal.

Let the components of the vector $(x_1, \ldots, x_\nu)$ be pairwise different from one another. If the rearrangement according to size $z_1 < \ldots < z_\nu$ of those numbers gives $x_k = z_{r_k}$, then we say that x_k *has rank* r_k, $\operatorname{rank} x_k = r_k$.

Let

$$\omega_{r_1\ldots r_m} = \{(x_1, \ldots, x_\nu) \in \boldsymbol{R}_\nu |\ x_j \neq x_k,\ \ j \neq k;\ \ \operatorname{rank} x_k = r_k\ (k = 1, \ldots, m)\},$$
$$(r_1, \ldots, r_m) \in \Pi_m^{(\nu)},$$

where $\Pi_m^{(\nu)}$ denotes the set of all $r_1 < \ldots < r_m$ chosen without repetition from the elements $1, \ldots, \nu$.

Obviously, these sets and ω_0 are mutually disjoint and their union is equal to $\boldsymbol{R}_\nu$.

Let the random vector variable $\zeta_\nu = (\xi_1, \ldots, \xi_\nu)$ on the probability space $(\Omega, \mathscr{A}, P)$ be given. Let the joint distribution function of the components of ζ_ν be continuous in each of the variables. Then $P(\zeta_\nu \in \omega_0) = 0$. Let

$$P(\zeta_\nu \in \omega_{r_1\ldots r_m}) = p_{r_1\ldots r_m}, \qquad (r_1, \ldots, r_m) \in \Pi_m^{(\nu)}, \tag{1.1}$$

where

$$p_{r_1\ldots r_m} \geqslant 0, \qquad \sum_{(r_1,\ldots,r_m)\in\Pi_m^{(\nu)}} p_{r_1\ldots r_m} = 1.$$

Let the probabilities (1.1) be given in the cases

$$\nu = m+n; \qquad m = 1, 2, \ldots; \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

We denote the totality of these probabilities by $\mathscr{P}$.

Let the triangular matrices

$$A_k = \begin{bmatrix} a_{11}^{(k)} & & & \\ a_{21}^{(k)} & a_{22}^{(k)} & & \\ \multicolumn{4}{c}{\dotfill} \\ a_{\nu 1}^{(k)} & a_{\nu 2}^{(k)} & \ldots & a_{\nu\nu}^{(k)} \\ \multicolumn{4}{c}{\dotfill} \end{bmatrix} \qquad (k = 1, 2, \ldots) \tag{1.3}$$

with real elements be given.

DEFINITION 1.1. By the *linear rank statistic* $\{A_k, \mathscr{P}\}$ we mean the stochastic process

$$\xi_{m,n} = \eta_{\nu m}^{(1)} + \ldots + \eta_{\nu m}^{(m)},$$

provided

$$P(\eta_{\nu m}^{(1)} = a_{\nu r_1}^{(1)}, \ldots, \eta_{\nu m}^{(m)} = a_{\nu r_m}^{(m)}) = p_{r_1\ldots r_m}, \qquad (r_1, \ldots, r_m) \in \Pi_m^{(\nu)},$$

where the integers ν, m, and n are defined by (1.2).

DEFINITION 1.2. We say that the linear rank statistic $\{A_k, \mathscr{P}\}$ *has asymptotic distributions* if for any integer $m \geqslant 1$ there exists a random variable ξ_m such that $\xi_{m,n} \Rightarrow \xi_m$, $n \to \infty$.

DEFINITION 1.3. We say that the linear rank statistic $\{A_k, \mathscr{P}\}$ *has a doubly asymptotic distribution* if there exists a random variable ξ such that $\xi_{m,n} \Rightarrow \xi$, if $n \to \infty$, then $m \to \infty$.

Clearly, a linear rank statistic having asymptotic ξ_m $(m = 1, 2, \ldots)$ distributions is doubly asymptotically distributed if and only if $\xi_m \Rightarrow \xi$.

2

In the sequel we investigate only the special cases of the probabilities (1.1) suitable for getting an asymptotic theorem for the random variable $\xi_{m,n}$.

If $A = (a_{jk})$ is a square matrix of order n with complex elements, then the permanent sum for A, denoted by $\operatorname{Per} A$, is defined as follows:

$$\operatorname{Per} A = \sum_{(i_1,\ldots,i_n)} a_{1i_1} \ldots a_{ni_n},$$

where $(i_1, \ldots, i_n)$ runs over the full symmetric group.

(a) Let the triangular stochastic matrix

$$\mathscr{S} = \begin{bmatrix} p_{11} & & & \\ p_{21} & p_{22} & & \\ \multicolumn{4}{c}{\dotfill} \\ p_{\nu 1} & p_{\nu 2} & \ldots & p_{\nu\nu} \\ \multicolumn{4}{c}{\dotfill} \end{bmatrix} \tag{2.1}$$

be given, i.e. let

$$p_{\nu j} \geqslant 0, \qquad \sum_{j=1}^{\nu} p_{\nu j} = 1 \qquad (j = 1, \ldots, \nu;\ \nu = 1, 2, \ldots).$$

Denote by p_ν and $\bar{p}_\nu$ the maximum and the minimum of the elements $p_{\nu 1}, \ldots, p_{\nu\nu}$, respectively.

The matrix $\mathscr{S}$ is said to be a matrix *of type R* if $\lim\limits_{\nu\to\infty} \bar{p}_\nu = 0$, *of Poisson type* if $\lim\limits_{\nu\to\infty} \nu p_\nu = \lim\limits_{\nu\to\infty} \nu \bar{p}_\nu = 1$, and *symmetric* if $p_{\nu j} = p_{\nu\nu-j+1}$ $(j = 1, \ldots, \nu;\ \nu = 1, 2, \ldots)$. If $\mathscr{S}$ is a stochastic matrix of the Poisson type, then it is also of type R, but not conversely.

Let the stochastic matrices

$$\mathscr{S}_k = \begin{bmatrix} p_{11}^{(k)} & & & \\ p_{21}^{(k)} & p_{22}^{(k)} & & \\ \multicolumn{4}{c}{\dotfill} \\ p_{\nu 1}^{(k)} & p_{\nu 2}^{(k)} & \ldots & p_{\nu\nu}^{(k)} \\ \multicolumn{4}{c}{\dotfill} \end{bmatrix} \qquad (k = 1, 2, \ldots) \tag{2.2}$$

be given. Denote by $B_{m\nu}$ the matrix $(\nu = m+n)$

$$B_{m\nu} = \begin{bmatrix} p_{\nu 1}^{(1)} \ldots p_{\nu\nu}^{(1)} \\ \ldots\ldots\ldots \\ p_{\nu 1}^{(m)} \ldots p_{\nu\nu}^{(m)} \end{bmatrix} = (B^{(1)} \ldots B^{(\nu)}),$$

where $B^{(r)}$ is obviously the rth column of matrix $B_{m\nu}$.

We introduce the following matrix operation:

$$G(B_{m\nu}) = \sum_{(k_1,\ldots,k_m)\in\Pi_m^{(\nu)}} \operatorname{Per}(B^{(k_1)} \ldots B^{(k_m)}).$$

Let the stochastic matrices (2.2) and matrices (1.3) be given.

DEFINITION 2.1. By the *linear rank statistic* $\{A_k, \mathcal{S}_k\}$ we mean the linear rank statistic $\{A_k, \mathcal{P}\}$ generated by

$$p_{r_1\dots r_m} = \frac{\operatorname{Per}(B^{(r_1)}\dots B^{(r_m)})}{G(B_{m\nu})}, \qquad (r_1, \dots, r_m) \in \Pi_m^{(\nu)},$$

provided

$$\lim_{n\to\infty} G(B_{m\nu}) = 1, \tag{2.3}$$

where integers ν, m and n are defined by (1.2).

We can prove ([3], Theorem 1) that if the matrices $\mathcal{S}_k$ $(k = 1, 2, \dots)$ are of type R, then limit (2.3) exists.

DEFINITION 2.2. We say that the linear rank statistic $(A_k, \mathcal{S}_k)$ is a linear rank statistic *of type* R, or *of Poisson type*, or *symmetric* if all elements of sequence $\{\mathcal{S}_k\}$ are stochastic matrices of type R, of Poisson type, or symmetric, respectively.

In the limit theory of linear rank statistics $\{A_k, \mathcal{S}_k\}$ as we have defined them, a role of fundamental importance is played by the following theorem ([3], Theorem 3).

THEOREM 2.1. *Let the linear rank statistic* $\{A_k, \mathcal{S}_k\}$ *be given. If* $\varphi_{m,n}(t)$ *denotes the characteristic function of random variable* $\xi_{m,n}$, *then uniformly at* $t \in \boldsymbol{R}_1$

$$\lim_{n\to\infty}\left[\varphi_{m,n}(t) - \frac{\operatorname{Per}\Phi_\nu(t)}{m!\,G(B_{m\nu})}\right] = 0 \qquad (m = 1, 2, \dots)$$

with $(\nu = m+n)$

$$\Phi_\nu(t) = \begin{bmatrix} \varphi_{11}^{(\nu)}(t) \dots \varphi_{1m}^{(\nu)}(t) \\ \dots\dots\dots\dots \\ \varphi_{m1}^{(\nu)}(t) \dots \varphi_{mm}^{(\nu)}(t) \end{bmatrix},$$

where $\varphi_{jk}^{(\nu)}(t)$ *is the characteristic function of the random variable which takes the values* $a_{\nu 1}^{(j)}, \dots, a_{\nu\nu}^{(j)}$ *with probabilities* $p_{\nu 1}^{(k)}, \dots, p_{\nu\nu}^{(k)}$, *respectively.*

By using the Continuity Theorem of the sequences of characteristic functions ([6], Theorem 3.6.1), the following theorems are direct consequences of Theorem 2.1 ([3], Theorems 4 and 5).

THEOREM 2.2. *The linear rank statistic* $\{A_k, \mathcal{S}_k\}$ *has asymptotic distributions if and only if there exists a limit of the sequence*

$$\left\{\frac{1}{m!}\operatorname{Per}\Phi_\nu(t)\right\}_{n=0}^{\infty}, \qquad t \in \boldsymbol{R}_1 \ (m = 1, 2, \dots) \tag{2.4}$$

and this limit is continuous at the origin. In this case uniformly at $t \in \boldsymbol{R}_1$

$$\lim_{n\to\infty}\varphi_{m,n}(t) = \frac{1}{m!}\lim_{n\to\infty}\Phi_\nu(t) \qquad (m = 1, 2, \dots).$$

THEOREM 2.3. *The linear rank statistic* $\{A_k, \mathcal{S}_k\}$ *has a doubly asymptotic distribution if and only if sequence* (2.4) *has a limit as* $n \to \infty$ *and then* $m \to \infty$, *and this limit is continuous at the origin. In this case uniformly at* $t \in \boldsymbol{R}_1$

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\varphi_{m,n}(t) = \lim_{m\to\infty}\frac{1}{m!}\lim_{n\to\infty}\Phi_\nu(t).$$

(b) Let $\mathcal{S}_k = \mathcal{S}$ $(k = 1, 2, \dots)$, where $\mathcal{S}$ is a stochastic matrix of the form (2.1). The corresponding linear rank statistic will be denoted by $\{A_k, \mathcal{S}\}$. We can prove ([3], Corollary 1) that $\{A_k, \mathcal{S}\}$ is a linear rank statistic if and only if $\mathcal{S}$ is a stochastic matrix of type R.

Since in this case matrix $B_{m\nu}$ consists of columnwise equal elements, the corresponding theorems of 2.1, 2.2, and 2.3 have a simpler form ([3], Theorems 6, 7, 8).

Let $\eta_j^{(\nu)}$ be the random variable which takes values $a_{\nu 1}^{(j)}, \dots, a_{\nu\nu}^{(j)}$ with probabilities $p_{\nu 1}, \dots, p_{\nu\nu}$, respectively. From Theorem 2.2 we get the following result, very useful in applications ([3], Theorem 9):

THEOREM 2.4. *If for each matrix* A_k *from the linear rank statistic* $\{A_k, \mathcal{S}\}$ *there exists a random variable* η_k *such that* $\eta_k^{(\nu)} \Rightarrow \eta_k$, $n \to \infty$, *then this linear rank statistic has asymptotic* $\eta_1 + \dots + \eta_m$ *distributions* $(m = 1, 2, \dots)$ *and the random variables* $\eta_1, \eta_2, \dots$ *are independent.*

In applications the special case $\mathcal{S} = \mathcal{S}_0$ plays an important role, where $\mathcal{S}_0$ is the so-called *arithmetic mean matrix*, i.e.

$$\mathcal{S}_0 = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \dots & \dots & \dots & \\ \frac{1}{\nu} & \frac{1}{\nu} & \dots & \frac{1}{\nu} \\ \dots & \dots & \dots & \end{bmatrix}.$$

It is easily seen that $\{A_k, \mathcal{S}_0\}$ is a linear rank statistic of Poisson type and symmetric.

The basis of the applicability of linear rank statistic $\{A_k, \mathcal{S}_0\}$ is the following. If the joint distribution function of the identically distributed random variables $\xi_1, \dots, \xi_\nu$ is symmetric for one of its variables, and is continuous in each of the variables, then ([7], 363, Satz 10) probabilities (1.1) are equal to $\binom{n+m}{m}^{-1}$. The conditions listed will be satisfied if $\xi_1, \dots, \xi_\nu$ are identically distributed, independent random variables with continuous distribution function.

3

In this section we deal with applications of the theorems of Section 2. Especially we give methods of constructing linear rank statistics with given asymptotic.

(a) First we give necessary and sufficient conditions for a simple linear rank statistic to have a doubly asymptotic normal distribution.

DEFINITION 3.1. We say that the linear rank statistic $\{A_k, \mathscr{S}_k\}$ is a *simple* linear rank statistic if $A_k = \alpha_k A_0$ $(k = 1, 2, \ldots)$, where $\alpha_k \neq 0$ is a real number and

$$A_0 = \begin{bmatrix} 1 & & & \\ 1 & 2 & & \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \ldots & \nu \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

THEOREM 3.1. *Let $\{\alpha_k A_0, \mathscr{S}_k\}$ be a symmetric simple linear rank statistic of Poisson type. Then*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} P\left(\frac{\xi_{m,n} - E(\xi_{m,n})}{D(\xi_{m,n})} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

if and only if the sequence $\{\alpha_k\}$ satisfies the condition

(3.1) $$\lim_{m\to\infty} \frac{\alpha_1^4 + \ldots + \alpha_m^4}{(\alpha_1^2 + \ldots + \alpha_m^2)^2} = 0.$$

Let $A_k = A_0$, $\mathscr{S}_k = \mathscr{S}_0$ $(k = 1, 2, \ldots)$. Let $\{A_0, \mathscr{S}_0\}$ be the corresponding linear rank statistic. $\{A_0, \mathscr{S}_0\}$ is the so-called *Wilcoxon statistic*, which plays an important role in the theory of rank tests. Obviously, the Wilcoxon statistic is a simple, symmetric linear rank statistic of Poisson type and satisfies condition (3.1). Therefore, on the basis of Theorem 3.1, we infer that the Wilcoxon statistic is doubly asymptotically normally distributed.

(b) In what follows we construct a linear rank statistics with given asymptotic distributions. They play an important role in the rank test theory. Namely, to use such linear rank statistic in the rank tests theory, it is sufficient for only one of the sample sizes to be large enough.

The decomposition of interval [0, 1] into disjoint subintervals realized by the points of division

(3.2) $$0 = x_{\nu 0} < x_{\nu 1} < \ldots < x_{\nu\nu} \quad (\nu = 1, 2, \ldots)$$

will be called a *distinguished decomposition sequence* if matrix (2.1), formed by the numbers

(3.3) $$p_{\nu j} = x_{\nu j} - x_{\nu j-1} \quad (j = 1, \ldots, \nu;\ \nu = 1, 2, \ldots),$$

is a stochastic matrix of type R.

On the basis of Theorem 2.4 we have the following theorem ([3], Theorem 14):

THEOREM 3.2. *Let* (3.2) *be a distinguished decomposition sequence of the interval* [0, 1], *and let $\mathscr{S}$ be the stochastic matrix of type R formed by the elements* (3.3). *Denote by $J_{\nu j}^{(k)}$ an arbitrary point in the interval determined by the points $x_{\nu j-1}$ and $x_{\nu j}$.*

If $\{f_k(x)\}$ is a sequence of Riemann-integrable functions on the interval [0, 1] *and the elements of matrix A_k are given by $a_{\nu j}^{(k)} = f_k(J_{\nu j}^{(k)})$, then the linear rank statistic $\{A_k, \mathscr{S}\}$ is asymptotically $f_1(\eta_1) + \ldots + f_m(\eta_m)$ $(m = 1, 2, \ldots)$ distributed, where $\eta_1, \eta_2, \ldots$ are independent random variables uniformly distributed in the interval* [0, 1].

We obtain hence the following corollary.

COROLLARY 3.1. *If $A_k = A$ $(k = 1, 2, \ldots)$ with*

$$A = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\nu} & \frac{2}{\nu} & \ldots & 1 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

then the linear rank statistic $\{A, \mathscr{S}_0\}$ is asymptotically $\eta_1 + \ldots + \eta_m$ $(m = 1, 2, \ldots)$ distributed, where $\eta_1, \eta_2, \ldots$ are independent random variables, uniformly distributed in the interval [0, 1].

The corollary just formulated is the case of a modified Wilcoxon statistic. Since $E(\eta_1) = 1/2$, $D^2(\eta_1) = 1/12$, $\eta_1 + \ldots + \eta_m$ is asymptotically normally distributed with expectation $m/2$ and with variance $m/12$.

(c) Let us suppose that the density function $p(x)$ of the random variable ξ defined on the interval $[a, b]$ is positive outside a set of measure zero. Let $\{\omega_\nu(x)\}$ be the sequence of orthogonal polynomials belonging to the density function $p(x)$. As is known, the roots $x_{\nu 1}, \ldots, x_{\nu\nu}$ of polynomial $\omega_\nu(x)$ fall into the interval $[a, b]$ and have multiplicity one. Let

$$l_{\nu j}(x) = \frac{\omega_\nu(x)}{(x - x_{\nu j})\omega'(x_{\nu j})}, \qquad C_{\nu j} = \int_a^b p(x) l_{\nu j}(x)\, dx.$$

On the basis of Theorem 2.4 we can prove the following statement ([3], Theorem 13).

THEOREM 3.3. *Matrix* (2.1) *formed with the quantities $p_{\nu j} = C_{\nu j}$ is a stochastic matrix of type R. Moreover, if $\{f_k(x)\}$ is a sequence of functions defined and continuous in the interval $[a, b]$, and the elements of matrix A_k are given by $A_{\nu j}^{(k)} = f_k(x_{\nu j})$, then the linear rank statistic $\{A_k, \mathscr{S}\}$ is asymptotically $f_1(\xi_1) + \ldots + f_m(\xi_m)$ $(m = 1, 2, \ldots)$ distributed, where $\xi_1, \xi_2, \ldots$ are independent random variables with a common density function $p(x)$.*

Similar results can be found in Chapter 4 of paper [3].

(d) We say that $T_n(f) = (\varphi(k-l))_{k,l=0}^{n}$ is the *Toeplitz matrix* of order $n+1$ generated by the Lebesgue integrable function $f(x)$ defined on interval $[-\pi, \pi]$ if

$$\varphi(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-txi} f(x)\, dx, \qquad t \in \boldsymbol{R}_1.$$

On the basis of Theorem 2.4 we get the following theorems ([1], Theorems 2.2, 2.4).

THEOREM 3.4. *Let* λ_{nk} $(k = 1, \ldots, n)$ *be the eigenvalues of the Toeplitz matrix of order n generated by the Lebesgue integrable function* $f(x)$ *defined in the interval* $[-\pi, \pi]$. *Let* $\mathscr{F}_k(x)$ $(k = 1, 2, \ldots)$ *be a continuous function defined on the narrowest interval containing the range of* $f(x)$. *If*

$$a_{\nu j}^{(k)} = \mathscr{F}_k(\lambda_{\nu j}) \quad (j = 1, \ldots, \nu;\ \nu, k = 1, 2, \ldots),$$

then the linear rank statistic $\{A_k, \mathscr{S}_0\}$ *is asymptotically* $\mathscr{F}_1(f(\eta_1)) + \ldots + \mathscr{F}_m(f(\eta_m))$ $(m = 1, 2, \ldots)$ *distributed, where* $\mathscr{S}_0$ *is the arithmetic mean matrix, and* $\eta_1, \eta_2, \ldots$ *are independent, uniformly distributed random variables in the interval* $[-\pi, \pi]$.

THEOREM 3.5. *If the expectation of the random variable* ξ *exists, and if its distribution function* $\mathscr{F}(x)$ *is continuous, strictly monotonously increasing in some interval* $[a, b]$, *and* $\mathscr{F}(a) = 0$, $\mathscr{F}(b) = 1$, *where* $a = -\infty$, $b = \infty$ *are also permitted, then the linear rank statistic defined in Theorem* 3.4 *with* $f(x) = \mathscr{F}^{-1}((x+\pi)/2\pi)$ *is asymptotically* $\mathscr{F}_1(f(\xi_1)) + \ldots + \mathscr{F}_m(f(\xi_m))$ $(m = 1, 2, \ldots)$ *distributed, where* $\xi_1, \xi_2, \ldots$ *are independent random variables with a common distribution function* $\mathscr{F}(x)$.

More similar results can be found in Chapters 2 and 3 of paper [1].

(d) Finally we give a theorem ([2], Theorem 3) to construct a linear rank statistics with a given asymptotic with the aid of pseudo-random numbers. More about this topic can be found in papers [2] and [5].

The following theorem is founded upon Theorem 2.4 and upon the well-known theorem of H. Weyl.

THEOREM 3.6. *If* $\{f_j(x)\}$ *is a sequence of Lebesgue-integrable functions defined on the interval* $[0, 1)$, *if moreover*, $x_j \in [0, 1)$, $\theta_j \in [0, 1)$ *is irrational and if*

$$x_{jk} = x_{jk-1} + \theta_j - [x_{jk-1} + \theta_j] \quad (k = 1, 2, \ldots;\ x_{0j} = x_j),$$

then the linear rank statistic $\{A_k, \mathscr{S}_0\}$ *generated by the elements*

$$a_{\nu j}^{k} = f_k(x_{kj-1}) \quad (j = 1, \ldots, \nu)$$

is asymptotically $f_1(\eta_1) + \ldots + f_m(\eta_m)$ $(m = 1, 2, \ldots)$ *distributed, where* $\eta_1, \eta_2, \ldots$ *are independent random variables uniformly distributed on the interval* $[0, 1)$.

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