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SEQUENTIAL STATISTICAL STRUCTURES

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1. Introduction

A sequential statistical procedure is characterized by the fact that the information of the sampling process at a given time n influences further sampling. This may be only the determination of the stopping point, but also the design of further observations (with fixed or random stopping), the determination of grouping intervals, etc.

In sequential sampling we deal with an on-line procedure of observations and actions by the statistician. For this purpose it is necessary to have

simple statistical decision procedures (tables, schemes) for decision making at every time point or

special numerical tools.

In both these directions many efforts have been made recently.

In view of practice there are

problems in which the sequential approach is necessary (for example, detection of a change-point in the distribution of quality parameters of produced units, decision about replacement of a technological equipment),

problems in which sequential statistical methods may be preferable (for example, in quality control of production),

problems in which the sequential approach is not applicable (for example, in certain biological trials).

Mathematical criteria of preferability of some statistical method are the sampling costs and the certainty of statistical decisions. These questions will be discussed in another paper. In this paper some basic concepts on the foundations of the sequential statistical approach will be given, concerning the determination of stopping rules, the selection of the observation structure and criteria of sufficiency in sequential statistical structures.

2. Observation scheme

Let $(\Omega, \mathfrak{S}, \mathfrak{P})$ be a given statistical structure. Ω is the sample space, consisting of points $(x_1, x_2, \ldots)$, $(x(t), t \in T)$ or others. (In the following we restrict ourselves to the discrete case.) $\mathfrak{S}$ is a σ-field on Ω, pointed out by the statistician. $\mathfrak{P} = \{P_\theta, \theta \in \Theta\}$ denotes a parametric family of probability measures.

In the sequential case we fix on $\mathfrak{S}$ a monotonic sequence of sub-σ-fields:

$$\{\mathfrak{S}_n, n \in N\}, \ \mathfrak{S}_n \subseteq \mathfrak{S}, \ \mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}, \quad n \in N.$$

Let

$$\mathfrak{S}_\infty := \sigma\{\bigcup_{n=1}^{\infty} \mathfrak{S}_n\}.$$

We connect with this sequence a time scale: At the point n we are able to observe events on $\mathfrak{S}_n$. (If a sequence of random variables $X_1, X_2, \ldots$ is observed, then $\mathfrak{S}_n$ may be the σ-field on Ω, induced by $X_1, X_2, \ldots, X_n$.)

3. Stopping time

Any stopping time τ must be a random variable on $(\Omega, \mathfrak{S}, \mathfrak{P})$ with values in N.

(a) *$\tau \equiv n$ for some $n \in N$ and all $\omega \in \Omega$.* This leads to a sample with a fixed sample size n.

(b) *τ is an arbitrary random time*, i.e. a random variable on $(\Omega, \mathfrak{S}, \mathfrak{P})$ with values in $\bar{N} = N \cup \{\infty\}$, without any relation to the sequence $\{\mathfrak{S}_n, n \in \bar{N}\}$. Such a time is not useful in practice, because the observation of $\{\tau = n\}$ requires the full observation of events from $\mathfrak{S}$.

(c) *τ is a Markov time in relation to the sequence $\{\mathfrak{S}_n, n \in N\}$.*

Definition. A function $\tau(\cdot)$ on Ω with values in $\bar{N}$ is called a *Markov time in relation to the sequence* $\{\mathfrak{S}_n, n \in N\}$ if, for all $n \in \bar{N}$, we have $\{\tau = n\} \in \bar{\mathfrak{S}}_n$. ($\bar{\mathfrak{S}}_n$ is the σ-field consisting of $\mathfrak{S}_n$ and all $A \in \mathfrak{S}$ with $P_\theta(A) = 0$ for all $\theta \in \Theta$.)

Remark. This definition is a mild generalization of the known definition ([7]). The observation of events in $\mathfrak{S}_n$ leads with $\mathfrak{P}$-probability 1 to the realisation of $\{\tau = n\}$ or $\{\tau > n\}$.

(d) *τ is a randomized time.* First we will give an extremely simple characterization of the randomized time, pointed out by R. Döhler [4].

Definition. A *random variable on* $(\Omega, \mathfrak{S}, \mathfrak{P})$ with values in $\bar{N}$ is a randomized time in relation to the sequence $\{\mathfrak{S}_n, n \in N\}$ if the event $\{\tau = n\}$ and the σ-field $\mathfrak{S}_\infty$ are conditionally independent under the condition $\mathfrak{S}_n$, i.e. with P_θ-probability 1 we have

$$P_\theta(\{\tau = n\} \cap A_\infty \mid \mathfrak{S}_n) = P_\theta(\tau = n \mid \mathfrak{S}_n) \cdot P_\theta(A_\infty \mid \mathfrak{S}_n),$$

$$\forall n \in N, \ \forall A_\infty \in \mathfrak{S}_\infty, \ \forall \theta \in \Theta.$$

We can prove that this is identical to

$$P_\theta(\tau = n \mid \mathfrak{S}_\infty) = P_\theta(\tau = n \mid \mathfrak{S}_n).$$

This equation shows that a randomized time is characterized by the fact that the conditional probability of $\{\tau = n\}$ is fixed after the observation of $\mathfrak{S}_n$ and is not changed by further observations.

Remarks 1. The given definition is identical to that given by D. Siegmund (see. [8]). But D. Siegmund uses an auxiliary sequence of σ-fields.

2. The Markov time is a special randomized time. For a Markov time τ we have

$$P_\theta(\tau = n \mid \mathfrak{S}_n) = P_\theta(\tau = n \mid \bar{\mathfrak{S}}_n) = \chi_{\{\tau = n\}} = P_\theta(\tau = n \mid \mathfrak{S}_\infty)$$

because of $\{\tau = n\} \in \bar{\mathfrak{S}}_n$, $\{\tau = n\} \in \mathfrak{S}_\infty$.

3. Another relation to the Markov time can be obtained in the following way.

Definition. Let τ be an arbitrary random time on $(\Omega, \mathfrak{S}, \mathfrak{P})$. $A \in \mathfrak{S}$ is called a *Markovian event for τ in relation to* $\{\mathfrak{S}_n, n \in N\}$ if

$$\{\tau = n\} \cap A \in \bar{\mathfrak{S}}_n, \quad \forall n \in \bar{N}.$$

We can obtain the following statements:

(1) *The random time τ is a Markov time iff Ω is a Markovian event. The class $\mathfrak{S}_\tau^M$ of all Markovian events in this case is a σ-field.*

(2) *The random time τ is a Markov time iff $\mathfrak{S}_\tau^M = \bar{\mathfrak{S}}_\tau$, where*

$$\mathfrak{S}_\tau = \sigma\{\mathfrak{S}_n \cap \{\tau = n\}, \ n \in \bar{N}\}.$$

(3) *The random time τ is a Markov time in relation to $\{\mathfrak{S}_n, n \in N\}$ iff for all $n \in N$ and $\theta \in \Theta$*

$$P_\theta(\tau = n \mid \sigma\{\mathfrak{S}_n \cap \{\tau \geqslant n\}, \ n \in N\}) \in \{0, 1\}, \quad P_\theta\text{-a.e.}$$

4. All interesting properties of a randomized time (such as the martingal properties for randomized stopped martingals, etc.) can be proved by using only the definition property. But we shall return to the statistical aspects.

4. Stopping rules (in the sense of R. Bahadur)

In the practical use of a randomized time the conditional probabilities $P_\theta(\tau = n \mid \mathfrak{S}_n)$ for all $n \in N$ must be given. This can be performed in a constructive way by using stopping rules in the sense of R. Bahadur.

Definition. A sequence of random variables $\{G_n, n \in N\}$ on $(\Omega, \mathfrak{S}, \mathfrak{P})$ is called a *stopping rule g if*

(1) $0 \leqslant G_n \leqslant 1$,

(2) $G_n(\cdot)$ is $\mathfrak{S}_n$-measurable, $n \in N$.

(In the paper of R. Bahadur [1] the G_n are Borelian functions on $X_1, X_2, \ldots, X_n$.)

By using the stopping rule g, we can realize an auxiliary random variable Z_n:

$$Z_n = \begin{cases} 1 & \text{with probability } 1-g_n \to \text{continuation},\\ 0 & \text{with probability } \quad g_n \to \text{stopping}. \end{cases}$$

Remark. A stopping rule g fixes a randomized time τ. For this τ the conditional probabilities $P_\theta(\tau = n \mid \mathfrak{S}_n)$ are independent of θ. (But the absolute probabilities $P_\theta(\tau = n)$ depend on θ. This is one of the reasons for studying randomized times: to realize τ with a prescribed function $P_\theta(\tau = n)$ of θ or $E_\theta \tau$.)

DEFINITION. A randomized time τ is called *parameter-free* if for the conditional probabilities $P_\theta(\tau = n \mid \mathfrak{S}_n)$ exist variants independent of θ. We use the notation $P_.(\tau = n \mid \mathfrak{S}_n.)$

THEOREM. *Any parametr-free randomized time τ can be realized by a stopping rule g.*

Proof. The proof is based on

$$G_n(\omega) = \begin{cases} 0 & \text{for } \omega \text{ with } P_.(\tau \geqslant n \mid \mathfrak{S}_n) = 0,\\ \dfrac{P_.(\tau = n \mid \mathfrak{S}_n)}{P_.(\tau \geqslant n \mid \mathfrak{S}_n)} & \text{otherwise}. \end{cases}$$

We can choose such variants of the conditional probabilities that $\{G_n,\ n \in N\}$ will be a stopping rule.

5. Sample space

Given the basic statistical structure, the sequence $\{\mathfrak{S}_n, n \in N\}$ and the randomized time τ in relation to $\{\mathfrak{S}_n\}$. The time τ restricts the σ-field of observable events. For example, instead of $X_1, X_2, \ldots$ we observe only $X_1, X_2, \ldots, X_\tau$. What, in these cases, is the σ-field of observable events for a given τ?

τ a Markov time. Let

$$\mathfrak{S}_\tau^M = \{A \in \mathfrak{S}\colon\ \{\tau \geqslant n\} \cap A \in \mathfrak{S}_n,\ \forall n \in N\};$$

$\mathfrak{S}_\tau^M$ is a σ-field. And conversely, if $\mathfrak{S}_\tau^M$ is a σ-field, then τ is a Markov time.

The σ-field $\mathfrak{S}_\tau^M$ is called the *observable σ-field to τ*. It contains especially the events $\{\tau = n\}$, $\{\tau \geqslant n\}$.

τ any randomized time. The σ-field of observable events must contain all events of the form $\{\tau \geqslant n\} \cap A_n$, $A_n \in \mathfrak{S}_n$, i.e. if we observe up to time n, we observe all events as the minimum up to n.

DEFINITION. Let τ be a randomized time. The σ-field

$$\mathfrak{S}_\tau = \sigma\{A_n \cap \{\tau \geqslant n\},\ A_n \in \mathfrak{S}_n,\ n \in N\}$$

is called the *σ-field of observable events for the time τ and the sequence* $\{\mathfrak{S}_n, n \in \bar{N}\}$.

COROLLARIES. (i) *We have*

$$\mathfrak{S}_\tau \equiv \sigma\{A_n \cap \{\tau = n\},\ A_n \in \mathfrak{S}_n,\ n \in \bar{N}\}.$$

Any event $A_\tau \in \mathfrak{S}_\tau$ has the representation $A_\tau = \bigcup_{n \in \bar{N}} (A_n \cap \{\tau = n\})$ with certain $A_n \in \mathfrak{S}_n$.

(ii) *τ is $\mathfrak{S}_\tau$-measurable. Any $\mathfrak{S}_\tau$-measurable random variable Y has the representation $Y = \sum_{n \in \bar{N}} \chi_{\{\tau=n\}} Y_n$ with $\mathfrak{S}_n$-measurable functions Y_n.*

DEFINITION. Let τ be a randomized time in relation to $\{\mathfrak{S}_n, n \in N\}$ and $\mathfrak{S}_\tau$ the σ-field of observable events. Then

$$(\Omega, \mathfrak{S}_\tau, \mathfrak{P}_\tau)$$

is called a *sequential statistical structure.* Here $\mathfrak{P}_\tau$ is the family $\{P_{\theta,\tau},\ \theta \in \Theta\}$ of measures restricted from measures $P_\theta \in \mathfrak{P}$ on $\mathfrak{S}$ to $\mathfrak{S}_\tau$.

Remark. The restriction $P_{\theta,\tau}$ from the P_θ can be obtained by using an extension of the space Ω by the results of the auxiliary experiments. We set up

$\Omega^0 := \Omega \times Z,$

$\quad Z$ the set of all sequences of 0 and 1,

$\quad \mathfrak{Z}$ σ-field on Z induced by the subsets $\{Z_n = i\}$, $i = 0, 1$;

$\mathfrak{S}^0 := \mathfrak{S} \otimes \mathfrak{Z},$

$\quad Q^\omega$ measure on $\mathfrak{Z}$ given by

$$Q^\omega(Z_n = 0) = G_n(\omega), \quad Q^\omega(Z_n = 1) = 1 - G_n(\omega),$$

$$Q^\omega\Big(\bigcap_{r=1}^{n} \{Z_r = i_r\}\Big) = \prod_{r=1}^{n} Q^\omega(Z_r = i_r), \quad i_r = 0, 1;$$

$$P_\theta^0(A \times B) := \int_A Q^\omega(B)\, dP_\theta, \quad A \in \mathfrak{S},\ B \in \mathfrak{Z},$$

can be extended to a measure P_θ^0 on $\mathfrak{S}^0$.

Thus we have the extended basic statistical structure $(\Omega^0, \mathfrak{S}^0, \mathfrak{P}^0)$, $\mathfrak{P}^0 = \{P_\theta^0,\ \theta \in \Theta\}$. The sequence $\{\mathfrak{S}_n,\ n \in \bar{N}\}$ is also changed by setting $\mathfrak{S}_n^0 := \mathfrak{S}_n \times Z$, $n \in \bar{N}$.

We define a new random variable τ^0 on $(\Omega^0, \mathfrak{S}^0, \mathfrak{P}^0)$ by

$$\tau^0(\omega, z_1, z_2, \ldots) = \begin{cases} \min[n\colon z_n = 0], \\ \infty \quad \text{if} \quad z_n = 1,\ n \in N. \end{cases}$$

τ^0 is a randomized time on $(\Omega^0, \mathfrak{S}^0, \mathfrak{P}^0)$ in relation to the sequence $\{\mathfrak{S}_n^0,\ n \in \bar{N}\}$. $(\Omega^0, \mathfrak{S}_{\tau^0}^0, \mathfrak{P}_{\tau^0}^0)$ is the extended sequential statistical structure. For the restrictions of the measures P_θ^0 to P_{θ,τ^0}^0 we have

$$P_{\theta,\tau^0}^0(A_n^0 \cap \{\tau^0 = n\}) = \int_{A_n} G_n \prod_{j=1}^{n-1} (1 - G_j)\, dP_\theta, \quad A_n \in \mathfrak{S}_n,$$

$$P_{\theta,\tau^0}^0(A_\infty^0 \cap \{\tau^0 = \infty\}) = \int_{A_\infty} \prod_{j=1}^{\infty} (1 - G_j)\, dP_\theta, \quad A_\infty \in \mathfrak{S}_\infty.$$

For any set $A_{\tau^0}^0 \in \mathfrak{S}_{\tau^0}^0$ we get the measure $P_{\theta,\tau^0}^0(A_{\tau^0}^0)$, using the representation

$$A_{\tau^0}^0 = \bigcup_{n \in \bar{N}} (A_n^0 \cap \{\tau^0 = n\}), \quad A_n^0 = A_n \times Z,\ A_n \in \mathfrak{S}_n.$$

6. Sufficiency

The concept of sufficiency is of principal importance in concrete decision problems.

Let $(\Delta, \mathfrak{D})$ be a decision space in which $d \in \Delta$ are the decisions and $\mathfrak{D}$ is a σ-field on Δ. Under a decision rule $S(\cdot,\cdot)$ we understand a transition probability on $\Omega \times D$, i.e. $S(\omega, \cdot)$ is for all $\omega \in \Omega$ a probability measure on $\mathfrak{D}$ and $S(\cdot, D)$ is a $\mathfrak{S}$-measurable function on Ω. To emphasize this we use the notation $S^{\mathfrak{S}}(\omega, D)$.

For comparison of decision rules the following definition is suitable.

DEFINITION. (a) $\lambda_0(D, S) := \int_\Omega S(\omega, D)\,dP_0$ is called the "image" of the decision rule S (J.-R. Barra [3]).

(b) The decision rules S_1 and S_2 are equivalent if

$$\lambda_\theta(D, S_1) = \lambda_\theta(D, S_2), \quad \forall D \in \mathfrak{D},\ \forall \theta \in \Theta.$$

(By equivalent decision rules the sets $D \in \mathfrak{D}$ get in the mean the same probabilities.)

The following question arises. Given a statistical structure $(\Omega, \mathfrak{S}, \mathfrak{P})$. When is it admissible to replace a decision rule $S^{\mathfrak{S}}(\cdot,\cdot)$ based on $\mathfrak{S}$ by a decision rule $S^{\mathfrak{T}}(\cdot,\cdot)$ based on a σ-field $\mathfrak{T}$ with $\mathfrak{T} \subseteq \mathfrak{S}$? The following theorem is well-known ([1]):

THEOREM. *Let $(\Omega, \mathfrak{S}, \mathfrak{P})$ be a statistical structure and let $\mathfrak{T}$, $\mathfrak{T} \subseteq \mathfrak{S}$, be a sufficient sub-$\sigma$-field for $(\mathfrak{S}, \mathfrak{P})$. Let $S^{\mathfrak{S}}(\cdot,\cdot)$ be a decision rule based on $\mathfrak{S}$. If, for all $D \in \mathfrak{D}$, there exists a variant*

$$E_\cdot(S^{\mathfrak{S}}(\omega, D) \mid \mathfrak{T}) := S^{\mathfrak{T}}(\omega, D)$$

being a probability measure on $(\Delta, \mathfrak{D})$ for all $\omega \in \Omega$, then the decision rule $S^{\mathfrak{T}}$ is equivalent to the rule $S^{\mathfrak{S}}$.

Remark. The last condition is satisfied, for example, if Δ is a Euclidean space and $\mathfrak{D}$ is the σ-field of Borel sets or if Δ is a complete separable metric space, $\mathfrak{D}$ the σ-field of Borel sets and the structure $(\Omega, \mathfrak{S}, \mathfrak{P})$ is dominated.

The theorem is also true in the sequential case if we replace the statistical structure by $(\Omega, \mathfrak{S}_\tau, \mathfrak{P}_\tau)$ and study the σ-field $\mathfrak{T} \subseteq \mathfrak{S}_\tau$. But here the following question arises: Given a sequence $\{\mathfrak{T}_n, n \in N\}$ of sub-σ-fields in $\mathfrak{S}$ with

$$\mathfrak{T}_n \subseteq \mathfrak{S}_n, \quad n \in \bar{N},$$

$$\mathfrak{T}_n \text{ sufficient (minimal-sufficient) for } (\mathfrak{S}_n, \mathfrak{P}^{\mathfrak{S}_n}).$$

($\{\mathfrak{T}_n, n \in N\}$ is called a *sufficient sequence of* σ-fields, it needs not to be a monotonic sequence.)

Does it follow from this that $\mathfrak{T}_\tau$ is a sufficient (minimal-sufficient) σ-field for $(\mathfrak{S}_\tau, \mathfrak{P}_\tau)$? We have the following assertions [5]:

(1) *If $\{\mathfrak{T}_n, n \in \bar{N}\}$ is a sufficient sequence of σ-fields and τ is a parameter-free randomized time, then $\mathfrak{T}_\tau$ is sufficient for $(\mathfrak{S}_\tau, \mathfrak{P}_\tau)$.*

(2) *From the minimal sufficiency of $\{\mathfrak{T}_n, n \in N\}$ the minimal-sufficiency of $\mathfrak{T}_\tau$ does not follow in general.*

Further investigations may be carried out for the case of $(\Omega, \mathfrak{S}, \mathfrak{P})$ being a finite or infinite product space.

PROBLEM. *Let τ be a parameter-free randomized time associated with the stopping rule g in relation to the sequence $\{\mathfrak{S}_n, n \in N\}$, $g = \{G_n(\omega), n \in N\}$. Let $(\mathfrak{T}_n, n \in N\}$ be a sufficient sequence for $\{\mathfrak{S}_n, n \in N\}$. Is it admissible to replace the stopping rule g by a rule $g^* = \{G_n^*(\omega), n \in N\}$ in which the G_n^* are only $\mathfrak{T}_n$-measurable and the sequential statistical structure will be the same?* The answer is positive if the sequence $\{\mathfrak{T}_n, n \in N\}$ is transitive.

DEFINITION. The sequence of sub-σ-fields $\{\mathfrak{T}_n, n \in \bar{N}\}$ for $\{\mathfrak{S}_n, n \in N\}$ is transitive if for all $\theta \in \Theta$:

(1) $\mathfrak{T}_n \subseteq \mathfrak{S}_n$, $n \in \bar{N}$,

(2) $\mathfrak{T}_{n+1}$ and $\mathfrak{S}_n$ are conditionally independent under the condition $\mathfrak{T}_n$,

(3) $\mathfrak{T}_\infty$ and $\mathfrak{S}_n$ are conditionally independent under the condition $\mathfrak{T}_n$.

Remark. For a sequence of random variables $X_1, X_2, \ldots$ the sequence of statistics $\{T_n(X_1, X_2, \ldots, X_n), n \in N\}$ is transitive if $T_{n+1} = f(T_n, X_{n+1})$, $n \in N$. In this case the sequence of induced by $\{T_n, n \in \bar{N}\}$ σ-fields is transitive.

THEOREM [4]. *Let $\{\mathfrak{T}_n, n \in \bar{N}\}$ be a sufficient sequence for $\{\mathfrak{S}_n, n \in \bar{N}\}$. For a given stopping rule g let*

$$G_n^*(\omega) = \begin{cases} 0 & \text{if } E_\cdot(\prod_{j<n}(1-G_j) \mid \mathfrak{T}_n) = 0, \\ \dfrac{E_\cdot(G_n \prod_{j<n}(1-G_j) \mid \mathfrak{T}_n)}{E_\cdot(\prod_{j<n}(1-G_j) \mid \mathfrak{T}_n)} & \text{otherwise.} \end{cases}$$

If and only if the sequence $\{\mathfrak{T}_n, n \in \bar{N}\}$ is transitive, the stopping rules g and $g^* = \{G_n^*(\omega), n \in N\}$ are equivalent in the sense that they induce the same measures on the σ-field $\mathfrak{T}^0_{\tau 0}$.

Remark. Even in the case of Markov time τ for which $G_n(\omega) \in \{0, 1\}$, $n \in N$ we have $G_n^*(\omega) \in [0, 1]$, i.e. g^* determines a randomized time. Thus the randomized times are inevitable for the formulation of the theorem. The simplification of the stopping rule by the sufficiency approach leads to the necessity of randomization.

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ON MULTIPLE TEST PROCEDURES

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Introduction

In many applications the statistical analysis is characterized by the fact that a number of detail questions should be answered and an overall view should be created by the totality of answers to the detail questions. Here are some examples of such situations:

A. The distribution of a random variable depends on a number of background variables. For each background variable the detail question is if the distribution of the random variable is influenced by this background variable. And the totality of the answer to these questions creates a picture of the dependence on the background variables. This kind of problems appears in many contexts.

B. In a comparison of some multidimensional random variable for two cases (e.g. treated and non-treated patients in a medical investigation) we may be interested in differences in the different components of the variable. These are the detail questions. But we are also probably interested in the differences in general, i.e. the totality of differences in all the components.

C. In an analysis of a stationary time series we may be interested in detail questions concerning the correlation at different time distances. But we may also be interested in getting a general picture of the dependence.

More examples of the same kind from different fields of applications are easily found. The examples are illustrations of *multiple statistical inference problems* where we have to take into consideration that we both want to answer detail questions and get a general view by the totality of answers to the detail questions. To make a test with conventional level of significance for each detail question is not good from an overall point of view. If we, for instance, make 40 independent tests of different detail hypotheses with level 0.05, we have a probability of only $0.95^{40} \approx 0.13$ that all hypotheses would be accepted if they were true. And there would be difficulties in getting a general view of the investigation if just a few hypotheses were rejected. The aim of these notes is to study the problem of constructing tests in such a way that their totality will give a general view in a reasonable way.