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ABOUT SECRETARY PROBLEMS

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0. Introduction

This paper deals with a generalization of the secretary problem where m of n $(1 \leqslant m \leqslant n)$ candidates are to be chosen without the possibility of return to already rejected objects. At each selection of a candidate the observation process stops. Therefore the optimal decision rule is a multiple stopping rule. In the first part of the paper we give a general solution of this problem. We then deal with secretary problems with interview cost. Finally the case $n \to \infty$ will be studied.

1. Denotations and general solution

Stopping problems have been considered by many authors in connection with the so-called secretary problem. There are for example Chow, Robbins, Siegmund [3] and Bartoszyński, Govindarajulu [1]. Here we deal with a similar problem which requires multiple stopping:

Given is a set of n objects such that we can decide between every two of them which is the better one. Without loss of generality we order the objects so that the best gets rank 1, the second rank 2 and so on. As in the secretary problem, we assume that all permutations of the n objects occur with equal probability. The objective is to minimize a given cost function which depends on the choice of m from among those n $(1 \leqslant m \leqslant n)$ objects. First of all the quality of the objects is unknown. But we have the possibility to observe them sequentially. For any object observed we have to decide on the strength of a comparison with already inspected objects whether to select it or not. A return to already rejected objects is not possible. The problem is to find a decision rule, i.e. a multiple stopping rule which minimizes the given cost function.

The following generalized version of the secretary problem is a special case of the above-mentioned problem which will be considered in the second part of the paper:

n candidates for m vacant secretarial positions will be interviewed one at a time in a random order. After every interview one can either decide to employ the candidate or to reject her. A rejected candidate is later no longer available. The objective is to choose good secretaries and is expressed in a cost function which sums up the rank numbers of the employed secretaries and the cost of the interviews. The problem is to determine a multiple stopping rule which minimizes this cost function.

We use the following notations:

$\boldsymbol{R}^1$ — set of real numbers.

N — set of observations; $N := \{1, 2, \ldots, n\}$.

R_i — absolute rank of the ith observed object; $i \in N$.

Y_i — relative rank of the ith observed object; $i \in N$. Y_i is the absolute rank of the ith object with respect to the first i objects.

V — set of relative ranks; $V := \{1, 2, \ldots, n\}$.

J_i — number of objects which are still to be chosen in the observations $i, i+1, \ldots, n$; $i \in N$.

M — set of possible numbers J_i; $M := \{1, 2, \ldots, m\}$; $\overline{M} := \{0, 1, \ldots, m\}$.

$D_i(y, j)$ — decision variable for the choice of the ith observed object with relative rank $Y_i = y$ if $J_i = j$ objects are still to be chosen; $(i, y, j) \in N \times V \times \overline{M}$. This means:

$$D_i(y, j) = \begin{cases} 1 & \text{Take the } i\text{th object!} \\ 0 & \text{Don't take the } i\text{th object!} \end{cases}$$

We set $D_i(y, 0) := 0$, $i \in N$, $y \in V$.

D — set of decisions; $D := \{0, 1\}$.

S_i — i-rank policy: $S_i | \{i, i+1, \ldots, n\} \times V \times M \to D$; $i \in N$. We have $S_i(k, y, j) = D_k(y, j)$. We denote a 1-rank policy as a multiple stopping rule.

$\overline{R}_i$ — set of all i-rank policies.

$w(i, y, j, S_i)$ — cost function at observation i for the expected cost in the time interval $i, i+1, \ldots, n$ if $Y_i = y$, $J_i = j$ and the i-rank policy S_i is used; $S_i \in \overline{R}_i$, $(i, y, j) \in N \times V \times M$, $y \leqslant i$.

$\overline{w}(i, j, S_i)$ — mean cost function; $S_i \in \overline{R}_i$, $(i, j) \in N \times M$.

(1) $$\overline{w}(i, j, S_i) := E_{Y_i} w(i, Y_i, j, S_i),$$

$u(h)$ — terminal cost; $h \in \overline{M}$. After the inspection of all n objects a penalty $u(h)$ will be realized which depends on the number $h = J_n - D_n(Y_n, J_n)$.

(2) $$\overline{w}(n+1, h, S_{n+1}) := u(h).$$

$v(i)$ — stopping cost; $i \in N$. Stopping cost $v(i)$ will be realized at the first observation i, where $J_i = 0$, which means that the observation process is stopped before i.

(3) $$\overline{w}(i, 0, S_i) := v(i).$$

$w^*(i, j)$ — value function at observation i if $J_i = j$ objects are still to be chosen; $i \in \{1, 2, \ldots, n+1\}$, $j \in \overline{M}$. We define

$$w^*(i, j) := \min_{S_i \in \overline{R}_i} \overline{w}(i, j, S_i) \quad \text{for all } (i, j) \in N \times M,$$
$$w^*(n+1, h) := u(h) \quad \text{for all } h \in \overline{M},$$
$$w^*(i, 0) := v(i) \quad \text{for all } i \in N.$$

Definition. An i-rank policy $S_i \in \overline{R}_i$ will be called *optimal* if for all $j \in M$

$$\overline{w}(i, j, S_i) = w^*(i, j); \quad i \in N.$$

The solution of the given problem consists in the determination of an optimal multiple stopping rule S_1^* which minimizes the value of the mean cost function $\overline{w}(1, \ldots, S_1)$ with respect to all $S_1 \in \overline{R}_1$. We solve the problem by Bellman's backward induction (dynamic programming). The applicability of this method is guaranteed by the following two assumptions:

A1: A transformation f, $f | N \times V \times M \times D \times \boldsymbol{R}^1 \to \boldsymbol{R}^1$, determines the cost function $w(i, y, j, d * S_{i+1})$ for all $(i, y, j) \in N \times V \times M$, $y \leqslant i$, i-rank policies $d * S_{i+1}$(1) and given $\overline{w}(i+1, j-d, S_{i+1})$ by the equation $w(i, y, j, d * S_{i+1}) = f(i, y, j, d, \overline{w}(i+1, j-d, S_{i+1}))$.

A2: The equation

$$\min_{S_{i+1} \in \overline{R}_{i+1}} f(i, y, j, d, \overline{w}(i+1, j-d, S_{i+1})) = f(i, y, j, d, \min_{S_{i+1} \in \overline{R}_{i+1}} \overline{w}(i+1, j-d, S_{i+1}))$$

is valid for all $(i, y, j, d) \in N \times V \times M \times D$, $y \leqslant i$.

These assumptions describe a large class of cost functions which includes also nonadditive ones. Now we can formulate the general solution:

Theorem. (i) *The value function is computed for all $(i, j) \in N \times M$ recursively in backward steps by the Bellman equation*

(4) $$w^*(i, j) = \frac{1}{i} \sum_{y=1}^{i} \min\big(f(i, y, j, 1, w^*(i+1, j-1)), f(i, y, j, 0, w^*(i+1, j))\big)$$

beginning with the terminal cost

(5) $$w^*(n+1, h) = u(h), \quad h \in \overline{M},$$

and the stopping cost

(6) $$w^*(k, 0) = v(k), \quad k \in N.$$

(1) The i-rank policy $d * S_{i+1}$ decides at observation i with $D_i(Y_i, J_i) \equiv d$ and at the remaining ones it behaves like S_{i+1}.

(ii) *A multiple stopping rule $S_1 \in \overline{R}_1$ is optimal iff S_1 fulfils for all $(k, y, j) \in N\times V\times M$, $y \leqslant k$, the condition*

$$(7)\qquad S_1(k, y, j) = \begin{cases} 0 & \text{if} \quad A > B, \\ a(k, y, j) & \text{if} \quad A = B, \\ 1 & \text{if} \quad A < B \end{cases}$$

with

$$A := f\big(k, y, j, 1, w^*(k+1, j-1)\big),$$
$$B := f\big(k, y, j, 0, w^*(k+1, j)\big)$$

and arbitrary $a(k, y, j) \in D$.

For the proof see Platen [5]. The general solution given above enables us to consider a large class of secretary problems. For example the nonadditive problem of maximizing the probability to choose the m best secretaries from n is solved by using this result in Platen [5].

There remains the problem of determining optimal multiple stopping rules for special cost functions. Since we are interested in secretary problems with interview cost which were already considered in a special case by Bartoszyński, Govindarajulu [1] we shall deal in the following section with additive cost functions.

2. Additive cost functions

We denote by $e(i, r, j, d)$ the elementary cost at observation $i \in N$ if $R_i = r$, $J_i = j$ and $D_i(Y_i, J_i) = d$. After finishing the inspection, which means $J_i = 0$, we set $e(i, r, 0, d) \equiv 0$. Then the random cost in the interval $i, i+1, \ldots, n$ with the use of the i-rank policy S_i and with $J_i = j$ is denoted by

$$(8)\qquad E_{i,j}^{S_i} := \sum_{k=i}^{n} e\big(k, R_k, J_k, S_i(k, Y_k, J_k)\big) + u\big(J_n - D_n(Y_n, J_n)\big).$$

Then $E\,E_{1,m}^{S_1}$ will be the expected sum of the elementary and the terminal costs if the multiple stopping rule S_1 is used. Our aim is to minimize $E\,E_{1,m}^{S_1}$ with respect to all $S_1 \in \overline{R}_1$. If we define for all $S_i \in \overline{R}_i$, $(i, y, j) \in N\times V\times M$, $y \leqslant i$

$$(9)\qquad w(i, y, j, S_i) := E(E_{i,j}^{S_i} | Y_i = y),$$

an optimal multiple stopping rule will do this. But we have to prove the assumptions A1 and A2.

Beginning with A1, we get from (9) and (8)

$$\begin{aligned} w(i, y, j, d * S_{i+1}) &= E\big(e(i, R_i, j, d) + E_{i+1, j-d}^{S_{i+1}} | Y_i = y\big) \\ &= E\big(e(i, R_i, j, d) | Y_i = y\big) + E(E_{i+1, j-d}^{S_{i+1}} | Y_i = y). \end{aligned}$$

Because of the fact that $E_{i+1, j-d}^{S_{i+1}}$ does not depend on Y_i we have

$$E(E_{i+1, j-d}^{S_{i+1}} | Y_i = y) = E E_{i+1, j-d}^{S_{i+1}} = \overline{w}(i+1, j-d, S_{i+1}).$$

Further, we define for all $(i, y, j, d) \in N\times V\times M\times D$, $y \leqslant i$

$$(10)\qquad g(i, y, j, d) := E\big(e(i, R_i, j, d) | Y_i = y\big).$$

We now get

$$\begin{aligned} (11)\qquad w(i, y, j, d * S_{i+1}) &= g(i, y, j, d) + \overline{w}(i+1, j-d, S_{i+1}) \\ &= f\big(i, y, j, d, \overline{w}(i+1, j-d, S_{i+1})\big) \end{aligned}$$

with

$$(12)\qquad \overline{w}(k, 0, S_k) = v(k) = 0, \quad k \in N,$$

and

$$(13)\qquad \overline{w}(n+1, h, S_{n+1}) = u(h), \quad h \in \overline{M}.$$

Therefore assumption A1 is fulfilled. The proof of A2 is easy to see. If we finally put (11)–(13) into (4)–(7), then the solution of our problem will be obtained.

We shall now define an important class of multiple stopping rules.

Definition. A multiple stopping rule $S_1 \in \overline{R}_1$ will be called *simple* if it can be characterized by the condition

$$(14)\qquad S_1(i, y, j) = \begin{cases} 1 & \text{if} \quad y \leqslant z_n(i, j), \\ 0 & \text{if} \quad y > z_n(i, j), \end{cases}$$

for all $(i, y, j) \in N\times V\times M$, $y \leqslant i$ and with $z_n(i, j) \in \{0, 1, \ldots, n\}$ being an (n, m)-matrix of integers. The following lemma gives a sufficient condition for the existence of optimal simple multiple stopping rules.

Lemma. *Let the elementary cost $e(i, r, j, d)$ be monotone increasing with respect to $r \in \{1, 2, \ldots, n\}$ for all $(i, j, d) \in N\times V\times D$; then the simple multiple stopping rule S_1^* characterized by the (n, m)-matrix of integers*

(15)

$$z_n^*(i, j) = \begin{cases} \text{greatest } y \in \{1, 2, \ldots, i\} & \text{with} \quad g(i, y, j, 1) - g(i, y, j, 0) \leqslant B, \\ 0 & \text{if} \quad g(i, 1, j, 1) - g(i, 1, j, 0) > B, \end{cases}$$

$$B := w^*(i+1, j) - w^*(i+1, j-1),\ \textit{will be optimal.}$$

For the proof see Platen [5].

Now we consider a class of cost functions with interview cost c_i, $i \in N$, which fulfils the assumption of the lemma. We suppose that for all $i \in N$ and $r \in \{1, 2, \ldots, n\}$

$$(16)\qquad e(i, r, j, d) = \begin{cases} rd + c_i & \text{if} \quad j \in M, \\ 0 & \text{if} \quad j = 0, \end{cases}$$

and for all $h \in \overline{M}$

$$(17)\qquad u(h) = hx, \quad x \in \{0, 1, 2, \ldots\}.$$

This means that we are interested in minimizing the sum of absolute ranks of selected objects and the cost of the interviews performed. Since the best objects have low ranks, an optimal multiple stopping rule will choose good objects. If there are selected h objects less than m until the nth observation, the terminal cost $u(h) =$

$= hx$ will also appear in the sum. Therefore $u(h)$ may be interpreted as a penalty in the case where less than m objects are chosen. From (10) and (16) we get by the statement $E(R_i|Y_i = y) = y(n+1)/(i+1)$ (see Platen [5]) for all $(i, y) \in N\times V$, $y \leqslant i$,

$$g(i, y, j, d) = \begin{cases} dy(n+1)/(i+1)+c_i & \text{if } j\in M, \\ 0 & \text{if } j = 0. \end{cases} \tag{18}$$

Using (11) and (18), we obtain from (4) the Bellman equation for all $(i, j) \in N\times M$ of the form

$$w^*(i,j) = \frac{1}{i}\sum_{y=1}^{i} \min\big(y(n+1)/(i+1)+c_i+w^*(i+1, j-1),\ c_i+w^*(i+1, j)\big) \tag{19}$$

with $w^*(n+1, h) = hx$, $h\in \overline{M}$ and $w^*(k, 0) = 0$, $k\in N$. In the same way we get from (15) for all $(i, j)\in N\times M$

$$z_n^*(i,j) = \begin{cases} \text{greatest } y\in\{1, 2, \dots, i\} & \text{with } y\leqslant p_n(i,j), \\ 0 & \text{if } 1 > p_n(i,j) \end{cases} \tag{20}$$

with

$$p_n(i,j) := \big(w^*(i+1, j)-w^*(i+1, j-1)\big)(i+1)/(n+1). \tag{21}$$

Because of (14) it is optimal to decide with

$$D_i(y,j) = \begin{cases} 1 & \text{if } y\leqslant z_n^*(i,j), \\ 0 & \text{if } y > z_n^*(i,j). \end{cases} \tag{22}$$

In a special case it is now easy to compute recursively the value function and, further, the matrix of integers $z_n^*(i,j)$ which characterizes an optimal multiple stopping rule.

It is intuitively clear that if the interviews are too expensive it will be best to choose the first m objects in every case without looking at their quality. The following lemma will give an assertion in this direction.

LEMMA. *If*

$$c_i \geqslant (n+1)/\big(i(i+1)\big) \tag{23}$$

for all $i\in N$, $x\geqslant (n+1)/2+c_1$ *and* $m = 1$, *then it will be optimal to stop at the first observed object.*

Proof. This proof is similar to one for another cost function in Bartoszyński, Govindarajulu [1].

We denote by $\overline{S}_1$ a stopping rule (which $\overline{S}_1$ is indeed because of $m = 1$) which only stops at the first observation. But $\tilde{S}_1$ will be an arbitrary stopping rule which never stops at the first observation. Because of $P(Y_1 = 1) = 1$ we get from (9) using (8), (10) and (18)

$$w(1, 1, 1, \overline{S}_1) = g(1, 1, 1, 1)+c_1 = (n+1)/2+c_1.$$

On the other hand, we have

$$\begin{aligned} w(1, 1, 1, \tilde{S}_1) &= \sum_{k=2}^{n}\sum_{y=1}^{k} P(A_{k,y}^{\tilde{S}_1})\Big(y\frac{n+1}{k+1}+\sum_{i=1}^{k} c_i\Big)+P(A_{n+1}^{\tilde{S}_1})x \\ &= \sum_{k=2}^{n}\sum_{y=1}^{k} P(A_{k,y}^{\tilde{S}_1})\Big(\frac{n+1}{k+1}+\sum_{i=1}^{k} c_i\Big)+P(A_{n+1}^{\tilde{S}_1})x+ \\ &\quad +\sum_{k=1}^{n}\sum_{y=2}^{k} P(A_{k,y}^{\tilde{S}_1})(y-1)\frac{n+1}{k+1} \\ &\geqslant \sum_{k=2}^{n}\sum_{y=1}^{k} P(A_{k,y}^{\tilde{S}_1})\Big(\frac{n+1}{k+1}+\sum_{i=1}^{k} c_i\Big)+P(A_{n+1}^{\tilde{S}_1})x, \end{aligned}$$

where $A_{k,y}^{\tilde{S}_1}$ denotes the event of stopping at k with $Y_k = y$ by $\tilde{S}_1$ and $A_{n+1}^{\tilde{S}_1}$ means that no object is chosen by $\tilde{S}_1$.

Because of the assumption we have for $k = 2, 3, \dots, n$

$$\Big(\frac{n+1}{k+1}+\sum_{i=1}^{k} c_i\Big)-\Big(\frac{n+1}{k}+\sum_{i=1}^{k-1} c_i\Big) = c_k-\frac{n+1}{(k+1)k}\geqslant 0$$

and therefore

$$\frac{n+1}{k+1}+\sum_{i=1}^{k} c_i \geqslant \frac{n+1}{2}c_1.$$

Using this, we get

$$w(1, 1, 1, \tilde{S}_1)\geqslant \Big(\sum_{k=2}^{n}\sum_{y=1}^{k} P(A_{k,y}^{\tilde{S}_1})+P(A_{n+1}^{\tilde{S}_1})\Big)\Big(\frac{n+1}{2}+c_1\Big).$$

But by the definition of $\tilde{S}_1$ we obtain

$$\sum_{k=2}^{n}\sum_{y=1}^{k} P(A_{k,y}^{\tilde{S}_1})+P(A_{n+1}^{\tilde{S}_1}) = 1,$$

and finally $w(1, 1, 1, \tilde{S}_1)\geqslant w(1, 1, 1, \overline{S}_1)$ which proves the lemma.

3. The case $n\to\infty$

In order to consider the asymptotic behaviour of optimal multiple stopping rules we norm the observation time by n. Further, we denote the value function at a normed time i/n for all $i\in\{1, 2, \dots, n+1\}$, $j\in\overline{M}$ and $n = 1, 2, \dots$ by

$$w_n^j(i/n) := w^*(i,j).$$

Let $c_n(i/n) := c_i$; then we can write the Bellman equation (19) for all $(i,j) \in N\times M$, $n = 1, 2, \ldots$, as the following difference equation:

(24) $$\big(w_n^j((i+1)/n)-w_n^j(i/n)\big)/(1/n)$$
$$= -\frac{n}{i}\Big(\sum_{y=1}^{i}\min\big(y(n+1)/(i+1)+c_n(i/n)+w_n^{j-1}((i+1)/n),$$
$$c_n(i/n)+w_n^j((i+1)/n)\big)-w_n^j((i+1)/n)\Big)$$
$$= -\frac{n}{i}\sum_{y=1}^{i}\min\big(y(n+1)/(i+1)+w_n^{j-1}((i+1)/n)-w_n^j((i+1)/n),\,0\big)-nc_n(i/n)$$
$$= \frac{n}{i}\sum_{y=1}^{i}\big(w_n^j((i+1)/n)-w_n^{j-1}((i+1)/n)-y(n+1)/(i+1)\big)^+-nc_n(i/n),\ (^2)$$

where $w_n^h((n+1)/n) = hx$, $h\in\overline{M}$ and $w_n^0(k/n) = 0$, $k\in N$. If we assume that for all $t\in[0,1]$ the sequence $\{n\cdot c_n([tn]/n)\}_{n=1,2,\ldots}$(³) converges to a continuous function $c(t)$ with $\int_0^1 c(t)\,dt<\infty$, then the sequence $\{w_n^j(t)\}_{n=1,2,\ldots}$ of linear interpolated solutions of the difference equation (24) converges for all $t\in(0,1]$ to a continuous function $w^j(t)$ determined by the analogous differential equation

(25) $$w^j(t)' = \frac{1}{t}\sum_{y=1}^{\infty}\big(w^j(t)-w^{j-1}(t)-y/t\big)^+-c(t),$$

where $w^h(1) = hx$, $h\in\overline{M}$ and $w^0(t)\equiv 0$, $j\in M$. This proposition follows directly from a theorem in Platen [5], which is also valid for many other cost functions. Analogously to (20) and (21) we define for all $t\in[0,1]$ and $j\in M$

(26) $$p^j(t) := \big(w^j(t)-w^{j-1}(t)\big)t$$

as the limit of the sequence $\{p_n([tn],j)\}_{n=1,2,\ldots}$ and in the same way

(27) $$z^j(t) := \begin{cases}\text{greatest } y\in\{1,2,\ldots\} & \text{with}\quad y\leqslant p^j(t),\\ 0 & \text{if}\quad 1>p^j(t).\end{cases}$$

Because of (22) for great n it is asymptotically optimal to decide with

(28) $$D_i(y,j) = \begin{cases}1 & \text{if}\quad y\leqslant z^j(i/n),\\ 0 & \text{if}\quad y>z^j(i/n),\end{cases}$$

$(i,y,j)\in N\times V\times M$, $y\leqslant i$. As examples show, this asymptotically optimal multiple stopping rule is already good enough for practical purposes in the case $n = 100$.

In the following we will try to determine the function $z^j(t)$. Obviously it follows from (26) for $j\in M$

(29) $$p^j(0) = 0 \quad\text{and}\quad p^j(1) = x.$$

(²) $(a)^+ := \max(a, 0)$.

(³) $[a]$:= greatest integer less than a.

So we can define for all $a\in[0,x]$ and $j\in M$

(30) $$t^j(a) := \max\{t\in[0,1]:\ p^j(t)\leqslant a\}.$$

On the other hand, we obtain from (25) by (26) for all $t\in(0,1]$ and $j\in M$

(31) $$w^j(t)' = \frac{1}{t}\sum_{y=1}^{\infty}\big(p^j(t)-y\big)^+/t-c(t) = \sum_{y=1}^{[p^j(t)]}\big(p^j(t)-y\big)/t^2-c(t)$$
$$= [p^j(t)]\big(p^j(t)-([p^j(t)]+1)/2\big)/t^2-c(t).$$

Putting this in the derivative of (26), we get for $j = 1$ and $t\in(0,1]$ the differential equation

(32) $$p^1(t)' = p^1(t)/t+\big(w^1(t)'-w^0(t)'\big)t$$
$$= p^1(t)/t+[p^1(t)]\big(p^1(t)-([p^1(t)]+1)/2\big)/t-c(t)t$$

and for $j = 2, \ldots, m$ a similar but more complicated equation. Since we are interested in an analytical solution of (32), we restrict ourselves to stopping problems with interview costs of the form

(33) $$c_i = c_n(i/n) = \begin{cases}c(n+1)/(i(i+1)) & \text{if}\quad i\geqslant\varepsilon n,\\ c(n+1)/(\varepsilon n(\varepsilon n+1)) & \text{if}\quad i<\varepsilon n,\end{cases}$$

$0\leqslant c<1$, $0<\varepsilon<1$, $n = 1, 2, \ldots$, $i\in N$. This means that we have to pay the same for every interview at $i<\varepsilon n$. A look at (18) shows that we do not have a trivial stopping problem. It is easy to see that the sequence $\{n\cdot c_n([tn]/n)\}_{n=1,2,\ldots}$ converges for all $t\in[0,1]$ to the continuous function

(34) $$c(t) = \begin{cases}c/t^2 & \text{if}\quad t\geqslant\varepsilon,\\ c/\varepsilon^2 & \text{if}\quad t<\varepsilon,\end{cases}$$

whith $\int_0^t c(t)\,dt = c(2/\varepsilon-1)<\infty$. Therefore our assumptions on the interview cost are fulfilled and we can solve the differential equation (32). If we assume $\varepsilon\leqslant t^1(1)$, we get for $t\in[t^1(a), b(a)]$, where $[p^1(t)] = a$, $a = 1, 2, \ldots, x-1$

(35) $$p^1(t) = \frac{a}{2}+\frac{c}{a+1}+\left(\frac{t}{t^1(a)}\right)^{a+1}\left(\frac{a}{2}-\frac{c}{a+1}\right).$$

Because of $\left(\dfrac{a}{2}-\dfrac{c}{a+1}\right)>0$ we have $p^1(t)'>0$ for $t\in[t^1(1),1]$ (for $t\in(0,t^1(1))$ this is also true). Taking into consideration the continuity of $p^1(t)$, we obtain from (30)

$$p^1(t^1(a+1)) = a+1$$

and from (35)

$$p^1(t^1(a+1)) = \frac{a}{2}+\frac{c}{a+1}+\left(\frac{t^1(a+1)}{t^1(a)}\right)^{a+1}\left(\frac{a}{2}-\frac{c}{a+1}\right).$$

It follows that

$$\frac{t^1(a+1)}{t^1(a)} = \left(\frac{a-\dfrac{2c}{a+1}+2}{a-\dfrac{2c}{a+1}}\right)^{1/(a+1)},$$

and we get for all $a \in \{1, 2, \ldots, x-1\}$

$$t^1(a) = \prod_{K=a}^{x-1}\left(\frac{K-\dfrac{2c}{K+1}}{K-\dfrac{2c}{K+1}+2}\right)^{1/(K+1)}. \tag{36}$$

Obviously this formula does not depend on ε. Further, we can see that the product in (36) will converge if the terminal cost x tends to infinity, i.e. if we are always interested in choosing an object by an optimal stopping rule. In the case of infinite terminal cost and no interview cost, i.e. $x = \infty$, $c = 0$, we have the well-known result of Chow, Moriguti, Robbins, Samuels [2]. Taking into consideration the properties of $p^1(t)$, it is easy to see from (27) that for all $t \in [t^1(a), t^1(a+1)]$, $a \in \{0, 1, \ldots, x-1\}$, $z^1(t) = [p^1(t)] = a$. Because of (28) for great n it will be asymptotically optimal to choose the ith object with relative rank $Y_i = y$ if $i/n \geqslant t^1(y)$.

References

[1] R. Bartoszyński and Z. Govindarajulu, *The secretary problem with interview cost* (to appear).

[2] Y. S. Chow, S. Moriguti, H. Robbins, and E. Samuels, *Optimum selection based on relative rank*, Israel J. Math. 2 (1964), pp. 81–90.

[3] Y. S. Chow, H. Robbins and D. Siegmund, *Great expectations: The theory of optimal stopping*, Boston 1971.

[4] P. H. Müller and E. Platen. *Das asymptotische Verhalten optimaler Rangstrategien beim "Problem der besten Wahl"*, Wiss. Z. Techn. Univ. Dresden 24.5 (1975), pp. 933–937.

[5] E. Platen, *Sequentielle Rangauswahlprobleme*, Dissertation, Techn. Univ. Dresden 1975.

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ASYMPTOTIC NORMALITY AND CONVERGENCE RATES OF LINEAR RANK STATISTICS UNDER ALTERNATIVES*

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1. Introduction

Let $\{X_{Ni}, i \geqslant 1\}$ be a sequence of independent random variables with continuous cdfs (cumulative distribution functions) $\{F_{Ni}, i \geqslant 1\}$ respectively. Consider a linear rank statistic S_N given by

$$S_N = \sum_{i=1}^{N} c_{Ni} a_N(R_{Ni}) \tag{1.1}$$

where R_{Ni} is the rank of X_{Ni} in $(X_{N1}, \ldots, X_{NN})$, $(c_{N1}, \ldots, c_{NN})$ are known (regression) constants, and $a_N(1), \ldots, a_N(N)$ are "scores" generated by a known real-valued function $\varphi(t)$, $0 < t < 1$, in either of the following ways:

$$a_N(i) = \varphi(i/(N+1)), \quad 1 \leqslant i \leqslant N, \tag{1.2}$$

$$a_N(i) = E\varphi(U_N^{(i)}), \quad 1 \leqslant i \leqslant N, \tag{1.3}$$

where $U_N^{(i)}$ is the ith order statistic in a sample of size N from the rectangular distribution over $(0, 1)$.

We make the following assumptions:

(IA) $\max\limits_{1 \leqslant i \leqslant N} |c_{Ni}|/s_N = O(N^{-1/2})$,

(IB) $|\varphi^{(i)}(t)| \leqslant k[t(1-t)]^{\delta-i-1/2}$, $i = 0, 1$; $\delta > 0$, K a generic constant, s_N^2 is approximate variance of S_N and is given by (1.7) and (1.8) below.

Our main results are the following:

THEOREM 1.1. *Let the scores $a_N(i)$, $1 \leqslant i \leqslant N$ be defined as in* (1.2). *Then, under assumptions* (IA) *and* (IB),

$$\sup_x \left| P\left(\frac{S_N-\mu_N}{s_N} \leqslant x\right) - \Phi(x)\right| \to 0 \quad \textit{as} \quad N \to \infty,\ s_N \neq 0, \tag{1.4}$$

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