It follows that

$$\frac{t^1(a+1)}{t^1(a)} = \left(\frac{a-\dfrac{2c}{a+1}+2}{a-\dfrac{2c}{a+1}}\right)^{1/(a+1)},$$

and we get for all $a \in \{1, 2, \ldots, x-1\}$

$$t^1(a) = \prod_{K=a}^{x-1}\left(\frac{K-\dfrac{2c}{K+1}}{K-\dfrac{2c}{K+1}+2}\right)^{1/(K+1)}. \tag{36}$$

Obviously this formula does not depend on ε. Further, we can see that the product in (36) will converge if the terminal cost x tends to infinity, i.e. if we are always interested in choosing an object by an optimal stopping rule. In the case of infinite terminal cost and no interview cost, i.e. $x = \infty$, $c = 0$, we have the well-known result of Chow, Moriguti, Robbins, Samuels [2]. Taking into consideration the properties of $p^1(t)$, it is easy to see from (27) that for all $t \in [t^1(a), t^1(a+1)]$, $a \in \{0, 1, \ldots, x-1\}$, $z^1(t) = [p^1(t)] = a$. Because of (28) for great n it will be asymptotically optimal to choose the ith object with relative rank $Y_i = y$ if $i/n \geqslant t^1(y)$.

References

[1] R. Bartoszyński and Z. Govindarajulu, *The secretary problem with interview cost* (to appear).

[2] Y. S. Chow, S. Moriguti, H. Robbins, and E. Samuels, *Optimum selection based on relative rank*, Israel J. Math. 2 (1964), pp. 81–90.

[3] Y. S. Chow, H. Robbins and D. Siegmund, *Great expectations: The theory of optimal stopping*, Boston 1971.

[4] P. H. Müller and E. Platen. *Das asymptotische Verhalten optimaler Rangstrategien beim "Problem der besten Wahl"*, Wiss. Z. Techn. Univ. Dresden 24.5 (1975), pp. 933–937.

[5] E. Platen, *Sequentielle Rangauswahlprobleme*, Dissertation, Techn. Univ. Dresden 1975.

Presented to the semester
MATHEMATICAL STATISTICS
September 15–December 18, 1976

MATHEMATICAL STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 6
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1980

ASYMPTOTIC NORMALITY AND CONVERGENCE RATES OF LINEAR RANK STATISTICS UNDER ALTERNATIVES*

MADAN L. PURI and NAVARATNA S. RAJARAM

Indiana University and Kent State University Bloomington, Ind., U.S.A.

1. Introduction

Let $\{X_{Ni}, i \geqslant 1\}$ be a sequence of independent random variables with continuous cdfs (cumulative distribution functions) $\{F_{Ni}, i \geqslant 1\}$ respectively. Consider a linear rank statistic S_N given by

$$S_N = \sum_{i=1}^{N} c_{Ni} a_N(R_{Ni}) \tag{1.1}$$

where R_{Ni} is the rank of X_{Ni} in $(X_{N1}, \ldots, X_{NN})$, $(c_{N1}, \ldots, c_{NN})$ are known (regression) constants, and $a_N(1), \ldots, a_N(N)$ are "scores" generated by a known real-valued function $\varphi(t)$, $0 < t < 1$, in either of the following ways:

$$a_N(i) = \varphi(i/(N+1)), \quad 1 \leqslant i \leqslant N, \tag{1.2}$$

$$a_N(i) = E\varphi(U_N^{(i)}), \quad 1 \leqslant i \leqslant N, \tag{1.3}$$

where $U_N^{(i)}$ is the ith order statistic in a sample of size N from the rectangular distribution over $(0, 1)$.

We make the following assumptions:

(IA) $\max_{1 \leqslant i \leqslant N} |c_{Ni}|/s_N = O(N^{-1/2})$,

(IB) $|\varphi^{(i)}(t)| \leqslant k[t(1-t)]^{\delta-i-1/2}$, $i = 0, 1$; $\delta > 0$, K a generic constant, s_N^2 is approximate variance of S_N and is given by (1.7) and (1.8) below.

Our main results are the following:

THEOREM 1.1. *Let the scores $a_N(i)$, $1 \leqslant i \leqslant N$ be defined as in* (1.2). *Then, under assumptions* (IA) *and* (IB),

$$\sup_x \left| P\left(\frac{S_N-\mu_N}{s_N} \leqslant x\right) - \Phi(x)\right| \to 0 \quad as \quad N \to \infty,\ s_N \neq 0, \tag{1.4}$$

* Work supported by the Air Force Office of Scientific Research, A.F.S.C.U.S.A.F., Grant No. AFOSR76-2927. Reproduction in whole or part permitted for any purpose of the United States Government.

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt,$$

(1.5) $$\mu_N = \sum_{i=1}^{N} c_{Ni} \int_{-\infty}^{\infty} \varphi(H(x))\, dF_{Ni}(x),$$

(1.6) $$H(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(x),$$

(1.7) $$s_N^2 = \sum_{i=1}^{N} s_{Ni}^2; \quad s_{Ni}^2 = \mathrm{Var}(A_{Ni}(X_{Ni})),$$

(1.8) $$A_{Ni}(x) = \frac{1}{N} \sum_{j=1}^{N} (c_{Ni} - c_{Nj}) \int \{I_{[x \leqslant y]} - F_{Ni}(y)\} \varphi'[H(y)]\, dF_N(y),$$

and

(1.9) $$I_{[x \leqslant y]} = \begin{cases} 0 & \text{if} \quad y < x, \\ 1 & \text{if} \quad y \geqslant x. \end{cases}$$

COROLLARY 1.1. *Let $\varphi(t) = F^{-1}(t)$, where F is a cdf. Let the scores $a_N(i)$, $1 \leqslant i \leqslant N$ be given by* (1.3). *Then, under assumptions* (IA) *and* (IB), *the conclusions of Theorem* 1.1 *hold.*

THEOREM 1.2. *Let the scores $a_N(i)$, $1 \leqslant i \leqslant N$ be given by* (1.2) *and assumption* (IA) *be satisfied. Let*

(1.10) $$\sup_x |\varphi'(x)| = ||\varphi'|| < \infty;$$

then

(1.11) $$\sup_x \left| P\left(\frac{S_N - \mu_N}{s_N} \leqslant x\right) - \Phi(x) \right| \leqslant c_1 l_{3,N} + \Delta_N,$$

where

(1.12) $$l_{3,N} = \varrho_N^3 / s_N^3, \quad c_1 = 0.7985,$$

(1.13) $$\varrho_N^3 = \sum_{i=1}^{N} \varrho_{Ni}^3, \quad \varrho_{Ni}^3 = E|A_{Ni}(X_{Ni})|^3, \quad \text{and} \quad \Delta_N \to 0.$$

Furthermore,

(1.14) $$N^{1/2} \Delta_N = O_p ||\varphi'|| s_N^{-1} \sum_{i=1}^{N} |c_{Ni}|.$$

Remark. Theorem 1.1 has been proved by Hájek [7] (1968). His conditions on the score generating function φ are milder than ours but the conclusion of our Theorem 1.1 is sharper in the sense that the centering constant μ_N appears naturally in place of ES_N given by Hájek [7]. Corollary 1.1 is an extension of a similar result proved by Chernoff and Savage [5] (1958) for the two sample problem and serves to remove some of the complications encountered in Hoeffding [8] (1973). Theorem 1.2 is related to the results of Bickel [4] (1972), Jurečkova-Puri [9] (1975), and Bergström–Puri [2] (1976). However, the bounds obtained in these papers are non-random and thus sharper than ours. On the other hand, our conditions on the score generating function φ are milder. We believe that Theorem 1.2 is true even when condition (1.10) is replaced by the assumption (IB). At the present time, the theory of asymptotic expansion for sums of dependent random variables is still at a rudimentary stage (Stein [12] (1970)) and it is doubtful if the random term Δ_N can be removed without additional assumptions on the underlying distributions.

2. Asymptotic normality of S_n

The proof of Theorem 1.1 (and Corollary 1.2) will be along the lines of the Chernoff–Savage theorem (1958) as given in Puri and Sen [11] (1971) with some modifications necessitated by greater generality of the present problem.

First we introduce the following notations:

(2.1) $$H_N(x) = \frac{1}{N} \sum_{i=1}^{N} I_{[X_{Ni} \leqslant x]},$$

(2.2) $$H(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(x),$$

(2.3) $$C_N(x) = \sum_{i=1}^{N} C_{Ni} I_{[X_{Ni} \leqslant x]},$$

(2.4) $$C(x) = \sum_{i=1}^{N} C_{Ni} F_{Ni}(x).$$

Note that the functions $H_N(x)$ and $C_N(x)$ are stochastic variables whereas $H(x)$ and $C(x)$ are non-random though depending on N.

Then the following inequalities are obvious:

(2.5) $$|C_N(x)| \leqslant N \max_{1 \leqslant i \leqslant N} |C_{Ni}| H_N(x),$$

(2.6) $$|C(x)| \leqslant N \max_{1 \leqslant i \leqslant N} |C_{Ni}| H(x), \quad -\infty < x < \infty.$$

Proof of Theorem 1.1. We rewrite S_N defined in (1.1) as

(2.7) $$S_N = \int_{-\infty}^{\infty} \varphi\left(\frac{N}{N+1} H_N(x)\right) dC_N(x) = \mu_N + B_{1N} + B_{2N} + \sum_{i=1}^{3} D_{iN}$$

where μ_N is given by (1.5), and

(2.8) $$B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x))\, d(C_N(x) - C(x)),$$

(2.9) $$B_{2N} = \int_{-\infty}^{\infty} (H_N(x)-H(x))\varphi'(H(x))\,dC(x),$$

(2.10) $$D_{1N} = \frac{-1}{N+1}\int_{-\infty}^{\infty} H_N(x)\varphi'(H(x))\,dC_N(x),$$

(2.11) $$D_{2N} = \int_{-\infty}^{\infty} (H_N(x)-H(x))\varphi'(H(x))\,d(C_N(x)-C(x)),$$

(2.12) $$D_{3N} = \int_{-\infty}^{\infty}\Big\{\varphi\Big(\frac{N}{N+1}H_N(x)\Big)-\varphi(H(x))- -\Big(\frac{N}{N+1}H_N(x)-H(x)\Big)\varphi'(H(x))\Big\}dC_N(x).$$

The proof will be accomplished if we establish the following:

(a) $|\mu_N| < \infty$,

(b) $(B_{1N}+B_{2N})/s_N$ is asymptotically normal,

(c) $D_{iN} = o_p(s_N)$, $i = 1, 2, 3$.

Proof of (a). Using (2.4) and (2.6), we obtain

(2.13) $$|\mu_N| \leqslant N\max_{1\leqslant i\leqslant N}|C_{Ni}|\int_{-\infty}^{\infty}|\varphi(H(x))|\,dH(x) < \infty, \quad \text{by assumption (IB).}$$

Proof of (b). To prove (b), we shall verify the Liapounov condition for B_{1N}/s_n and B_{2N}/s_n. Integrating B_{2N} by parts, we obtain

(2.14) $$B_{2N} = [H_N(x)-H(x)]B^*(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}B^*(x)\,d[H_N(x)-H(x)]$$

where

$$B^*(x) = \int_{x_0}^{x}\varphi'(H(y))\,dC(y)$$

where x_0 is determined arbitrarily such that $H(x_0) > 0$.

We first consider $\int_{-\infty}^{\infty}B^*(x)\,d[H_N(x)-H(x)]$; we have

(2.15) $$\int_{-\infty}^{\infty}B^*(x)\,d[H_N(x)-H(x)] = \sum_{i=1}^{N}[B^*(X_{Ni})-EB^*(X_{Ni})]/N.$$

We verify the condition

$$\frac{1}{S_N^{2+\delta'}}\sum_{i=1}^{N}E[|B^*(X_{Ni})-EB^*(X_{Ni})|/N]^{2+\delta'}\to 0$$

as $N\to\infty$ for some $\delta' > 0$. To verify this, it suffices to show that

(2.16) $$\frac{1}{S_N^{2+\delta'}}\sum_{i=1}^{N}E[|B^*(X_{Ni})|/N]^{2+\delta'}\to 0 \quad \text{as} \quad N\to\infty.$$

Choose $\delta' > 0$ such that $(2+\delta')(\delta-\frac{1}{2}) > 1$. Then

(2.17) $$\frac{1}{s_N^{2+\delta'}}\sum_{i=1}^{N}E|B^*(X_{Ni})/N|^{2+\delta'}$$
$$= \frac{1}{s_N^{2+\delta'}N^{2+\delta'}}\sum_{i=1}^{N}\int_{-\infty}^{\infty}\Big|\int_{x_0}^{x}\varphi'(H(y))\,dC(y)\Big|^{2+\delta'}dF_{Ni}(x)$$
$$\leqslant \frac{1}{s_N^{2+\delta'}N^{2+\delta'}}\sum_{i=1}^{N}N^{2+\delta'}\max_{1\leqslant i\leqslant N}|C_{Ni}|^{2+\delta'}\int_{-\infty}^{\infty}\Big|\int_{x_0}^{x}\varphi'(H(y))H(y)\Big|^{2+\delta'}dF_{Ni}(x)$$
$$\leqslant \{\max_{1\leqslant i\leqslant N}|C_{Ni}|/s_N\}^{2+\delta'}\sum_{i=1}^{N}\int_{-\infty}^{\infty}\{|\varphi(H(x))|+|\varphi(H(x_0))|\}^{2+\delta'}dF_{Ni}(x)$$
$$= O(N^{-\delta'/2})\int_{-\infty}^{\infty}\{|\varphi(H(x))|+|\varphi(H(x_0))|\}^{2+\delta'}dH(x)\to 0 \quad \text{as} \quad N\to\infty.$$

We now show that

$$\frac{\beta(x)}{s_N}\Big|_{-\infty}^{\infty} = o_p(1), \quad \text{where} \quad \beta(x) = [H_N(x)-H(x)]B^*(x) \quad \text{(see (2.14)).}$$

We note that

$$\Big|\frac{\beta(x)}{s_N}\Big| = N^{1/2}|H_N(x)-H(x)|\;\frac{1}{N^{1/2}s_N}\Big|\int_{x_0}^{x}\varphi'(H(y))\,dc(y)\Big|$$
$$\leqslant N^{1/2}|H_N(x)-H(x)|\,O(1)\Big|\int_{x_0}^{x}\varphi'(H(y))\,dH(y)\Big|$$
$$\leqslant kN^{1/2}|H_N(x)-H(x)|\{H(x)(1-H(x))\}^{\delta-1/2}.$$

Now since $\forall\varepsilon > 0$, $\delta' > 0$, $\exists\, c(\varepsilon, \delta')$

(2.18) $$P\Big[\sup_x N^{1/2}\frac{|H_N(x)-H(x)|}{\{H(x)(1-H(x))\}^{\delta'-1/2}} > C(\varepsilon, \delta')\Big] < \varepsilon,$$

it follows that with probability $> 1-\varepsilon$,

$$\Big|\frac{\beta(x)}{s_N}\Big| \leqslant k\{H(x)(1-H(x))\}^{\delta-\delta'}C(\varepsilon, \delta')\to 0 \quad \text{as} \quad x\to\pm\infty$$

by choosing $\delta' < \delta$.

Thus the Liapounov condition (2.16) is satisfied for B_{2N}/s_N. The verification (of the Liapounov condition) for B_{1N}/s_N is similar, and the same is true for $(B_{1N}+B_{2N})/s_N$ by using the C_r-inequality. This proves (b).

Proof of (c).

$$(2.19)\qquad \left|\frac{D_{1N}}{s_N}\right| \leqslant \frac{1}{Ns_N}\left|\sum_{i=1}^{N}\varphi'(H(X_{Ni}))C_{Ni}\right| \leqslant \frac{1}{N}\sum_{i=1}^{N}V_{Ni}$$

where

$$(2.20)\qquad V_{Ni} = \left|\varphi'(H_{Ni}(X))\frac{c_{Ni}}{s_N}\right|.$$

To establish that $\frac{1}{N}\sum_{i=1}^{N}V_{Ni} \to 0$ in probability, it suffices to show that

$$(2.21)\qquad N^{-\alpha}\sum_{i=1}^{N}E|V_{Ni}|^{\alpha} < \infty \quad \text{for some } 0<\alpha<1 \text{ (cf. Loève [10], p. 241).}$$

Taking $\alpha = 2/3$,

$$\frac{1}{N^{2/3}}\sum_{i=1}^{N}E|V_{Ni}|^{2/3}$$
$$\leqslant \frac{K}{N^{2/3}}\max_{1\leqslant i\leqslant N}\left|\frac{C_{Ni}}{s_N}\right|^{2/3}\sum_{i=1}^{N}\int_{-\infty}^{\infty}\{H(x)(1-H(x))^{2\delta/3-1}dF_{Ni}(x)$$
$$\leqslant K\int_0^1\{u(1-u)\}^{2\delta/3-1}du < \infty \text{ uniformly in } N.$$

Now consider

$$D_{2N} = \int_{-\infty}^{\infty}(H_N(x)-H(x))\varphi'((H(x))d(C_N(x)-C(x)).$$

Noting (2.18), it follows that with probability $>1-\varepsilon$,

$$|H_N(x)-H(x)||\varphi'(H(x))| \leqslant \frac{K}{N^{1/2}}C(\varepsilon,\delta')\{H(x)(1-H(x))\}^{\delta-\delta'-1}.$$

Set $0<\delta^* = \delta-\delta'$, choose $\delta'<\delta$, and note that

$$(2.22)\qquad \frac{1}{s_N N^{1/2}}\int_{-\infty}^{\infty}\{H(x)(1-H(x))\}^{\delta^*-1}dC_N(x)$$
$$= \frac{1}{s_N N^{1/2}}\sum_{i=1}^{N}C_{Ni}\{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1}.$$

Setting

$$(2.23)\qquad V_{Ni} = \frac{N^{1/2}C_{Ni}}{s_N}\{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1},$$

we have to show that

$$(2.24)\qquad \frac{1}{N}\sum_{i=1}^{N}\{V_{Ni}-EV_{Ni}\} \to 0 \text{ in probability.}$$

This will follow if we show that for some $\alpha>0$,

$$(2.25)\qquad N^{-(1+\alpha)}\sum_{i=1}^{N}E|V_{Ni}|^{1+\alpha} \to 0.$$

Choose $\alpha>0$ such that $(1+\alpha)(\delta^*-1) > -1$ (i.e. $0<\alpha<\delta^*/(1-\delta^*)$). Then

$$(2.26)\qquad N^{-(1+\alpha)}\sum_{i=1}^{N}E|V_{Ni}|^{1+\alpha}$$
$$\leqslant \frac{1}{N^{\alpha}\cdot N}\left|\frac{N^{1/2}\max_{1\leqslant i\leqslant N}|C_{Ni}|}{s_N}\right|^{1+\alpha}\sum_{i=1}^{N}E\{H(X_{Ni})(1-H(X_{Ni}))\}^{(1+\alpha)(\delta^*-1)}$$
$$= O(1)\frac{1}{N^{\alpha}}\int_{-\infty}^{\infty}\{H(x)(1-H(x))\}^{(1+\alpha)(\delta^*-1)}dH(x) \to 0, \quad \text{as} \quad N\to\infty.$$

This proves (2.25).

Finally, consider

$$D_{3N} = \int_{-\infty}^{\infty}\left\{\varphi\left(\frac{N}{N+1}H_N(x)\right)-\varphi(H(x))-\left(\frac{NH_N(x)}{N+1}-H(x)\right)\varphi'(H(x))\right\}dC_N(x).$$

We have to show that $D_{3N}/s_N = o_p(1)$. Write

$$(2.27)\qquad C_{3N} = \frac{D_{3N}}{N^{1/2}s_N},$$

and note that

$$(2.28)\qquad |C_{3N}| \leqslant O(1)\int_{-\infty}^{\infty}\left|\varphi\left(\frac{N}{N+1}H_N(x)\right)-\varphi(H(x))-\right.$$
$$\left.-\left(\frac{NH_N(x)}{N+1}-H(x)\right)\varphi'(H(x))\right|dH_N(x),$$

since

$$\max_{1\leqslant i\leqslant N}\frac{|C_{Ni}|}{s_N N^{1/2}} = O(N^{-1}).$$

The proof that the right-hand side of (2.28) is $o_p(N^{-1/2})$ follows precisely as in Puri and Sen [11], pp. 401–405. Thus the proof of Theorem 1.1 follows.

Proof of Corollary 1.1. Write

$$\varphi_N(t) = \sum_{i=1}^{[Nt]}a_{N(i)}, \quad 0<t<1; \quad a_N(i) = E\varphi(U_N^{(i)}), \quad 1\leqslant i\leqslant N,$$

where $[\alpha]$ is the greatest integer $\leqslant \alpha$; and let

$$S_N^* = \int_{-\infty}^{\infty} \varphi_N\left(\frac{NH_N(x)}{N+1}\right) dC_N(x). \tag{2.29}$$

The proof of the corollary will be accomplished if we show that $(S_N - S_N^*)/s_N \to 0$ in probability as $N \to \infty$.

LEMMA 2.1. *Under the hypotheses of Theorem* 1.1 *and Corollary* 1.1,

$$\lim_{N\to\infty} \varphi_N(t) = \varphi(t), \tag{2.30}$$

$$\left|\int_{-\infty}^{\infty} \left\{\varphi_N\left(\frac{N}{N+1} H_N(x)\right) - \varphi\left(\frac{N}{N+1} H_N(x)\right)\right\} dC_N(t)\right| = o_p(s_N). \tag{2.31}$$

Proof. The proof of (2.30) is well known (see Puri and Sen [11], pp. 408–409); and so we prove (2.31).

Using (2.5), we obtain

$$\left|\int_{-\infty}^{\infty} \left\{\varphi_N\left(\frac{NH_N(x)}{N+1}\right) - \varphi\left(\frac{NH_N(x)}{N+1}\right)\right\} dC_N(x)\right| \tag{2.32}$$

$$\leqslant \max_{1\leqslant i\leqslant N} |C_{Ni}| \sum_{i=1}^{N} \left|\varphi_N\left(\frac{NH_N(X_{Ni})}{N+1}\right) - \varphi\left(\frac{NH_N(X_{Ni})}{N+1}\right)\right|$$

$$= \max_{1\leqslant i\leqslant N} |C_{Ni}| \sum_{i=1}^{N} \left|\varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right)\right|.$$

Now by assumption I(A),

$$\max_{1\leqslant i\leqslant N} \left|\frac{C_{Ni}}{s_N}\right| \sum_{i=1}^{N} \left|\varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right)\right| \tag{2.33}$$

$$= O(1) N^{-1/2} \sum_{i=1}^{N} \left|\varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right)\right|$$

which $\to 0$ as $N \to \infty$ (cf. Puri and Sen [11], pp. 409–411). This proves the lemma; and hence Corollary 1.1.

To compute the $\operatorname{var}(B_{1N} + B_{2N})$, note that

$$B_{1N} = \sum_{i=1}^{N} \varphi(H(X_{Ni})) C_{Ni} + \text{constant},$$

$$B_{2N} = -\frac{1}{N} \sum_{i=1}^{N} B^*(X_{Ni}) + \text{constant},$$

where

$$B^*(x) = \int_{x_0}^{x} \varphi'(H(y))\, dc(y) = \sum_{j=1}^{N} C_{Nj} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y).$$

Noting that

$$\varphi(H(x)) = \int_{x_0}^{x} \varphi'(H(y))\, dH(y) + \varphi(H(x_0)),$$

and setting

$$A_{Ni}(x) = \frac{C_{Ni}}{N} \sum_{j=1}^{N} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y) - \frac{1}{N} \sum_{j=1}^{N} C_{Nj} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y)$$

$$= \frac{1}{N} \sum_{j=1}^{N} (C_{Ni} - C_{Nj}) \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y),$$

we obtain

$$\operatorname{Var}(B_{1N} + B_{2N}) = \sum_{i=1}^{N} \operatorname{var}(A_{Ni}(X_{Nj})).$$

Proof of Theorem 1.2. Using (2.7), we have

$$\frac{S_N - \mu_N}{s_N} = \frac{T_N}{s_N} + \frac{D_N}{s_N}, \tag{2.34}$$

where

$$T_N = B_{1N} + B_{2N} \quad \text{and} \quad D_N = D_{1N} + D_{2N} + D_{3N}. \tag{2.35}$$

Write

$$F_N(x) = P\left(\frac{S_N - \mu_N}{s_N} \leqslant x\right), \qquad G_N(x) = P\left(\frac{T_N}{s_N} \leqslant x\right). \tag{2.36}$$

Then

$$F_N(x) - \Phi(x) = G_N\left(x - \frac{D_N}{s_N}\right) - \Phi(x) \tag{2.37}$$

$$= \left[G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)\right] + \left[\Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x)\right].$$

Since (by Polya's theorem), both $F_N(x) - \Phi(x)$ and $G_N(x - D_N/s_N) - \Phi(x - D_N/s_N)$ converge to 0 uniformly in x, it follows that the random quantity

$$\Delta_N = \sup_x \left|\Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x)\right| \quad \text{converges to zero.} \tag{2.38}$$

We now estimate

$$\sup_x \left|G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)\right| = \sup_x |G_N(x) - \Phi(x)|. \tag{2.39}$$

Observe that by the Berry–Esseen theorem (cf. Feller [6], p. 544),

$$\sup_x |G_N(x) - \Phi(x)| \leqslant \frac{C\varrho_N^3}{s_N^3} \tag{2.40}$$

where C can be taken to be 0.7975 (cf. Van Beek [1] (1972), and Bhattacharya–Ranga Rao [3] (1976)).

From (2.40) and (2.37), we have

$$\sup_x |F_N(x)-F(x)| \leqslant \frac{C\varrho_N^3}{s_N^3}+\Delta_N. \tag{2.41}$$

It remains to show that $N^{1/2}\Delta_N = O_p(||\varphi'||s_N^{-1}\sum_{i=1}^{N}|C_{Ni}|)$

From (2.38), since

$$\left|\Phi\left(x-\frac{D_N}{s_N}\right)-\Phi(x)\right| = \left|\frac{D_N}{s_N}\right|\Phi'\left(x-\frac{\alpha D_N}{s_N}\right) \quad \text{for some } \alpha,\ 0\leqslant\alpha\leqslant 1, \tag{2.42}$$
$$\leqslant \frac{1}{\sqrt{2\pi}}\left|\frac{D_N}{s_N}\right|,$$

it suffices to show that $N^{1/2}D_N = O_p(||\varphi'||s_N^{-1}\sum_{i=1}^{N}|C_{Ni}|)$.

Now re-arranging the terms of D_N, we obtain

$$D_N = \int_{-\infty}^{\infty}\left\{\varphi\left(\frac{N}{N+1}H_N(x)\right)-\varphi(H(x))\right\}dC_N(x)-\int_{-\infty}^{\infty}(H_N(x)-H(x))\,dC(x). \tag{2.43}$$

To simplify the proof, we drop the factor $N/(N+1)$ since it does not affect the conclusion. By the mean-value theorem,

$$\varphi(H_N(x))-\varphi(H(x)) = (H_N(x)-H(x))\varphi'(\xi_N(x)) \quad \text{for some } \xi_N(x). \tag{2.44}$$

Hence

$$\left|\frac{N^{1/2}}{s_N}\int_{-\infty}^{\infty}(H_N(x)-H(x))\varphi'(\xi_N(x))\,dC_N(x)\right| \leqslant \frac{||\varphi'||}{s_N}\int_{-\infty}^{\infty}N^{1/2}(H_N(x)-H(x))\,dC_N(x).$$

Let $\varepsilon > 0$ be given. Then from Puri and Sen [11], there exists a constant $C(\varepsilon)$ such that with probability $> 1-\varepsilon$,

$$\sup_x N^{1/2}|H_N(x)-H(x)| < C(\varepsilon).$$

Hence, with probability $> 1-\varepsilon$,

$$\left|\frac{N^{1/2}}{s_N}\int_{-\infty}^{\infty}(H_N(x)-H(x))\varphi'(\xi_N(x))\,dC_N(x)\right| \leqslant \frac{C(\varepsilon)}{s_N}||\varphi'||\sum_{i=1}^{N}|C_{Ni}|. \tag{2.45}$$

The proof of

$$\frac{1}{s_N}\left|\int_{-\infty}^{\infty}N^{1/2}(H_N(x)-H(x))\varphi'(H(x))\,dC(x)\right| \leqslant \frac{C(\varepsilon)}{s_N}||\varphi'||\sum_{i=1}^{N}|C_{Ni}| \tag{2.46}$$

in probability is identical.

(2.45) and (2.46) establish the theorem (1.2).

References

[1] P. van Beek, *An application of the Fourier method to the problem of sharpening the Berry-Esseen inequality*, Z. Wahrscheinlichkeits theorie verv. Geb. 23 (1972), pp. 187–197.

[2] H. Bergström and M. L. Puri, *Convergence and remainder terms in linear rank statistics*, Ann. Statist. 5 (1977), pp. 671–680.

[3] R. N. Bhattacharya and R. Ranga Rao, *Normal approximation and asymptotic expansions*, John Wiley, New York 1976.

[4] P. J. Bickel, *Edgeworth expansions in nonparametric statistics*, Ann. Statist. 2 (1974), pp. 1–20.

[5] H. Chernoff and I. R. Savage, *Asymptotic normality and efficiency of certain non-parametric test statistics*, Ann. Math. Statist. 29 (1958), pp. 972–996.

[6] W. Feller, *An introduction to Probability theory and its applications*, vol. 2. John Wiley, New York 1971.

[7] J. Hájek, *Asymptotic normality of simple linear rank statistics under alternatives*, Ann. Math. Statist. 39 (1968), pp. 325–346.

[8] W. Hoeffding, *On the centering of a simple linear rank statistics*, Ann. Statist. 1 (1973), pp. 56–66.

[9] J. Jurečková and M. L. Puri, *Order of normal approximation for rank test statistics distribution*, Ann. Prob. 3 (1975), pp. 526–533.

[10] M. Loève, *Probability theory*, 3rd ed., Van Nostrand, New York 1963.

[11] M. L. Puri and P. K. Sen, *Non-parametric methods in multivariate analysis*, John Wiley, New York 1971.

[12] C. Stein, *A bound for error in the normal approximation to the distribution of a sum of dependent random variables*, Proc. 6th Berkeley Symp. Math. Stat. and Prob., vol. 2, 1970.

Presented to the semester
MATHEMATICAL STATISTICS
September 15–December 18, 1976