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ON SOME PROBLEMS RELATED TO FUNDAMENTAL CYCLE SETS OF A GRAPH: RESEARCH NOTES

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1. Introduction

The paper is concerned with some structural features of a fundamental cycle set graph and mutual connections between the adjacent fundamental cycle graphs. It seems that making use of the properties described in this paper we shall be able to improve the methods for solving some extremal problems related to cycles of a graph.

Before getting into the details, we shall begin with a brief summary of some definitions and notations which will be used in this paper. Other terms not defined here can be found in [3].

Let $G = (V(G), E(G))$ be a *simle graph*, i.e., without loops and multiple edges. In what follows, the set of all the edges and the set of all the vertices of a graph F are denoted by $E(F)$ and $V(F)$, respectively. We consider in this paper labelled graphs only. Two graphs G_1 and G_2 satisfy the relation $G_1 = G_2$ if they are the isomorphic labelled graphs. A *simple path* from v to w is a sequence of distinct vertices and edges leading from v to w. A closed simple path is a *cycle*. With every graph G we can associate the vector space of all cycles and unions of edge-disjoint cycles called the *cycle space* of G. A *cycle basis* of G is defined as a basis for the cycle space of G which consists entirely of cycles. There are special cycle bases of a graph which can be derived from spanning trees of G (in the sequel, a spanning tree of G will be called simply a tree of G). Let t be a tree of G. Then, the set of cycles $c(t)$ obtained by inserting each of the remaining edges of G into t is a *fundamental cycle set* of G with respect to t. Two cycle bases c_1 and c_2 satisfy the relation $c_1 = c_2$ if there exists a one-to-one correspondence φ: $c_1 \leftrightarrow c_2$ between the cycles of c_1 and the cycles of c_2, where the elements in c_1 and c_2 are considered as labelled graphs which are cycles. Paper [9] contains some necessary and sufficient conditions for a cycle basis to be a fundamental cycle set.

Two trees t_1 and t_2 of a graph G are said to be *adjacent* if there exist edges

$e_1 \in t_1 - t_2$ and $e_2 \in t_2 - t_1$ such $t_2 = t_1 - e_1 \cup e_2$. The *tree graph* T of a connected graph G is defined as a graph in which each vertex corresponds to a tree of G and two vertices of T are adjacent if they correspond to adjacent trees of G.

Let c be a cycle basis of a graph G. The intersection graph $B(G, c)$ of c over the set of edges of G is called a *cycle graph* of G with respect to c. If c is a fundamental cycle set, then $B(G, c)$ is called a *fundamental cycle graph*. Some necessary conditions for a graph to be a cycle graph or a fundamental cycle graph were presented in [8].

The *length* of a cycle basis $c = \{f_i\}$ is defined as follows

$$|c| = \sum_{f_i \in c} |f_i|,$$

where $|f_i|$ denotes the number of edges in f_i.

The problem of finding a minimum cycle basis and a minimum fundamental cycle set of a graph has been considered in [4], [6].

Enumeration of all the cycles of a graph using the vector space approach, finding a minimum cycle basis and a minimum fundamental cycle set of a graph, and finding the longest cycle of a graph are three problems related to cycle bases of a graph which still have some open question, see [2], [7]; [4], [5], and [2], [9], respectively.

It can easily be shown that there exists a one-to-one correspondence between cycles of a graph G and connected induced subgraphs of $B(G, c)$, where c is a cycle basis of G, but unfortunately this correspondence is not necessarily onto. Thus the following questions arise

PROBLEM 1.1. What cycle basis c of a graph G be chosen so that $B(G, c)$ has a minimum number of connected induced subgraphs.

PROBLEM 1.2. Let G be a given graph. Does there exist a cycle basis c of G such that there exists a one-to-one correspondence between the family of all cycles of G and the family of all induced subgraphs of $B(G, c)$?

Another problem related to a cycle basis of a graph appeared in [2].

PROBLEM 1.3. Let G be a given graph. It it possible to find a cycle basis of G such that for every cycle f of G, the basic cycles that comprise f can be ordered in such a way that all ring sums of the consecutive subsequences of the basic cycles are cycles?

The main purpose of this paper is to clarify some structural features of fundamental cycle set graphs (Section 3) and adjacent fundamental cycle graphs (Section 4).

2. Conjectures and counterexamples

One of the first questions appearing to someone who has introduced a new concept is how this new concept is related to other notions introduced so far. Usually some conjectures are formulated as the result of such an investigation.

In this section we present some counterexamples for the conjectures which are concerned with the concepts defined in Section 1 and with other notions related to trees and cycle bases.

It is easy to find a graph G such that its minimum cycle basis does not minimize the number of the (induced) connected subgraphs in the cycle graph of G ([7], [8]).

CONJECTURE 2.1 [7]. *The cycle graph of a graph G with the minimum number of edges has the minimum number of the induced connected subgraphs among all the cycle graphs of G.*

We can show only that in general this conjecture is not valid, i.e., if we take into consideration $\mathcal{G}_n$, the class of all graphs with n vertices, then the number of the induced connected subgraphs of a graph is not an increasing function of the number of edges of the graph. For instance, two graphs shown in Fig. 2.1 belong to $\mathcal{G}_5$, F has four edges and G has five edges but F has 15 induced connected subgraphs and G has only 13 ones.

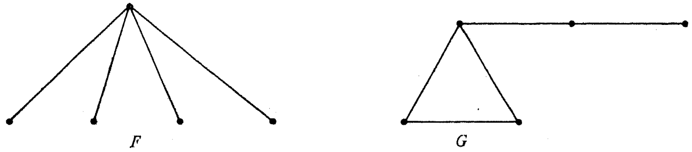

Fig. 2.1

It can be easily shown that graphs F and G are not the cycle graphs of the same graph so they do not form a counterexample for conjecture 2.1.

Paper [9] contains the counterexamples for the following conjecture of Dixon and Goodman (see [2]): for any cycle basis c of a given graph G and for every cycle f of G it is possible to order the basic cycles which comprise f in such a way that all ring sums of the consecutive subsequences of the basic cycles are cycles.

Papers [4], [6] which deal with the problem of finding a minimum cycle basis contain some examples showing that the extremum trees of a graph with respect

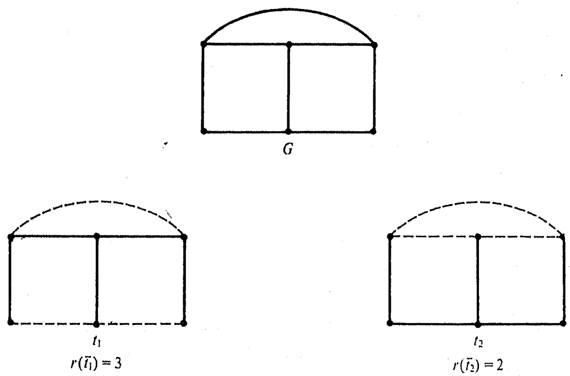

Fig. 2.2

to a properly defined weight function are not good approximations of the tree which generates the minimum fundamental cycle set. It has also been shown that a local neighbourhood search optimization algorithm fails to find an optimal solution to the minimum fundamental cycle set problem.

A *central tree* has been introduced by Deo [1] as the best starting tree for generating the rest of the trees in a graph. A central tree is a tree t_0 in G such that $r(\bar{t}_0) \leqslant r(\bar{t}_i)$ for every tree t_i in G, where $\bar{t}$ is the complement of t (i.e., $t \cup \bar{t} = G$) and $r(F)$ is the rank function of a graph (i.e., $r(F) = v(F) - p(F)$, where v and p denote the number of vertices in F and the number of connected subgraphs in F, resp.). Unfortunately, as it is illustrated in Fig. 2.2, in general a central tree does not generate a minimum fundamental cycle set. The minimum fundamental cycle set of the graph G is generated by non-central tree t_1 and the central tree t_2 of G generates the fundamental cycle set which is not minimal.

3. Adjacency of fundamental cycle sets of a graph

A local neighbourhood search type algorithm for finding a minimum fundamental cycle set of a graph has been presented in [4]. The algorithm starts with a fundamental cycle set corresponding to an arbitrary tree of a graph and then at every step the minimum fundamental cycle set among those which can be generated by the trees adjacent to the tree generating the current fundamental cycle set is chosen as a new, improved solution.

It is obvious that the adjacency of trees of a graph induces somehow the adjacency of fundamental cycle sets but we must be aware of the fact that even non-adjacent trees can generate the same fundamental cycle set.

Figure 3.1 shows a graph F and four of its trees. Tree t_1 is adjacent to t_2 and $c(t_1) = c(t_2)$, t_1 is adjacent to t_3 and $c(t_1) \neq c(t_3)$, and despite t_1 is not adjacent to t_4 we have $c(t_1) = c(t_4)$.

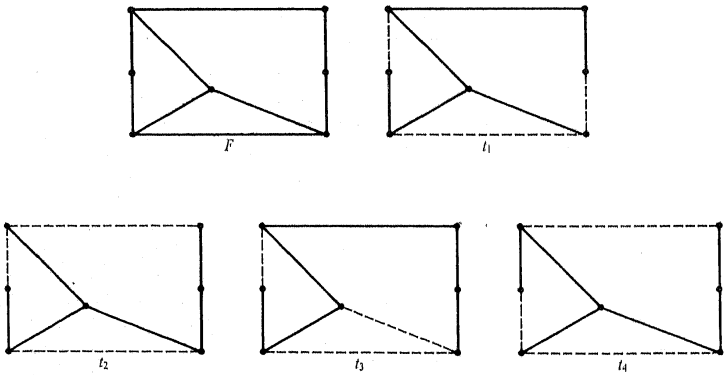

Fig. 3.1

Let t_1 and t_2 be two adjacent trees of a graph G, i.e., there exist two edges $e_1 \in t_1 - t_2$ and $e_2 \in t_2 - t_1$ such that $t_2 = t_1 - e_1 \cup e_2$, and let f_l denote the fundamental cycle in $c(t_1)$ such that $e_1, e_2 \in f_l$, i.e., e_1 belongs to the fundamental cycle of t_1 with respect to the chord e_2 and e_2 belongs to the fundamental cutset of t_1 with respect to the tree edge e_1. Then, the fundamental cycle set $c(t_2)$ of tree t_2 is of the form

$$(1) \qquad c(t_2) = f_l \cup \{f_j\colon f_j \in c(t_1),\ e_1 \notin f_j\} \cup \{g_i = f_i \oplus f_l\colon f_i \in c(t_1),\ e_1 \in f_i, f_i \neq f_l\},$$

where the third set on the right-hand side consists of the new fundamental cycles, i.e., which do not belong to $c(t_1)$ and the second one is the subset of $c(t_1)$ containing fundamental cycles which do not contain the tree edge e_1.

Let c_1 and c_2 be two cycle bases of a graph G which are not necessarily fundamental cycle sets. Then, c_1 and c_2 are said to be *adjacent* if there exist cycles $f \in c_1 - c_2$ and $g \in c_2 - c_1$ such that $c_2 = c_1 - f \cup g$. Now, we may ask which fundamental cycle sets generated by the adjacent trees are adjacent. The following theorem gives the answer to this question.

THEOREM 3.1. *Two fundamental cycle sets $c(t_1)$ and $c(t_2)$ generated by the adjacent trees t_1 and t_2 are adjacent if and only if e_1 belongs to exactly two fundamental cycles of $c(t_1)$.*

Proof. The theorem follows directly from (1). ∎

Figure 3.2 shows a graph and its two adjacent trees which generate the fundamental cycle sets that are not adjacent cycle bases.

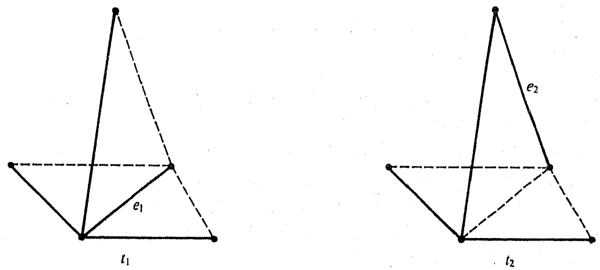

Fig. 3.2

As it was illustrated in Fig. 3.1, the same fundamental cycle set may be generated by different and even non-adjacent trees.

Let $I(c)$ denote the set of edges of a graph which belong to at least two cycles of a cycle basis c. To find all trees of a graph which generate the same fundamental cycle set we shall prove the following lemma.

LEMMA 3.1. *For any graph G and its subset of edges E_1 which contains no cycle, if there exists a fundamental cycle set c such that $I(c) = E_1$, then c is determined uniquely.*

Proof. Without loss of generality we may assume that a given graph G is biconnected so that every edge in G belongs to at least one cycle in G. We shall show now that the edges of G which do not belong to E_1 can be uniquely partitioned into vertex-disjoint series of edges (a *series of edges* is defined as an elementary path consisting only of vertices of degree 2 except its end-vertices).

Suppose that v is a vertex of G incident with no edge in E_1. Since every edge in G belongs to a cycle and the edges not in E_1 belong to exactly one cycle, v is of even degree, therefore the fundamental cycle set c induces the unique partition of the edges incident with v into pairs. Now, spliting v into two vertices in such a way that no pair of edges is split, we decrease the cyclomatic number of the graph despite the set of cycles remains the same.

Thus, the vertices incident with no edge in E_1 can be only of degree 2 so that the edges in G which do not belong to E_1 can be uniquely partitioned into vertex-disjoint (except end-vertices) series of edges. Since c is a fundamental cycle set, i.e., every cycle in c contains at least one edge which belongs to exactly one cycle in c, there exists a one-to-one correspondence between the cycles in c and the series of edges, which together with E_1 generate c uniquely. ■

It can be easily shown that in general the lemma does not remain valid when E_1 is any subset of edges of a graph and/or c is any cycle basis. In these cases there may exist no one-to-one correspondence between the series consisting of edges which do not belong to E_1 and the cycles in c.

COROLLARY 3.1. *Let c_1 and c_2 be two fundamental cycle sets of a graph G. Then, $I(c_1) = I(c_2)$ if and only if $c_1 = c_2$.* ■

COROLLARY 3.2. *Let t_1 be a tree of a graph G. Then for any other tree t of G we have $c(t_1) = c(t)$ if and only if $t \supseteq I(c(t_1))$.* ■

As indicated above, the set of edges of a graph which belong to at least two cycles of a fundamental cycle set of a graph plays an important role in finding all trees of the graph which generate the same fundamental cycle set. The next two lemmas provide some properties of I-sets corresponding to the fundamental cycle sets generated by the adjacent trees.

LEMMA 3.2. *Let t_1 and t_2 be two adjacent trees of a graph G, i.e., $t_2 = t_1 - e_1 \cup e_2$. Then $I(c(t_2)) = I(c(t_1))$ if $e_1 \notin I(c(t_1))$ and $e_2 \in I(c(t_2))$ otherwise.*

Proof. The first part follows from the fact that the fundamental cycle sets generated by trees t_1 and $t_2 = t_1 - e_1 \cup e_2$ are isomorphic if $e_1 \notin I(c(t_1))$. Otherwise, if $e_1 \in I(c(t_1))$, then e_2 belongs to the cycle $f_l \in c(t_1) \cap c(t_2)$ which contains e_1 and e_2, and therefore e_2 belongs to every cycle (at least to one) $f \in c(t_1), f \neq f_l$ which contains e_1, since $e_2 \in f \oplus f_l$. ■

LEMMA 3.3. *Let t_1 and t_2 be two adjacent trees of a graph G, i.e., $t_2 = t_1 - e_1 \cup e_2$ and let $f_l \in c(t_1)$ and $e_1, e_2 \in f_l$. Then*

$$I(c(t_2)) = f_l \cap \Big[\bigcup_{f_i^1 \in U_1} f_i^1 \cup \bigcup_{f_j^1 \in V_1} \overline{f_j^1}\Big] \cup \overline{f_l} \cap I(c(t_1)),$$

where $U_1 = \{f_i^1 \in c(t_1)\colon e_1 \notin f_i^1\}$ and $V_1 = \{f_j^1 \in c(t_1)\colon e_1 \in f_j^1, f_j^1 \neq f_l\}$.

Proof. The set $I(c)$ is the union of all edges which belong to exactly two cycles of c, so that

$$I(c(t_2)) = \bigcup_{f_i^2, f_j^2 \in c(t_2)} f_i^2 \cap f_j^2$$

and applying (1) we can partition this set as follows

$$I(c(t_2)) = \bigcup_{f_i^2, f_j^2 \in U_2} f_i^2 \cap f_j^2 \cup \bigcup_{\substack{f_i^2 \in U_2 \\ f_j^2 \in V_2}} f_i^2 \cap f_j^2 \cup \bigcup_{f_i^2, f_j^2 \in V_2} f_i^2 \cap f_j^2 \cup \bigcup_{f_i^2 \in U_2} f_l \cap f_i^2 \cup \bigcup_{f_j^2 \in V_2} f_l \cap f_j^2$$

where U_2 and V_2 denote the second and the third set on the right-hand side of (1). Since $U_1 = U_2$ and $V_2 = \{f_j^1 \oplus f_l\colon f_j^1 \in V_1\}$, we obtain

$$\begin{aligned}
I(c(t_2)) = & \bigcup_{f_i^1, f_j^2 \in U_1} f_i^1 \cap f_j^1 \cup \bigcup_{\substack{f_i^1 \in U_1 \\ f_j^1 \in V_1}} f_j^1 \cap (f_j^1 \oplus f_l) \cup \\
& \cup \bigcup_{f_i^1, f_j^1 \in V_1} (f_i^1 \oplus f_l \cap f_j^1 \oplus f_l) \cup \bigcup_{f_i^1 \in U_1} f_l \cap f_i^1 \cup \bigcup_{f_j^1 \in V_1} f_l \cap f_j^1 \oplus f_l \\
= & (f_l \cup \overline{f_l}) \cap \bigcup_{f_i^1, f_j^1 \in U_1} f_i^1 \cap f_j^1 \cup \overline{f_l} \cap \bigcup_{\substack{f_i^1 \in U_1 \\ f_j^1 \in V_1}} f_i^1 \cap f_j^1 \cup f_l \cap \bigcup_{\substack{f_i^1 \in U_1 \\ f_j^1 \in V_1}} f_i^1 \cap \overline{f_j^1} \cup \overline{f_l} \cap \\
& \cap \bigcup_{f_i^1, f_j^1 \in V_1} f_i^1 \cap f_j^1 \cup f_l \cap \bigcup_{f_i^1, f_j^2 \in V_1} \overline{f_i^1} \cap \overline{f_j^1} \cup f_l \cap \bigcup_{f_i^1 \in U_1} f_i^1 \cup f_l \cap \bigcup_{f_j^1 \in V_1} \overline{f_j^1} \\
= & f_l \cap \Big[\bigcup_{f_i^1, f_j^1 \in U_1} f_i^1 \cap f_j^1 \cup \bigcup_{\substack{f_i^1 \in U_1 \\ f_j^1 \in V_1}} f_i^1 \cap \overline{f_j^1} \cup \bigcup_{f_i^1, f_j^1 \in V_1} \overline{f_i^1} \cap \overline{f_j^1} \cup \bigcup_{f_i^1 \in U_1} f_i^1 \cup \bigcup_{f_j^1 \in V_1} \overline{f_j^1}\Big] \cup \\
& \cup \overline{f_l} \cap \Big[\bigcup_{f_i^1, f_j^1 \in U_1} f_i^1 \cap f_j^1 \cup \bigcup_{\substack{f_i^1 \in U_1 \\ f_j^1 \in V_1}} f_i^1 \cap f_j^1 \cup \bigcup_{f_i^1, f_j^1 \in V_1} f_i^1 \cap f_j^1\Big].
\end{aligned}$$

Applying the absorption law $x \cup xy = x$ to the first term and adding the empty set $\overline{f_l} \cap \big[\bigcup_{f_i^1 \in U_1 \cup V_1} f_l \cap f_i^1\big]$ to the second one, we obtain finally

$$I(c(t_2)) = f_l \cap \Big[\bigcup_{f_i^1 \in U_1} f_i^1 \cup \bigcup_{f_j^1 \in V_1} \overline{f_j^1}\Big] \cup \overline{f_l} \cap I(c(t_1)). \ ■$$

As shown above, a fundamental cycle set of a graph may be generated by more than one tree of a graph. This fact leads to the following definition of the adjacency of fundamental cycle sets of a graph.

Two fundamental cycle sets c_1 and c_2 of a graph G are said to be *tree-adjacent* if there exist two adjacent trees t_1 and t_2 of G such that $c_1 = c(t_1)$ and $c_2 = c(t_2)$. Thus, we can define a *fundamental cycle set graph* F of a graph G as a graph in which each vertex corresponds to a fundamental cycle set of G and each edge corresponds to a pair of tree-adjacent fundamental cycle sets of G. The rest of this section is intended to clarify some structural features of the fundamental cycle set graph and some relations between the tree graph T and the fundamental cycle set graph F of a graph G.

Let $\{e_1, e_2, \ldots, e_n\}$ be a subset of edges of a graph G. Then, following the idea of Kishi and Kajitani [5] we can define a subset of trees of G

$$T\begin{bmatrix} e_1, e_2, \ldots, e_n \\ x_1, x_2, \ldots, x_n \end{bmatrix},$$

where $x_i = 0$ or 1 $(i = 1, 2, \ldots, n)$ as the collection of the trees which contain e_i if $x_i = 1$ but do not if $x_i = 0$. Let $t \in T$; then all vertices of the tree graph which correspond to trees in

$$T\begin{bmatrix} I(c(t)) \\ 1 \end{bmatrix} \tag{2}$$

are condensed into one vertex in the fundamental cycle set graph, since all trees which contain edges $I(c(t))$ generate the same fundamental cycle set.

Let F_c denote the local subgraph of F with respect to a fundamental cycle set c which is the subgraph defined by the collection of all the fundamental cycle sets whose distance from c is 1. Note that a subset of trees of the form (2) may contain also trees which do not belong to T_t, the local subgraph of the tree graph T with respect to t.

In order to describe the structure of local subgraphs of F we procede similarly as it has been done in the case of local subgraphs of T, see [5]. First, we define τ- and γ-sets.

Let c_0 be a fundamental cycle set of a graph G and x_i be an edge which does not belong to $I(c_0)$. Then, the subset of fundamental cycle sets $\tau_{c_0}(x_i)$ is defined as the collection of all the fundamental cycle sets of G whose I-sets contain x_i and whose distance from c_0 is 1. Similarly, the subset of fundamental cycle sets $\gamma_{c_0}(a_j)$ for $a_j \in I(c_0)$ is defined as the collection of all the fundamental cycle sets whose I-sets do not contain a_j and whose distance from c_0 is 1.

It is easily shown that any τ- or γ-set of a fundamental cycle set c_0 forms the complete subgraph of the local subgraph F_{c_0} of F with respect to c_0. Moreover, applying Lemma 3.2, it can be shown that

$$\tau_{c_0}(x_i) \cap \gamma_{c_0}(a_j) = \begin{cases} c(t_{ij}), \\ \emptyset, \end{cases}$$

whether the fundamental cutset of a tree which contains $I(c_0)$ with respect to a_j contains x_i or not, where t_{ij} can be obtained from a tree t_0 which generates c_0 by removing a_j and adding x_i.

Let us consider a graph G shown in Fig. 3.3 and its tree $t = \{a_1, a_2, a_3, a_4, x_1, x_2, x_6\}$. In this case we have $I(c(t)) = \{a_1, a_2, a_3, a_4\}$ and notice that $\gamma_{c(t)}(a_2) = \gamma_{c(t)}(a_3)$, $\tau_{c(t)}(x_1) = \tau_{c(t)}(x_2) = \tau_{c(t)}(x_3)$, $\tau_{c(t)}(x_5) = \tau_{c(t)}(x_6)$.

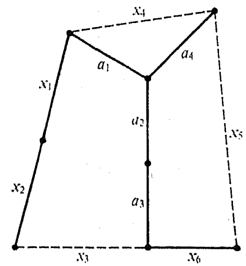

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Fig. 3.3

The following theorem is a counterpart of Theorem 4 in [5] dealing with the decomposition of a local subgraph.

THEOREM 3.2. *Let c be a fundamental cycle set of G, let $I(c) = \{a_1, a_2, \ldots, a_p\}$ denote the set of edges of G which belong to at least two fundamental cycles of c, and let $\{x_1, x_2, \ldots, x_q\}$ be the set of edges of G which do not belong to $I(c)$. Then the sets of all the vertices and the edges of the local subgraph F_c can be partitioned as follows*:

$$V(F_c) = \bigcup_{i=1}^{q} V[\tau_c(x_i)] = \bigcup_{j=1}^{p} V[\gamma_c(a_j)],$$

$$E(F_c) = \bigcup_{i=1}^{q} E[\tau_c(x_i)] \cup \bigcup_{j=1}^{p} E[\gamma_c(a_j)],$$

where

$$V[\tau_c(x_i)] \cap V[\tau_c(x_k)] = \emptyset \quad or \quad V[\tau_c(x_i)] = V[\tau_c(x_k)] \quad (i \neq k),$$
$$E[\tau_c(x_i)] \cap E[\tau_c(x_k)] = \emptyset \quad or \quad E[\tau_c(x_i)] = E[\tau_c(x_k)] \quad (i \neq k),$$

whether x_i and x_k are contained in the same fundamental cycle set or not,

$$V[\gamma_c(a_i)] \cap V[\gamma_c(a_k)] = \emptyset \quad or \quad V[\gamma_c(a_i)] = V[\gamma_c(a_k)] \quad (i \neq k),$$
$$E[\gamma_c(a_i)] \cap E[\gamma_c(a_k)] = \emptyset \quad or \quad E[\gamma_c(a_i)] = E[\gamma_c(a_k)] \quad (i \neq k),$$

whether a_i and a_k belong to the same subset of fundamental cycles of c or not, and

$$E[\tau_c(x_i)] \cap E[\gamma_c(a_j)] = \emptyset. \ \blacksquare$$

The proof of the theorem can be easily derived from the preceding consideration so it is omitted.

It is well known that a tree graph contains a hamiltonian cycle. First, the existence has been proved by induction and then the investigation of some topological features of a tree graph led to two procedures for generating a hamiltonian cycle in a tree graph, see [5]. We conjecture that a fundamental cycle set graph is also hamiltonian.

In fact, the existence of a hamiltonian cycle in a fundamental cycle set graph is of a little use to our problems presented in Section 1, since we are more interested in finding a special cycle basis or a special fundamental cycle set of a graph than in enumerating all cycle bases or fundamental cycle sets of a graph.

EXAMPLE 3.1. Consider the graph G shown in Fig. 3.4 (a) (see also [5]). The tree graph of G has 12 vertices and the fundamental cycle set graph F of G has only 4 ones. Figure 3.4 (b) shows one of the hamiltonian cycles of the tree graph of G obtained by the procedure presented by Kishi and Kajitani. (Different figures in vertices correspond to different fundamental cycle sets of G generated by trees associated with vertices.) Figure 3.4 (c) is self-explained.

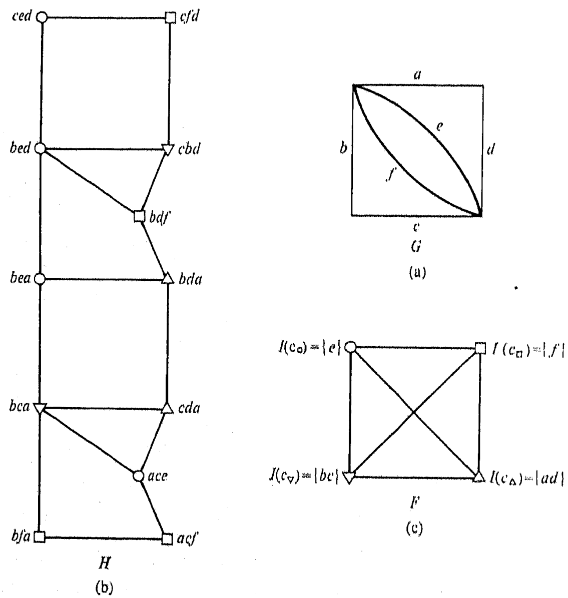

Fig. 3.4

4. Adjacency of fundamental cycle graphs

This section is concerned with the mutual connections between two *adjacent fundamental cycle graphs* of a graph, i.e., between two cycle graphs which correspond to the tree-adjacent fundamental cycle sets. Section 2 contains the counterexample for the conjecture that in the class of all graphs with the same number of vertices the number of the connected subgraphs of a graph is a non-decreasing function of the

number of edges of the graph. Despite this fact we hope that this conjecture is valid in the class of all fundamental cycle graphs of a graph.

Let t_1 and t_2 be two adjacent trees of a graph G, i.e., there exist $e_1 \in t_1 - t_2$ and $e_2 \in t_2 - t_1$ such that $t_2 = t_1 - e_1 \cup e_2$. Suppose that $e_1, e_2 \in f_l$, where $f_l \in c(t_1)$, $c(t_2)$ then

$$c(t_2) = f_l \cup \{g_i = f_l \oplus f_i \colon f_i \in c(t_1),\ e_1 \in f_i\} \cup$$
$$\cup \{f_i \colon f_i \in c(t_1), f_l \cap f_i = \emptyset\} \cup \{f_i \colon f_i \in c(t_1), f_l \cap f_i \neq \emptyset,\ e_1 \notin f_i\}$$
$$= f_l \cup N_1 \cup N_2 \cup N_3.$$

One can easily show that we must take into consideration the labelled fundamental cycle graph $B(G, c(t_1))$ to be able to transform it into a fundamental cycle graph of G with respect to the tree-adjacent fundamental cycle set $c(t_2)$. Let r_i and s_{ij} denote the labels of vertices and edges of $B(G, c)$, resp., defined as follows $r_i = r(f_i) = \{f_i\}$ for the vertex corresponding to cycle f_i and $s_{ij} = s((f_i, f_j)) = \{f_i \cap f_j\}$ for the edge corresponding to a non-empty intersection of two cycles (though labels s_{ij} are redundant, since they can be easily derived from the vertex labels, we introduced them for the sake of simplicity).

In the rest of this section we describe $B(G, c(t_2))$ in terms of the labelled fundamental cycle graph $B(G, c(t_1))$, where t_1 and t_2 are two adjacent trees of G.

Let us notice that $c(t_2) - c(t_1) = N_1$ so that only those vertices of $B(G, c(t_2))$ change their labels and neighbours.

The following steps constitute an algorithm for transforming $B(G, c(t_1))$ into $B(G, c(t_2))$, r'_i, s'_{ij} denote the labels of vertices and edges of the latter graph.

Step 1. *Labeling of vertices.*

$$r'_i = \{g_i\} = \{f_l \oplus f_i\} = f_l \cup f_i - f_l \cap f_i = r_l \cup r_i - s_{li} = r_i \cup s'_{il}$$

for every vertex corresponding to the fundamental cycle in N_1 (see Step 2 (a)), and

$$r'_j = r_j,$$

otherwise.

Step 2. *Labeling of edges.*

(a) Labeling of the edges incident with f_l. Since $g_i \cap f_l \neq \emptyset$ $(g_i \in N_1)$ and other fundamental cycles of $B(G, c(t_2))$ remain the same, f_l does not change its neighbours. We must modify only the labels of edges connecting f_l with N_1.

$$s'_{il} = g_i \cap f_l = f_l - (f_i \cap f_l) = r_l - s_{il}.$$

(b) Edges connecting vertices in N_1. Let us consider two fundamental cycles g_i and g_j $(g_i, g_j \in N_1)$. We have

$$g_i \cap g_j = f_l \oplus f_i \cap f_j \oplus f_l = (f_l \cap \bar{f_i} \cup \bar{f_l} \cap f_i) \cap (f_j \cap \bar{f_l} \cup \bar{f_j} \cap f_l)$$
$$= [f_i \cap f_j - f_l] \cup [f_l - (f_i \cup f_j)].$$

Notice that f_l non $\subseteq f_i \cup f_j$, since cycle basis $c(t_1)$ is a fundamental cycle set, i.e., every cycle in $c(t_1)$ contains an edge which does not belong to other cycles in $c(t_1)$

(see [9]). Therefore $g_i \cap g_j \neq \emptyset$, i.e., there exists the edge in $B(G, c(t_2))$ between g_i and g_j and it has the following label

$$s'_{ij} = s_{ij} - r_l \cup r_l - (r_l \cup r_j) = s_{ij} \cup r_l - (s_{il} \cup s_{jl}).$$

(c) Edges between N_1 and N_2. If $g_i \in N_1$ and $f_j \in N_2$, then

$$s'_{ij} = g_i \cap f_j = f_l \oplus f_i \cap f_j = f_l \cap \overline{f_i} \cap f_j \cup f_i \cap \overline{f_l} \cap f_j,$$

and since $f_l \cap f_j = \emptyset$ for $f_j \in N_2$, we obtain

$$s'_{ij} = f_i \cap f_j = s_{ij}.$$

(d) Edges between N_1 and N_3. Let $g_i \in N_1$ and $f_j \in N_3$. In this case we have

$$s'_{ij} = g_i \cap f_j = f_l \oplus f_i \cap f_j = f_i \cap f_j - f_l \cup f_j \cap f_l - f_i,$$

and we shall prove that $s'_{ij} = \emptyset$ if and only if $f_i \cap f_j = f_j \cap f_l$. Suppose that $s'_{ij} = \emptyset$, i.e., $f_i \cap f_j - f_l = f_j \cap f_l - f_i = \emptyset$. Hence $f_l \supseteq f_i \cap f_j$ and $f_i \supseteq f_j \cap f_l$, and therefore $f_l \cap f_j \supseteq f_i \cap f_j$ and $f_i \cap f_j \supseteq f_j \cap f_l$. Thus, $f_i \cap f_j = f_j \cap f_l$. On the other hand, if $f_i \cap f_j = f_j \cap f_l$, then $f_i \cap f_j - f_l = f_j \cap f_l - f_l = \emptyset$ and $f_l \cap f_j - f_i = f_j \cap f_i - f_i = \emptyset$ so that $s'_{ij} = \emptyset$.

Let us suppose that $f_i \cap f_j = \emptyset$; then $s'_{ij} = f_j \cap f_l \cap \overline{f_i} = f_j \cap f_l = s_{jl} \neq \emptyset$, so that a new edge between g_i and f_l appears in $B(G, c(t_2))$.

In the opposite case, i.e., if $f_i \cap f_j \neq \emptyset$ we have: if $s_{ij} = s_{jl}$, then $s'_{ij} = \emptyset$ and

$$s'_{ij} = f_l \oplus f_i \cap f_j = [(f_i \cup f_l) - (f_i \cap f_l)] \cap f_j = (f_i \cup f_l) \cap f_l - (f_i \cap f_l)$$
$$= f_i \cap f_j \cup f_l \cap f_j - (f_i \cap f_l) = s_{ij} \cup s_{lj} - s_{il} \neq \emptyset$$

if $s_{ij} \neq s_{jl}$. ∎

The procedure presented above describes the elementary transformation of a fundamental cycle graph into a cycle graph corresponding to a tree-adjacent fundamental cycle set. We hope that the method can be generalized to produce a fundamental cycle graph which corresponds to a fundamental cycle set having special properties required by an extremal problem considered related to a cycle basis of a graph.

Added in proof. The conjecture, that every fundamental cycle set graph is hamiltonian has been settled by the author in affirmative (see M. M. Sysło, *On same problems related to fundamental cycle sets of a graph*, in: G. Chartrand, Y. Alavi, D. Goldsmith, L. Lesniak-Foster, D.R. Lick (eds.), *The Theory and Applications of Graphs*, J. Wiley, 1981).

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