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# SZ.-NAGY–FOIAŞ THEORY AND SIMILARITY FOR A CLASS OF TOEPLITZ OPERATORS

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## 1. Introduction

**1.1.** In this paper we determine explicitly the Sz.-Nagy–Foiaş characteristic function of a Toeplitz operator of the form $T_{\varphi/\psi}$, where $\varphi$ and $\psi$ are finite Blaschke products, $\psi$ having one zero. We use this to prove a similarity theorem (Theorem 2, below) for $T_{\varphi/\psi}$. The reason for considering Toeplitz operators of this special form is to compare Theorem 2 with a similarity theorem from [1], restated here as Theorem 1. These two theorems occupied my two lectures to the Spectral Semester at the Stefan Banach Center.

In Section 1.2, we introduce Toeplitz operators and the similarity problem and in Section 1.3, we discuss the Sz.-Nagy–Foiaş characteristic function.

**1.2.** Let $L^2$ denote the $L^2$ space of Lebesgue measure on $[0, 2\pi]$ and $H^2$ the $L^2$ closure of the polynomials in $e^{it}$. For a bounded measurable function $F$ on $[0, 2\pi]$, the Toeplitz operator $T_F$ is defined on $H^2$ by

$$T_F x = PFx$$

where $P$ is the projection of $L^2$ on $H^2$.

If $F$ is reasonably smooth (for example rational), the spectral theory of $T_F$ is well known [3]. The essential spectrum of $T_F$ is the curve $\Gamma$: $t \to F(e^{it})$ and for $\lambda \notin \Gamma$, the index of $T_F$ is minus the winding number of $\Gamma$ around $\lambda$. Either the kernel or the cokernel of $T_F - \lambda I$ is always 0, so that the index describes completely the multiplicity of $\lambda$ as an eigenvalue. Moreover, $T_F$ has no eigenvalues in the essential spectrum [1].

In [1], the following similarity theorem was proved for $T_F$.

THEOREM 1. *Suppose that $F(z)$ is a rational function with no poles on $|z| = 1$. Suppose that the curve $\Gamma$ is a simple closed curve of winding number $n$ about its interior points and suppose that $F(z)$ is $n$-to-1 in some annulus $r < |z| \leqslant 1$. Then $T_F$ is similar to $T_{\tau(z^n)}$ where $\tau$ is the Riemann mapping function from the unit disk to the interior of the curve $\Gamma$.*

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The point of Theorem 1 is that $T_{\tau(z^n)}$ is multiplication by $\tau(z^n)$ on $H^2$, $\tau$ being analytic, so $T_{\tau(z^n)}$ is a much simpler operator than $T_F$.

In this paper we consider $T_F$ where $F$ is of the form $F = \varphi/\psi$, where $\varphi$ and $\psi$ are finite Blaschke products, $\psi$ having only one factor:

$$\varphi(z) = \prod_1^n \frac{z-a_j}{1-\bar{a}_j z}, \quad \psi(z) = \frac{z-b}{1-\bar{b}z}, \quad |a_j|,\ |b| < 1. \tag{1}$$

Thus if $n = 1$, the spectrum of $T_F$ is the curve $\Gamma$ and if $n > 1$, the spectrum is the disk and every $\lambda$ of modulus less than 1 is an eigenvalue of $T_F^*$ of multiplicity $n-1$. In case $F'(z) \neq 0$ for $|z| = 1$, Theorem 1 applies and proves that $T_F$ is similar to $T_{z^{n-1}}$. Our methods allow us to examine exactly what happens if $F'(z)$ vanishes on the unit circle. We return to this matter in Part 3.

**1.3.** The Sz.-Nagy–Foiaş operator theory [4] associates with a contraction operator $T$ the operator-valued analytic function $\Theta_T$ given by

$$\Theta_T(\lambda) = -T + \lambda D_{T^*}(I-\lambda T^*)^{-1} D_T$$

where $D_T = (I-T^*T)^{1/2}$, $D_{T^*} = (I-TT^*)^{1/2}$ and where the values of $\Theta_T$ are operators from $\mathcal{D}_T$, the closure of the range of $D_T$, to $\mathcal{D}_{T^*}$, the closure of the range of $D_{T^*}$. Sz.-Nagy and Foiaş use $\Theta_T$ to construct a model for $T$ in case $T$ is completely nonunitary; that is, they construct, from $\Theta_T$, in a canonical way, an operator unitarily equivalent to $T$, and using this representation, many properties of $T$ can be studied. We mention here just one theorem of Sz-.Nagy and Foiaş, which we shall use in this paper: if $\Theta_T$ has a (bounded, analytic) left inverse in $|\lambda| < 1$, then $T$ is similar to an isometry ([5], Theorem 1.4).

In [2], Goor proved that a Toeplitz operator which is a contraction is completely nonunitary. Our approach to $T_F$, where $F = \varphi/\psi$ and (1) holds, is to compute $\Theta_T$ explicitly and prove similarity to an isometry using the above theorem of Sz.-Nagy and Foiaş. Along the way, it is interesting to have the characteristic function $\Theta_T$ for a naturally occurring operator, given explicitly.

## 2. Sz.-Nagy–Foiaş theory

In this part, we compute the Sz.-Nagy–Foiaş characteristic function $\Theta_T$ for $T = T_F$, where $F = \varphi/\psi$, given by (1). The computation is done in four steps in the next four sections, as follows.

2.1. Computation of $D_T$ and $\mathcal{D}_T$,

2.2. Computation of $(I-\lambda T^*)^{-1}\mathcal{D}_T$,

2.3. Computation of $D_{T^*}$ and $\mathcal{D}_{T^*}$,

2.4. Completion of computation of $\Theta_T$.

**2.1.** First we state without proof the simple



LEMMA 1. *If $g(z) \in H^2$, $\zeta$ is a complex number with $|\zeta| < 1$, and $P$ is the projection of $L^2$ on $H^2$, then*

$$P\bar{g}(e^{it})(1-\bar{\zeta}e^{it})^{-1} = \bar{g}(\zeta)(1-\bar{\zeta}e^{it})^{-1},$$

$$Pg(e^{it})(e^{it}-\zeta)^{-1} = [g(e^{it})-g(\zeta)](e^{it}-\zeta)^{-1}.$$

We can now compute $T_F^* T_F$; first on $\psi H^2$, then on $(\psi H^2)^\perp$. Let $x \in H^2$,

$$T_F^* T_F \psi x = T_F^* \varphi x = \psi x.$$

Since $(\psi H^2)^\perp$ is the one-dimensional span of $(1-\bar{b}z)^{-1}$, we need only compute

$$\begin{aligned} T_F^* T_F (1-\bar{b}z)^{-1} &= T_F^* P\varphi(e^{it}-b)^{-1} = T_F^*[\varphi(e^{it})-\varphi(b)](e^{it}-b)^{-1} \\ &= P(\psi/\varphi)[\varphi(e^{it})-\varphi(b)](e^{it}-b)^{-1} \\ &= P[1-\varphi(b)\bar{\varphi}(e^{it})](1-\bar{b}e^{it})^{-1} \\ &= (1-\bar{b}e^{it})^{-1}(1-|\varphi(b)|^2). \end{aligned}$$

The conclusion is

$$\mathcal{D}_T = \{(1-\bar{b}e^{it})^{-1}\},$$

$$D_T(1-\bar{b}e^{it})^{-1} = |\varphi(b)|(1-\bar{b}e^{it})^{-1}.$$

**2.2.** We begin this computation with another lemma.

LEMMA 2. *For each $\lambda$, $|\lambda| < 1$, the equation*

$$\lambda\bar{\varphi}(\bar{z})(1-bz)-(z-\bar{b}) = 0 \tag{2}$$

*has a unique solution $z = d(\lambda)$ in $|z| < 1$.*

The proof of the lemma could be based on Lemma 2.1 of [1], but we give a self-contained proof of existence; uniqueness will follow from the computation of $(I-\lambda T^*)^{-1}$.

*Proof.* Let $E = \{z\colon |z-\bar{b}|/|1-az| < |\lambda|\}$, and let $h(z)$ denote the inverse of the function $\lambda^{-1}(z-\bar{b})/(1-bz)$ on $E$. We have that $h$ maps $U = \{|z| < 1\}$ into $E$ and so $h(\bar{\varphi}(\bar{z}))$ maps $U$ into $E$; in particular $h(\bar{\varphi}(\bar{z}))$ maps $U$ into $U$. By the Brouwer Fixed Point Theorem, there is a solution $d(\lambda)$ to

$$h(\bar{\varphi}(\bar{d}(\lambda))) = \bar{d}(\lambda)$$

from which (2) follows.

To compute $(I-\lambda T_F^*)^{-1}\mathcal{D}_T$ (i.e. to compute $(I-\lambda T_F^*)^{-1}(1-\bar{b}z)^{-1}$) we first compute $(I-\lambda T_F^*)(1-d(\lambda)z)^{-1}$, where $d(\lambda)$ comes from Lemma 2:

$$\begin{aligned} (I-\lambda T_F^*)(1-de^{it})^{-1} &= (I-\lambda T_{\psi/\varphi})(1-de^{it})^{-1} \\ &= (1-de^{it})^{-1}-\lambda P\bar{\varphi}(e^{it})(e^{it}-b)(1-\bar{b}e^{it})^{-1}(1-de^{it})^{-1}. \end{aligned}$$

The last term is computed by expanding $(z-b)(1-\bar{b}z)^{-1}(1-dz)^{-1}$ in partial fractions and applying Lemma 1. We get

$$\begin{aligned} &P\bar{\varphi}(e^{it}-b)(1-\bar{b}e^{it})^{-1}(1-de^{it})^{-1} \\ &= (\bar{b}-d)P\bar{\varphi}[(1-|b|^2)(1-\bar{b}e^{it})^{-1}-(1-bd)(1-de^{it})^{-1}] \\ &= (\bar{b}-d)^{-1}[(1-|b|^2)\bar{\varphi}(b)(1-\bar{b}e^{it})^{-1}-(1-bd)\bar{\varphi}(\bar{d})(1-de^{it})^{-1}] \end{aligned}$$

and, by Lemma 2, this is

$$= (\bar b - d)^{-1}[(1-|b|^2)\bar\varphi(b)(1-\bar b e^{it})^{-1} - \lambda^{-1}(d-\bar b)(1-de^{it})^{-1}]$$
$$= (1-|b|^2)\bar\varphi(b)(\bar b - d)^{-1}(1-\bar b e^{it})^{-1} - \lambda^{-1}(1-de^{it})^{-1}.$$

This gives

$$(I-\lambda T_F^*)^{-1}(1-de^{it})^{-1} = -\lambda(1-|b|^2)\bar\varphi(b)(\bar b - b)^{-1}(1-\bar b e^{it})^{-1}$$

and we get for our result

$$(I-\lambda T_F^*)^{-1}(1-\bar b e^{it})^{-1} = \lambda^{-1}(d-\bar b)(1-|b|^2)^{-1}\bar\varphi(b)^{-1}(1-de^{it})^{-1}.$$

**2.3.** In analogy with $D_T$, we start the computation of $D_{T^*}$ by computing $T_F T_F^*$ on $\varphi x$, $x \in H^2$:

$$T_F T_F^* \varphi x = T_F \psi x = \varphi x.$$

Thus $\mathscr{D}_{T^*} \subset (\varphi H^2)^\perp$. In addition, the kernel $\ker T^*$ of $T^*$ is contained in $(\varphi H^2)^\perp$ and, by Section 1.2, $\dim\ker(T^*) = n-1$. Thus $T_F T_F^*$ must be a one-dimensional operator on the ($n$-dimensional) space $(\varphi H^2)^\perp$. We claim the range of $T_F T_F^*$ is the one-dimensional span of the vector $q_b = [\varphi(z)-\varphi(b)]/(z-b)$. We compute

$$T_F T_F^* q_b = T_F P\psi[1-\varphi(b)\bar\varphi(e^{it})]/(e^{it}-b)$$
$$= T_F P[1-\varphi(b)\bar\varphi(e^{it})]/(1-\bar b e^{it})$$
$$= T_F(1-\bar b e^{it})^{-1} - \varphi(b)\, T_F P\bar\varphi(e^{it})/(1-\bar b e^{it})$$
$$= \big(1-|\varphi(b)|^2\big)\, T_F(1-\bar b e^{it})^{-1}$$

by Lemma 1. Furthermore, this is

$$= \big(1-|\varphi(b)|^2\big)P\varphi/(e^{it}-b)$$
$$= \big(1-|\varphi(b)|^2\big)q_b$$

again by Lemma 1.

Summarizing the result of this section, we have

$$\mathscr{D}_{T^*} = (\varphi H^2)^\perp$$

and, for $x \in \mathscr{D}_{T^*}$,

$$(I-T_F T_F^*)^{1/2}x = \begin{cases} |\varphi(b)|x & \text{if } x = q_b, \\ x & \text{if } x \perp q_b. \end{cases}$$

**2.4.** From 2.2–2.3, we see that we shall need $Q(1-de^{it})^{-1}$, where $Q$ is the projection on the one-dimensional span of $q_b$. That is, we need

$$Q(1-de^{it})^{-1} = \big((1-de^{it})^{-1}, q_b\big)q_b/\|q_b\|^2 = \bar q_b(\bar d)q_b/\|q_b\|^2.$$

Now

$$\|q_b\|^2 = \|[1-\bar\varphi(e^{it})\varphi(b)]/(1-e^{-it}b)\|^2$$
$$= \|[1-\bar\varphi(b)\varphi(e^{it})]/(1-be^{it})\|^2 = [1-|\varphi(b)|^2]/(1-|b|^2),$$

since the vector in the last norm is the reproducing kernel in $(\varphi H^2)^\perp$. Thus

$$Q(1-de^{it})^{-1} = \bar q_b(\bar d)(1-|b|^2)[1-|\varphi(b)|^2]^{-1}q_b.$$



The projection of $(1-dz)^{-1}$ on $(\varphi H^2)^\perp$ is $[1-\bar\varphi(\bar d)\varphi(z)]/(1-dz)$, and so

$$(I-T_F T_F^*)^{1/2}(1-dz)^{-1} = (I-Q)[1-\bar\varphi(\bar d)\varphi(z)]/(1-dz) + |\varphi(b)|\,Q(1-dz)^{-1}$$
$$= [1-\bar\varphi(\bar d)\varphi(z)]/(1-dz) + \big(|\varphi(b)|-1\big)Q(1-dz)^{-1}$$
$$= [1-\bar\varphi(\bar d)\varphi(z)]/(1-dz) - \bar q_b(\bar d)(1-|b|^2)[1+|\varphi(b)|]^{-1}q_b.$$

Putting all these computations together, we get for $\Theta_T$:

$$\Theta_T(\lambda)(1-\bar b e^{it})^{-1} = -T_F(1-\bar b e^{it})^{-1} + \lambda D_{T^*}(I-\lambda T_F^*)^{-1}|\varphi(b)|(1-\bar b e^{it})^{-1}$$
$$= -q_b + |\varphi(b)|\bar\varphi(b)^{-1}(d-\bar b)(1-|b|^2)^{-1}D_{T^*}(1-de^{it})^{-1}$$
$$= -q_b + |\varphi(b)|\bar\varphi(b)^{-1}(d-\bar b)(1-|b|^2)^{-1}[1-\bar\varphi(\bar d)\varphi(e^{it})]/(1-de^{it}) -$$
$$- |\varphi(b)|\bar\varphi(b)^{-1}(d-\bar b)\bar q_b(\bar d)[1+|\varphi(b)|]^{-1}q_b$$
$$= |\varphi(b)|\bar\varphi(b)^{-1}(d-\bar b)(1-|b|^2)^{-1}[1-\bar\varphi(\bar d)\varphi(e^{it})]/(1-de^{it}) -$$
$$-\big(1+|\varphi(b)|\bar\varphi(b)^{-1}(d-\bar b)[1+|\varphi(b)|]^{-1}[\bar\varphi(\bar d)-\bar\varphi(b)]/(d-\bar b)\big)q_b$$
$$= \bar\varphi(b)^{-1}\left[\left(|\varphi(b)|\,\frac{d-\bar b}{1-|b|^2}\right)\frac{1-\bar\varphi(\bar d)\varphi(e^{it})}{1-de^{it}} - \left(\frac{\bar\varphi(b)+\bar\varphi(\bar d)\,|\varphi(b)|}{1+|\varphi(b)|}\right)q_b\right]$$

(recall that $d$ is a function of $\lambda$, as given in Lemma 2).

## 3. Similarity

**3.1.** Our computation of $\Theta_T$, along with a theorem of Sz.-Nagy and Foiaş, permit us to prove $T_F$ is similar to an isometry. Indeed, according to [5], Theorem 1.4, it suffices to show that $\Theta_T$ has a (bounded, analytic) left inverse in $|\lambda| < 1$. To construct such an inverse, let $Q$ denote the projection onto the span of $q_b = [\varphi(z)-\varphi(b)]/(z-b)$. We compute $Q\Theta_T(1-\bar b e^{it})^{-1} = \big(\Theta_T(1-\bar b e^{it})^{-1}, q_b\big)q_b/\|q_b\|^2$:

$$\left(|\varphi(b)|\,\frac{d-\bar b}{1-|b|^2}\,\frac{1-\bar\varphi(\bar d)\varphi(e^{it})}{1-de^{it}} - \frac{\bar\varphi(b)+\bar\varphi(\bar d)\,|\varphi(b)|}{1+|\varphi(b)|}\,q_b, q_b\right)$$
$$= (1-|b|^2)^{-1}\big[|\varphi(b)|(d-\bar b)\bar q_b(\bar d) - \big(\bar\varphi(b)+\bar\varphi(\bar d)\,|\varphi(b)|\big)\big(1-|\varphi(b)|\big)\big]$$

since, as in Section 2.3, $\|q_b\|^2 = \big(1-|\varphi(b)|^2\big)/(1-|b|^2)$. So

$$= (1-|b|^2)^{-1}[\bar\varphi(\bar d)\,|\varphi(b)|^2 - \bar\varphi(b)].$$

Thus

$$Q\Theta_T(\lambda)(1-\bar b e^{it})^{-1} = \bar\varphi(b)^{-1}(1-|b|^2)^{-1}\bar\varphi(b)[\bar\varphi(\bar d)\varphi(b)-1]q_b/\|q_b\|^2$$
$$= (1-|b|^2)^{-1/2}\big(1-|\varphi(b)|^2\big)^{-1/2}[\bar\varphi(\bar d)\varphi(b)-1]q_b/\|q_b\|$$

and

$$Q\Theta_T(\lambda)(1-|b|^2)^{1/2}(1-\bar b e^{it})^{-1} = \big(1-|\varphi(b)|^2\big)^{-1/2}[\bar\varphi(\bar d)\varphi(b)-1]q_b/\|q_b\|.$$

If $V$ is the isometry that sends $q_b/\|q_b\|$ to $(1-|b|^2)^{1/2}(1-\bar b e^{it})^{-1}$, then

$$\big(1-|\varphi(b)|^2\big)^{1/2}[\bar\varphi(\bar d)\varphi(b)-1]^{-1}VQ$$

is a left inverse for $\Theta_T$. We conclude that $T_F$ is similar to an isometry $U$ whose structure we investigate in what follows.

**3.2.** The shift part in the Wold decomposition of $U$ is obviously a unilateral shift of multiplicity $n-1$. The unitary part of $U$ is the same as the residual part of the minimal unitary dilation of $T_F$ ([4], p. 344, Theorem 1.2) which is absolutely continuous and has spectral multiplicity at most 1 ([4], p. 274, Theorem 6.3). We will therefore know the unitary part of $U$ if we can determine its spectrum.

We have $U = ST_FS^{-1}$, for some bounded, invertible $S$. Thus $U^* = S^{-1*}T_F^*S^*$, and we conclude that $A = S^{-1*}K$ where $A$ is the span of the eigenvectors for $U^*$ and $K$ is the span of the eigenvectors for $T_F^*$:

$$A = \bigvee_{|z|<1} \ker(U^*-zI), \qquad K = \bigvee_{|z|<1} \ker(T_F^*-zI).$$

It follows that $S$ maps the orthogonal complement of $K$ to the orthogonal complement of $A$ and implements a similarity between the unitary part of $U$ and the operator

$$T_F' = T_F|_{K^\perp}.$$

In the remainder of this section, we derive a criterion for the invertibility of $T_F'-\lambda I$.

LEMMA 3. *Let $\lambda$ be a complex number and $k \in H^2$, then $k \in R(T_F-\lambda I)$ (the range of $T_F-\lambda I$) if and only if there is a constant $c$ such that*

$$[(z-b)k(z)-c]/[\varphi(z)-\lambda\psi(z)] \in H^2. \tag{3}$$

*Proof.* We have $k \in R(T_F-\lambda I)$ if and only if there is $x \in H^2$ such that

$$k = (T_F-\lambda I)x = P(\varphi\overline{\psi}-\lambda)x = (\varphi\overline{\psi}-\lambda)x+y$$

for some $y \perp H^2$. But for this $y$

$$\psi y = \psi k-(\varphi-\lambda\psi)x \in H^2,$$

and we have $\psi y \in (\psi H^2)^\perp = H^2 \ominus \psi H^2$, and so $\psi y = c/(1-\overline{b}z)$. Thus

$$\psi k = (\varphi-\lambda\psi)x+c/(1-\overline{b}z),$$
$$(z-b)k = (\varphi-\lambda\psi)(1-\overline{b}z)x+c$$

which is obviously equivalent to (3).

COROLLARY 1. *An $H^2$ function $k$ belongs to $K^\perp$ if and only if for every $|\lambda|<1$, there is a $c$ such that* (3) *holds.*

*Proof.* We have

$$K^\perp = \Big[\bigvee_{|\lambda|<1} \ker(T_F^*-\overline{\lambda}I)\Big]^\perp = \bigcap_{|\lambda|<1} R(T_F-\lambda I).$$

COROLLARY 2. *For any complex $\lambda$, $T_F'-\lambda I$ has a bounded inverse if and only if for every $k \in K^\perp$, there is a $c$ such that* (3) *holds.*



*Proof.* If $T_F'-\lambda I$ is invertible, then

$$K^\perp = R(T_F'-\lambda I) \subset R(T_F-\lambda I)$$

and Lemma 3 implies the "only if" half of the corollary.

Conversely, it suffices to consider only $\lambda$ of modulus 1 since otherwise, $T_F'$ being similar to a unitary operator, $T_F'-\lambda I$ is automatically invertible. We claim that $(1-\overline{b}z)^{-1}$ times the left-hand member of (3) belongs to $K^\perp$. This will prove $T_F'-\lambda I$ maps $K^\perp$ onto $K^\perp$ and, since $T_F'-\lambda I$ is one-to-one ([1], Lemma 2.2) will show $T_F'-\lambda I$ is invertible.

By Corollary 1, the claim is equivalent to the existence of $c_\mu$, for every $|\mu|<1$, such that

$$\{(z-b)(1-\overline{b}z)^{-1}[(z-b)k(z)-c]/[\varphi-\lambda\psi]-c_\mu\}/\{\varphi-\mu\psi\} \in H^2. \tag{4}$$

Since $k \in K^\perp$ there is, by Corollary 1, a $c_\mu'$ such that

$$[(z-b)k(z)-c_\mu']/[\varphi-\mu\psi] \in H^2.$$

Setting $c_\mu = (c-c_\mu')/(\lambda-\mu)$, we see that for every $z$ such that $\varphi(z)-\mu\psi(z) = 0$, we have

$$[(z-b)k(z)-c]/[\varphi(z)/\psi(z)-\lambda] = [c_\mu'-c]/(\mu-\lambda)$$

and (4) follows.

*Remark.* An alternative approach to Corollary 1 is to note that for $|\lambda|<1$, the eigenvectors of $(T_F-\lambda I)^*$ take the form

$$k_\lambda = (\overline{\zeta}-\overline{b})(1-\overline{\zeta}z)^{-1}-(\overline{\xi}-\overline{b})(1-\overline{\xi}z)^{-1}$$

where $\zeta, \xi$ are complex numbers in $|z|<1$ satisfying $F(\zeta) = F(\xi) = \lambda$.

**3.3.** We are ready, in this section, to complete the determination of the spectrum of $T_F'$ and hence also the description of the isometry $U$.

As $t$ increases from 0 to $2\pi$, the argument of $F(e^{it}) = \varphi(e^{it})/\psi(e^{it})$ increases by $2\pi(n-1)$.

DEFINITION. *$F(z)$ backs up* at $z = e^{i\theta}$, if there is a closed interval $[\theta_1, \theta_2]$, $\theta_1 < \theta_2$, containing $\theta$ and such that the argument of $F(e^{it})$ is decreasing for $\theta_1 < t < \theta_2$.

LEMMA 4. *Let $\Sigma$ be the set where $F$ backs up. Then $F(\Sigma)$ is the spectrum of $T_F'$.*

*Proof.* Let $\lambda$ be a point of modulus 1 not belonging to $F(\Sigma)$, and suppose also that $F'(z) \neq 0$, for all $z$ of modulus 1 such that $F(z) = \lambda$. We claim first that for every $z_0$ of modulus 1 such that $F(z_0) = \lambda$, and for every $k \in K^\perp$, $k$ can be analytically continued across $z_0$.

For the proof, we first note that, since $F(z)$ assumes the value $\lambda$ exactly $n-1$ times on $|z| = 1$, there must exist $z_1$ in $|z|<1$ such that $F(z_1) = \lambda$. We can choose neighborhoods $D_1$ of $z_1$ and $D_0$ of $z_0$ such that $F$ is single valued in $D_0$ and $D_1$ and $F(D_0) = F(D_1)$. Thus there is a function $\nu(z)$ from $D_0$ to $D_1$ such that

$$F(\nu(z)) = F(z) \quad \text{for } z \in D_0.$$

Further, since $F'(z) \neq 0$ in $D_0$, $\nu(z)$ is analytic in $D_0$. Now let $k \in K^{\perp}$. By Corollary 1 of Section 3.2, $k(\nu(z)) = k(z)$ in $D_0 \cap \{|z| < 1\}$. But $\nu(z)$ is analytic in $D_0$ and $k$ is analytic in $D_1$ (we may assume, of course, $D_1 \subset \{|z| < 1\}$), and hence $k(\nu(z))$ extends to be analytic in $D_0$, and this provides a continuation of $k$ across $z_0$.

Now we can show that $T'_F - \lambda I$ is invertible. Indeed, let $k \in K^{\perp}$, and let $M$ be the set of $z$ in $|z| \leqslant 1$ where $F(z) = \lambda$. The function $k$ is analytic at each $z \in M$ by the claim just proved, and the values of $k(z)$ agree at each point in $M$; indeed Corollary 1 of Section 3.2 proves this if $\lambda$ is replaced by a sequence $\lambda_n$, with $|\lambda_n| < 1$ which may be taken to approach $\lambda$ as $n \to \infty$. Thus taking $c$ to be the common value of $k(z)$ for $z \in M$, we see by Corollary 2 of Section 3.2, that $(T'_F - \lambda I)^{-1}$ exists.

From the above we conclude that the spectrum of $T'_F$ is contained in $F(\Sigma) \cup \cup \{\lambda_1, \ldots, \lambda_p\}$ where $\lambda_1, \ldots, \lambda_p$ are points where $F(\lambda_j) = F(z_j)$ for some $z_j$ such that $F'(z_j) = 0$. But the operator $T'_F$ being similar to a unitary operator with absolutely continuous spectrum, there can be no isolated points in the spectrum of $T'_F$ and we have proved that the latter spectrum is contained in $F(\Sigma)$.

It remains only to prove $T'_F - \lambda I$ is not invertible if $\lambda \in F(\Sigma)$. Let $\Gamma$ be an arc in $F(\Sigma)$. $F(z)$ assumes every value $\lambda \in \Gamma$ exactly $n+1$ times on $|z| = 1$ (at least $n+1$ times since $F(e^{it})$ has winding number $n-1$ and backs up at $\lambda$; at most $n+1$ times since the equation $F(z) = \lambda$ is equivalent to a polynomial of degree $n+1$). Thus $F(z)$ omits $\Gamma$ in $|z| < 1$. Pick two points $z_1, z_2$ where $F$ backs up and where $\lambda_i = F(z_i) \in \Gamma$. The function

$$G(z) = [(F(z) - \lambda_1)/(F(z) - \lambda_2)]$$

is analytic in $|z| < 1$ and omits a ray from 0 to $\infty$. We may therefore set

$$G_\varepsilon(z) = \frac{G(z)^\varepsilon - G(b)^\varepsilon}{z - b}$$

for suitable $\varepsilon > 0$. Since $F(z) - \lambda_2$ can have only simple zeros on $|z| = 1$, $G_\varepsilon(z) \in H^2$ if $\varepsilon < 1/2$, and if $|z| < 1$,

$$(z-b)\,G_\varepsilon(z) - G(\lambda)^\varepsilon + G(b)^\varepsilon$$

vanishes wherever $\varphi - \lambda\psi = 0$, so by Corollary 1 of Section 3.2, $G_\varepsilon(z) \in K^{\perp}$. But $G_\varepsilon \notin R(T'_F - \lambda_1 I)$. Indeed, $\varphi - \lambda_1 \psi$ has a simple zero at $z = z_1$, but

$$(z-b)\,G_\varepsilon(z) - G(b)^\varepsilon$$

while vanishing at $z = z_1$ is only $O(z - z_1)^\varepsilon$, so

$$[(z-b)\,G_\varepsilon(z) - G(b)^\varepsilon]/(\varphi - \lambda_1 \psi)$$

cannot belong to $H^2$, and by Corollary 2 of Section 3.2, $T_F^* - \lambda_1 I$ is not invertible. This proves the lemma.

To summarize Part 2, we state

THEOREM 2. *Let $F(z) = \varphi(z)/\psi(z)$ where $\varphi$ and $\psi$ are finite Blaschke products with $n > 1$ and 1 zeros, respectively. Then $T_F$ is similar to $T_{z^{n-1}} \oplus V$, where $V$ is multiplication by $e^{it}$ on $L^2(F(\Sigma))$, $\Sigma$ the set where $F$ backs up. If $n = 1$, $T_F$ is similar to $V$.*



COROLLARY. *The minimal isometric dilation of $T_F$ is unitarily equivalent to $T_{z^n} \oplus V$.*

While I believe that, with appropriate modifications for multiplicity, Theorem 2 should generalize to the case where $\psi$ is an arbitrary finite Blaschke product. I doubt that the characteristic function of $T_F$ should admit such a simple description as given in Part 1 if $\psi$ has more than, say, two zeros.

Finally, it is worth noting that the computations of Part 2 and Section 3.1 remain valid for $T_{\varphi/\psi}$ if $\varphi$ is an arbitrary inner function and $\psi$ is a Blaschke product with one zero. We have not, however, determined the structure of the isometry to which $T_{\varphi/\psi}$ is similar, in this case.

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