SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

EMBEDDING $C(X)$ IN AN ALGEBRA OF ANALYTIC FUNCTIONS

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Let X be an infinite compact Hausdorff space, let $C(X, \mathbf{C})$ denote the algebra of continuous complex-valued functions on X with the usual pointwise operations, and, for $f \in C(X, \mathbf{C})$, let

$$|f|_X = \sup\{|f(x)|:\ x \in X\},$$

so that $|\cdot|_X$ is the uniform norm on X and $(C(X, \mathbf{C}), |\cdot|_X)$ is a commutative Banach algebra.

Now let $||\cdot||$ be any norm with respect to which $C(X, \mathbf{C})$ is a normed algebra. Then it was proved by Kaplansky in 1948 that

$$||f|| \geqslant |f|_X \quad (f \in C(X)),$$

and this suggests the conjecture that every such norm $||\cdot||$ on $C(X, \mathbf{C})$ is equivalent to the uniform norm. This occurs if and only if $||\cdot||$ is a complete norm on $C(X, \mathbf{C})$, and it is also easy to see that it holds if and only if every algebra homomorphism from $C(X, \mathbf{C})$ into a Banach algebra is necessarily continuous. In this latter formulation, we see that the question is one of *automatic continuity*.

The seminal paper on the automatic continuity of homomorphisms from $C(X, \mathbf{C})$ and other commutative Banach algebras is the 1960 paper of Badé and Curtis [1] in which, for example, it is proved that a homomorphism from $C(X, \mathbf{C})$ into a Banach algebra is necessarily continuous on a dense subalgebra of $C(X, \mathbf{C})$, and that there is a discontinuous homomorphism from $C(X, \mathbf{C})$ if and only if there exists a *radical homomorphism* from $C(X, \mathbf{C})$, that is, a non-zero homomorphism from a maximal ideal of $C(X, \mathbf{C})$ into a commutative radical Banach algebra.

Of course, the algebras $C(X, \mathbf{C})$ are exactly the commutative C^*-algebras. Let $B(H)$ denote the Banach algebra of bounded linear operators on a Hilbert space H. These are non-commutative C^*-algebras. In 1967, Johnson [7] proved that every homomorphism from certain C^*-algebras, including each $B(H)$, was necessarily continuous (see [9], 12.4). Thus, the automatic continuity problem was resolved for some non-commutative C^*-algebras, although left open for each infinite-dimensional commutative C^*-algebra.

Here, I briefly describe the following solution to Kaplansky's problem.

THEOREM 1. *Let X be an infinite compact Hausdorff space. Then, assuming the continuum hypothesis, there exists a discontinuous algebra monomorphism from $C(X, C)$ into a Banach algebra, and there exists an incomplete algebra norm on $C(X, C)$.*

This theorem is proved in [2], and applications to the construction of discontinuous homomorphisms from some other topological algebras are given in [3]. This problem was also solved independently by Jean Esterle: he describes his work in this volume [6]. See also the announcement [5]. The book of Sinclair [9] is an excellent survey of automatic continuity theory up to about 1973, and it includes proofs of the results of Kaplansky, Badé and Curtis, and Johnson described above. The article [4] is a further survey of automatic continuity theory, bringing the story up to 1977, and it includes a fairly long description of the two solutions to Kaplansky's problem, as well as some applications to other problems. It also discusses the role of the continuum hypothesis in the theorem, and points out that results of Woodin and Solovay show that there are models of set theory (in which the continuum hypothesis does not hold) in which every homomorphism from each $C(X, C)$ into a Banach algebra is necessarily continuous.

Now let $C(X)$ denote the algebra of continuous real-valued functions on X, let Ω be a connected open set in C, and let $\mathcal{O}(\Omega)$ be the algebra of analytic functions on Ω. Our present approach to the problem is based on the following question.

QUESTION. Is there an algebra homomorphism $\mu\colon C(X) \to \mathcal{O}(\Omega)$?

One would not expect there to be such a homomorphism, and in fact the only such maps are the zero homomorphism and the trivial ones of the form $\mu(f) = f(x_0)1$, for some $x_0 \in X$. For take $z \in \Omega$. Then there exists $x_0 \in X$ such that $(\mu f)(z) = f(x_0)$ for $f \in C(X)$. If $f \geqslant 0$ in $C(X)$ and $f(x_0) = 0$, then f is infinitely divisible in $C(X)$—for each $n \in N$, there exists $g \in C(X)$ with $g^n = f$. Hence $(\mu f)(z_0) = 0$ and μf is infinitely divisible in $\mathcal{O}(\Omega)$, and so $\mu f = 0$, by the basic properties of analytic functions. Now an arbitrary function $f \in C(X)$ with $f(x_0) = 0$ can be written $f = gh$ with $g(x_0) = 0$ and $g \geqslant 0$ (take $g = |f|^{1/2}$), so the remark follows.

Let me now describe a modification of $\mathcal{O}(\Omega)$. For $\sigma \geqslant 1$, let $\Omega_\sigma = \{z \in C\colon \operatorname{Re} z > 1, |z| > \sigma\}$, and let $\mathcal{O}_\infty = \bigcup \{\mathcal{O}(\Omega_\sigma)\colon \sigma \geqslant 1\}$. Thus, $\mathcal{O}_\infty$ is an algebra of analytic functions defined on "half-neighbourhoods of ∞". The domain of a function in $\mathcal{O}_\infty$ depends on the function.

THEOREM 2. *Assuming the continuum hypothesis, there is a non-trivial homomorphism $\mu\colon C(X) \to \mathcal{O}_\infty$.*

Actually, I shall indicate that there is a non-trivial homomorphism $\mu\colon M \to \mathscr{C}_\infty$, where M is a maximal ideal of $C(\beta N)$ and $\mathscr{C}_\infty$ will be described below. First, however, let us see how this homomorphism μ leads to the discontinuous monomorphism required in Theorem 1.

Let ω be a continuous function on $[0, \infty)$ such that $\omega(0) = 1$, $\omega(t) > 0$, and $\omega(s+t) \leqslant \omega(s)\omega(t)$ $(s, t \geqslant 0)$. Let

$$L^1(\omega) = \Big\{f\colon \|f\| = \int_0^\infty |f(t)|\omega(t)\,dt < \infty\Big\},$$

and, for $f, g \in L^1(\omega)$, let

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

It is easily checked that, with this convolution multiplication, $L^1(\omega)$ is a commutative Banach algebra. If $\omega(t)^{1/t} \to 0$ as $t \to \infty$, then $L^1(\omega)$ is a radical Banach algebra. For example, if we take $\omega(t) = \exp(-t^2)$, then we have a radical Banach algebra.

Thus, suppose that $L^1(\omega)$ is a radical Banach algebra. If $F \in \mathscr{C}_\infty$, then $|F(z)| \to 0$ sufficiently rapidly as $|z| \to \infty$ with $\operatorname{Re} z > 1$ so that the inverse Laplace transformation

$$(\mathscr{L}^{-1}F)(t) = \frac{1}{2\pi i}\int_{\tau - i\infty}^{\tau + i\infty} F(z)e^{zt}\,dt,$$

can be defined for suitable τ. The map $\mathscr{L}^{-1} \circ \mu\colon M \to L^1(\omega)$ is a homomorphism.

To begin a description of the proof of Theorem 2, consider first the domain of μ. Take $X = \beta N$, the Stone–Čech compactification of N, take $\mathfrak{p} \in \beta N \setminus N$, let $M_\mathfrak{p} = \{f \in C(\beta N)\colon f(\mathfrak{p}) = 0\}$, and let $J_\mathfrak{p} = \{f\colon f = 0 \text{ near } \mathfrak{p}\}$. Then $M_\mathfrak{p}$ is a maximal ideal and $J_\mathfrak{p}$ is a prime ideal of $C(X)$. Let $A = M_\mathfrak{p}/J_\mathfrak{p}$, so that A is an integral domain, and let $\pi\colon M_\mathfrak{p} \to A$ be the quotient map. We construct a monomorphism $\theta\colon A \to \mathscr{C}_\infty$, and then $\mathscr{L}^{-1} \circ \theta \circ \pi\colon M_\mathfrak{p} \to L^1(\omega)$ is a homomorphism with kernel $J_\mathfrak{p}$. Since $J_\mathfrak{p}$ is dense in $M_\mathfrak{p}$, this homomorphism is certainly discontinuous, and it is easy to construct from it the discontinuous monomorphism from $C(X, C)$ required in Theorem 1.

The algebra $C(X)$ is a partially ordered set: $f \leqslant g$ if $f(x) \leqslant g(x)$ for each $x \in X$. (It is for this reason that we are working with real-valued functions.) It is property of βN that, for each f, either $f \leqslant 0$ or $f \geqslant 0$ in some neighbourhood of $\mathfrak{p}$. Thus, the quotient order on A is a total order. The algebra A is real-closed, which means that $A(\sqrt{-1})$ is algebraically closed, and $(A, \leqslant)$ is an η_1-set, which means that, given countable subsets S and T of A with $S < T$, there exists $a \in A$ with $S < \{a\} < T$. Thus, the quotient field of A is a real-closed, totally ordered η_1-field of cardinality $\aleph_1$ (assuming the continuum hypothesis), and so it is a non-standard model of R.

I now describe $\mathscr{C}_\infty$. If $\sigma \geqslant 1$, we say that $F \in \mathscr{C}_\sigma$ if $F \in \mathcal{O}(\Omega_\sigma)$ and either $F = 0$ or F satisfies:

(1) $F = O(z^{-k})$ as $|z| \to \infty$, $\operatorname{Re} z > 1$, for each $k \in N$;
(2) $F(z) \neq 0$ for $z \in \Omega_\sigma$;
(3) $F(R) \subset R$.

Then $\mathscr{C}_\infty = \bigcup\{\mathscr{C}_\sigma : \sigma \geqslant 1\} \subset \mathcal{O}_\infty$. We say that $F \geqslant 0$ if $F(\boldsymbol{R}) \subset \boldsymbol{R}^+$.

The exact form of these conditions is important. Note first that, if $F \in \mathscr{C}_\infty$, then, by (1), $\mathscr{L}^{-1}(F)$ can be defined. Next, if $F \in \mathscr{C}_\infty$, then, by (2) and (3), $F = \exp G$ for a unique G with $G(\boldsymbol{R}) \subset \boldsymbol{R}$, so that $F^\alpha = \exp(\alpha G)$ can be defined for $\alpha > 0$, and $F^\alpha \in \mathscr{C}_\infty$. Also, if $F, G \in \mathscr{C}_\infty$, then $FG \in \mathscr{C}_\infty$. However, if $F, G \in \mathscr{C}_\infty$, it does not follow that $F+G \in \mathscr{C}_\infty$, for $F+G$ may not satisfy (2), so $\mathscr{C}_\infty$ is *not* a subalgebra of $\mathcal{O}_\infty$. But we do have the result that, if $F, G \in \mathscr{C}_\infty$ and $G = o(F)$, then $F+G = F(1+G/F)$ belongs to $\mathscr{C}_\infty$, for $F(1+G/F)$ belongs to $\mathscr{C}_\sigma$, for σ sufficiently large. It is at this point that we use the fact that the domains of the functions of $\mathscr{C}_\infty$ are not fixed.

Now think again about A. If $a, b \in A$, then $\lim a/b$ must exist in $\{-\infty\}\cup\boldsymbol{R}\cup\{\infty\}$. We say that $a \prec b$ if $\lim a/b = 0$ and that $a \sim b$ if $\lim a/b \in \boldsymbol{R}\setminus\{0\}$. Then $a|b$ in A if and only if $a \prec b$. It is rather easy to show that there is a set $(\Psi) \subset A$ such that: if $a, b \in (\Psi)$ and $\alpha > 0$, then ab, $a^\alpha \in (\Psi)$; if $a, b \in (\Psi)$ and $a \prec b$, then $a/b \in (\Psi)$; if $a, b \in (\Psi)$ and $a \neq b$, then $a \prec b$ or $b \prec a$; for each $a \in A$, there exists $b \in (\Psi)$ with $b \sim a$. Roughly, we are picking out from A a representative of each rate of decrease to zero.

The set $((\Psi); \preccurlyeq)$ is an η_1-set of cardinality $\aleph_1$. We start by mapping (Ψ) into $\mathscr{C}_\infty$, and in doing this we specify $\aleph_1$ functions in $\mathscr{C}_\infty$, which is a little unusual. Introduce

$$\varepsilon_n(z) = \frac{n}{z}(1-e^{-z/n}) \qquad (\operatorname{Re} z > 1).$$

Then ε_n has no zero, ε_n is close to 1 when $|z|$ is small compared with n, and $\varepsilon_n = O(z^{-1})$ as $|z| \to \infty$ for fixed n. Now let $\varepsilon_{\boldsymbol{n}}(z) = \prod_{j=1}^{\infty} \varepsilon_{n(j)}(z)$, where $\boldsymbol{n} = (n(j))$ belongs to a certain family, $\mathscr{S}$, of real-valued sequences. Actually, we work with $E_{\boldsymbol{n}}(z) = \varepsilon_{\boldsymbol{n}}(z^{1/2})$. Note first that $E_{\boldsymbol{n}} \in \mathscr{C}_\infty$. Next, we can introduce an order on the functions $E_{\boldsymbol{n}}$: let $\Lambda_{\boldsymbol{n}}$ be the inverse of the function $\log E_{\boldsymbol{n}}$, and say that $\boldsymbol{m} \succ \boldsymbol{n}$ if $\Lambda_{\boldsymbol{n}}(y) - \Lambda_{\boldsymbol{m}}(y) \to \infty$ as $y \to \infty$. This order is designed to give the important result:

THEOREM. *If* $\boldsymbol{m} \succ \boldsymbol{n}$, *then* $E_{\boldsymbol{n}}/E_{\boldsymbol{m}} \in \mathscr{C}_\infty$.

A rather long technical calculation shows that there is a totally ordered η_1-set in $(\mathscr{S}; \preccurlyeq)$ satisfying various extra conditions, and with this, a proof using transfinite induction shows:

THEOREM. *There is an injection* $\theta\colon (\Psi) \to (\{E_{\boldsymbol{n}}\})$ *such that* $\theta(ab) = (\theta a)(\theta b)$; $\theta(a^\alpha) = (\theta a)^\alpha$, *and such that*, if $a \prec b$, *then* $\theta a \prec \theta b$.

Let $\mathscr{Q}_0$ be the inverse-closed subalgebra of A generated by (Ψ). (A subalgebra B of A is *inverse-closed* if $b_1, b_2 \in B$, $b_1/b_2 \in A$ implies that $b_1/b_2 \in B$.) Then it is easy to see that there is a monomorphism $\theta\colon \mathscr{Q}_0 \to \mathscr{C}_\infty$ such that, if $a \geqslant 0$, then $\theta a \geqslant 0$: the point to note is that, if $a, b \in \mathscr{Q}_0$ and $a \prec b$, then $\theta a \prec \theta b$, so that $\theta a/\theta b$ and $\theta a + \theta b$ belong to $\mathscr{C}_\infty$.

DEFINITION. $(\mathscr{Q}; \theta; \mathscr{F})$ is a *triple* if: $\mathscr{Q}$ is an inverse-closed subalgebra of A; $\mathscr{F} \subset \mathscr{C}_\infty$; $\theta\colon \mathscr{Q} \to \mathscr{F}$ is an algebra isomorphism; if $q \geqslant 0$, then $\theta q \geqslant 0$; and $\mathscr{Q} \supset \mathscr{Q}_0$.

Clearly, we must now study the problem of extending a triple to one with larger initial term. Suppose that $(\mathscr{Q}; \theta; \mathscr{F})$ is a triple and that $a \in A\setminus\mathscr{Q}$. There are two cases according as a is algebraic or transcendental with respect to $\mathscr{Q}$.

Case 1. a is algebraic with respect to $\mathscr{Q}$.

It can be shown that one can suppose that a has a "minimal polynomial" of the special form

$$\mathfrak{q}(X) \equiv q_0 + X + (\gamma_2+q_2)X^2 + \ldots + (\gamma_n+q_n)X^n,$$

where $q_0, q_2, \ldots, q_n \in \mathscr{Q}$ and $\gamma_2, \ldots, \gamma_n \in \boldsymbol{R}$. (In fact, something rather more complicated is needlessly considered in [2].) We have that $\mathfrak{q}(a) = 0$, and a satisfies no equation of degree less than n. Let $Q_j = \theta(q_j)$. We have to find $w = F(z)$ so that

$$Q_0(z) + w + (\gamma_2 + Q_2(z))w^2 + \ldots + (\gamma_n + Q_n(z))w^n = 0. \tag{1}$$

Since $Q_j(z) \to 0$ as $|z| \to \infty$, this is nearly $w + \gamma_2 w^2 + \ldots + \gamma_n w^n = 0$ for $|z|$ large, an equation with one root at 0 and no other root in some neighbourhood of 0. Rouché's theorem and the implicit function theorem lead to a solution, $w = F(z)$, of (1), and it is easy to check that $F \in \mathscr{C}_\infty$. One can now make the required extension in the usual algebraic way. Here, it is necessary to check that the new range is contained in $\mathscr{C}_\infty$.

Case 2. a is transcendental with respect to $\mathscr{Q}$.

In this case, θa must be defined so that $q_1 < a < q_2$ with $q_1, q_2 \in \mathscr{Q}$ implies that $\theta q_1 < \theta a < \theta q_2$. I do not know how to do this in general. Consider the special case in which the position of a in the order is specified by countably many members of $\mathscr{Q}$. Then, given (Q_n), $(Q'_m) \subset \mathscr{F}$ with

$$Q_1 < Q_2 < \ldots < Q'_2 < Q'_1,$$

we must find $F \in \mathscr{C}_\infty$ with

$$Q_1 < Q_2 < \ldots < F < \ldots < Q'_2 < Q'_1. \tag{2}$$

My method for this is long and complicated — it involves a curious quasi-analytic algebra of functions which may have some interest in its own right — and I should like to find a shorter method. At the analogous point in his construction, Jean Esterle uses the Mittag-Leffler theorem, which is possible because he is working in a Banach algebra, but this does not seem to be available in my context because we are not working in a *metrizable* algebra.

After constructing an extension in case 1 and the special form of case 2, we apply Zorn's lemma to obtain a maximal triple, $(\mathscr{Q}_*; \theta_*; \mathscr{F}_*)$, say. Because we could not deal with the general case 2, we do not know that $\mathscr{Q}_*$ is equal to A. However, it is not hard to see that $\mathscr{Q}_*$ is a totally ordered, real-closed η_1-set of cardinality $\aleph_1$. Thus, by a recent result of Johnson [8], $\mathscr{Q}_*$ is isomorphic to A. This is sufficient for the result.

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Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS,
WARSAW 1982

ON RESONANCES IN MATHEMATICAL SCATTERING THEORY

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1. Resonance problem in mathematical scattering theory

Let H_0 and H be self-adjoint operators given in the separable Hilbert space $\mathfrak{H}$. Let $P_{ac}(H_0)$ be the orthoprojection onto the absolutely continuous subspace of H_0. The following strong limits are called *wave operators*, if they exist,

$$W_{\pm}(H, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0),$$

implying the definition of the scattering operator

$$S(H, H_0) = W_+^*(H, H_0)\, W_-(H, H_0)$$

and the scattering amplitude operator

$$T = S-1.$$

Using the direct integral decomposition of $P_{ac}(H_0)\,\mathfrak{H}$,

$$P_{ac}(H_0)\mathfrak{H} = \int_{\sigma_{ac}(H_0)} \oplus\mathfrak{H}_0(\lambda)d\lambda$$

where H_0 is represented by multiplication with λ in the separable Hilbert space $\mathfrak{H}_0(\lambda)$ and using the commutity of S with H_0, S can be represented in $\mathfrak{H}_0(\lambda)$ by the scattering matrix $S(\lambda)$. The same holds for T represented by the scattering amplitude $T(\lambda)$. Poles of $T(\lambda)$ meromorphically continued are called *resonances*.

On the other side, let H and H_0 be connected by

$$H = H_0+V$$

where V is also self-adjoint and bounded. Furthermore, let V be factorized by

$$V = B^*A$$

with bounded A and B. In perturbation theory (see e.g. [1]) poles of the factorized resolvent,

$$A(z-H)^{-1}B^*,$$

defined for $\operatorname{Im} z > 0$, meromorphically continued into the lower half plane are called *virtual poles*.