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SPECTRAL THEORY
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ON RESONANCES IN MATHEMATICAL SCATTERING THEORY

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1. Resonance problem in mathematical scattering theory

Let H_0 and H be self-adjoint operators given in the separable Hilbert space $\mathfrak{H}$. Let $P_{ac}(H_0)$ be the orthoprojection onto the absolutely continuous subspace of H_0. The following strong limits are called *wave operators*, if they exist,

$$W_{\pm}(H, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0),$$

implying the definition of the scattering operator

$$S(H, H_0) = W_+^*(H, H_0)\, W_-(H, H_0)$$

and the scattering amplitude operator

$$T = S-1.$$

Using the direct integral decomposition of $P_{ac}(H_0)\,\mathfrak{H}$,

$$P_{ac}(H_0)\mathfrak{H} = \int_{\sigma_{ac}(H_0)} \oplus\mathfrak{H}_0(\lambda)d\lambda$$

where H_0 is represented by multiplication with λ in the separable Hilbert space $\mathfrak{H}_0(\lambda)$ and using the commutity of S with H_0, S can be represented in $\mathfrak{H}_0(\lambda)$ by the scattering matrix $S(\lambda)$. The same holds for T represented by the scattering amplitude $T(\lambda)$. Poles of $T(\lambda)$ meromorphically continued are called *resonances*.

On the other side, let H and H_0 be connected by

$$H = H_0+V$$

where V is also self-adjoint and bounded. Furthermore, let V be factorized by

$$V = B^*A$$

with bounded A and B. In perturbation theory (see e.g. [1]) poles of the factorized resolvent,

$$A(z-H)^{-1}B^*,$$

defined for $\operatorname{Im} z > 0$, meromorphically continued into the lower half plane are called *virtual poles*.

The resonance problem [6] consists in finding relations between resonances and virtual poles. For certain local scattering systems this problem is solved in [2]:

THEOREM 1. (i) *The real virtual poles form locally the eigenvalue spectrum of* H.
(ii) *The scattering amplitude* $T(\lambda)$ *is locally meromorphically continuable.*
(iii) *The nonreal virtual poles are locally identical with the resonances.*

For proving Theorem 1 the following simplified assumptions are sufficient:

(1) The ranges of A^* and B^* are to be identical. The range of V is to be dense in $\mathfrak{H}$.

(2) The spectral discrete part of H_0 in a certain open interval Δ is to have finite multiplicity. The multiplicity of the absolutely continuous spectrum of H_0 with respect to Δ is to be constant and finite.

(3) Let $AP_0(\Delta)P_{ac}(H_0)(z-H_0)^{-1}P_{ac}(H_0)P_0(\Delta)B^*$, $P_0(\cdot)$ spectral measure of H_0, be holomorphically continuable across Δ into $G_+ = C_+ \cup G$, where C_+ is the upper half plane and G a certain region symmetric to the real line containing Δ.

(4) Let $\big(1-AP_{ac}(H_0)(z-H_0)^{-1}P_{ac}(H_0)B^*\big)^{-1}$ be bounded holomorphically in G_+.

(5) The representer of $P_{ac}(H_0)B^*f$ in $L^2(\Delta, \mathfrak{H}_0, d\lambda)$ is called $\big(P_{ac}(H_0)B^*f\big)(\lambda)$. Let $\big(P_{ac}(H_0)B^*f\big)(\lambda)$ be holomorphically continuable into G for such a set $\{f\}$ for which $\big(P_{ac}(H_0)B^*f\big)(\lambda)$ is fundamental for $\mathfrak{H}_0$.

These properties are e.g. satisfied for Hölder continuous potential functions exponentially decreasing to infinity [5].

2. Resonance model for many particle systems

In many particle systems the scattering events are multiplied. To list them channel Hamiltonians A_ϱ are introduced corresponding roughly spoken to the free evolution of certain cluster decompositions of the many particle system. The channel wave operators are defined as usual by

$$W_\pm(H, A_\varrho) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itA_\varrho}P_{ac}(A_\varrho)$$

where $P_{ac}(A_\varrho)$ are called channel projections. The partial scattering operator between two channels is

$$S_{\varrho\sigma} = W_+^*(H, A_\varrho)\, W_-(H, A_\sigma)$$

mapping $P_{ac}(A_\sigma)\mathfrak{H}$ into $P_{ac}(A_\varrho)\mathfrak{H}$. Let the direct integral decomposition of the absolutely continuous subspaces be given as above

$$P_{ac}(H)\mathfrak{H} = \int_{\sigma_{ac}(H)} \oplus \mathfrak{H}_\sigma(\lambda)\, d\lambda$$

and

$$P_{ac}(A_\varrho)\mathfrak{H} = \int_{\sigma_{ac}(A_\varrho)} \oplus \mathfrak{H}_\varrho(\lambda)\, d\lambda.$$

In correspondence to the channel wave operators $W_\pm(H, A_\varrho)$ the channel wave matrices $W_\varrho^\pm(\lambda)$ are defined in [3] mapping $\mathfrak{H}_\varrho(\lambda)$ into $\mathfrak{H}(\lambda)$. The poles of the partial scattering matrix $S_{\varrho\sigma}(\lambda)$ mapping $\mathfrak{H}_\sigma(\lambda)$ into $\mathfrak{H}_\varrho(\lambda)$ locally meromorphically continued are called *resonances*.

It is considered the following N-particle system. Let

$$H = -\sum_{i=1}^{3N} \frac{\Delta_i}{2m_i} + V(x)$$

be given in $\mathfrak{H} = L^2(R^{3N})$, Δ_i—Laplacian, m_i—mass. Let $V(x) \in L^1_{loc}(R^{3N})$, bounded and a sum of two-body-potentials

$$V(x) = \sum_{i<j} V_{ij}(|x_i - x_j|)$$

where each V_{ij} is to have a maximum for certain particle distances. Then there exists a region $D_0 \in R^{3N}$ on which $V(x) \geqslant M = \min\limits_{i,j}\big(\max\limits_{x\in R^{3N}} V_{ij}(x)\big)$. Cutting $V(x)$ on D_0 we obtain a new Hamiltonian

$$H_M = -\sum_{i=1}^{3N} \frac{\Delta_i}{2m_i} + V(x)\big(1-\chi_{D_0}(x)\big) + M\chi_{D_0}(x)$$

where $\chi_{D_0}(x)$ is the characteristic function of D_0.

Letting the projection $P = \chi_{D_0}(\cdot)$ and $\bar{P} = 1-P$ some powers of the resolvent of H_M tends to a pseudoresolvent $R_\infty(z)$, $z \in C-(0, \infty)$ in trace norm as $M \to \infty$ such that $R_\infty(z)/\bar{P}\mathfrak{H}$ is a resolvent of a self-adjoint operator H_∞ defined on $\bar{P}\mathfrak{H}$ [4]. Using Feynman–Kac-formula and the invariance principle of wave operators the following one-channel scattering operator exists

$$S_\infty = S(H, H_\infty) = W_+^*(H, H_\infty)\, W_-(H, H_\infty)$$

with the corresponding one-channel scattering matrix $S_\infty(\lambda)$.

THEOREM 2. *The multi-channel partial scattering matrix* $S_{\varrho\sigma}(\lambda)$ *can be factorized and expressed by means of the one-channel scattering matrix* $S_\infty(\lambda)$:

$$S_{\varrho\sigma}(\lambda) = W_{\varrho,\infty}^+(\lambda)^* S_\infty(\lambda)\, W_{\sigma,\infty}^-(\lambda),$$

$\lambda \in \sigma_{ac}(A_\varrho)\cap\sigma_{ac}(A_\sigma)$, *where* $W_{\varrho,\infty}^+(\lambda)$ *and* $W_{\sigma,\infty}^-(\lambda)$ *are the channel wave matrices corresponding to* $W_+(H_\infty, A_\varrho)$ *and* $W_-(H_\infty, A_\sigma)$, *respectively.*

In proving the existence of resonances this theorem can be helpful for reducing the multi-channel problem to a one-channel consideration.

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ON THE IDEAL STRUCTURE OF COMMUTATIVE BANACH ALGEBRAS

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All rings (in particular all algebras) in this paper are assumed commutative. An ideal I in a ring A is called *modular* if A/I has a unit. For a Banach algebra B, $\mathfrak{M}$ denotes its Gelfand space, the space of non-trivial complex homomorphisms of B. $b(x)$ is the map of $b \in B$ by the homomorphism $x \in \mathfrak{M}$. Thus $x \mapsto b(x)$, $x \in \mathfrak{M}$, is the Gelfand transform of b. The *hull* $h(I)$ of an ideal $I \subseteq B$ is the set of all $x \in \mathfrak{M}$ for which the kernel, i.e. the corresponding modular maximal ideal, contains I. The ideal $I \subseteq B$ is called *primary at* $x \in \mathfrak{M}$ if I is modular and $h(I) = \{x\}$, and *primary at* ∞ if $h(I) = \emptyset$.

The paper contains three independent results on the structure of the set of modular ideals in a Banach algebra. The first is a general result. It is rather a collection of observations, described in Theorems 1 and 2 and in the corollary to Theorem 2. For an arbitrary ideal A in a Banach algebra B, the theorems establish strong connections between the set of modular A-ideals and the set of modular B-ideals. Theorem 2 and its corollary give in addition results on the closely related question on the possibilities of representing an ideal in B as intersection of two ideals with disjoint hulls. Theorems 3 and 4 deal with closed primary ideals in a Banach algebra B with unit. In Theorem 3 it is assumed that the elements which are rational functions of a fixed set of elements $a_1, a_2, \ldots, a_n$ form a dense subspace. Identifying the Gelfand space with the joint spectrum of these elements, a complete description is given of the closed primary ideals at interior points. In Theorem 4 we specialize in a different way. Here we assume that $a \in B$ has the property that the closure I_n of the ideal generated by a^n has co-dimension n, $n \geqslant 1$. Under a supplementary condition on the norm, it is shown that all remaining closed primary ideals in I_1 are contained in $\bigcap_{n \geqslant 1} I_n$. Theorem 4 extends Theorem 1 in [3], and the basic idea of the proof is the same.

Elementary Banach algebra theory which can be found in [7] or [8] will be used freely without specific references.