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Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

ON THE IDEAL STRUCTURE OF COMMUTATIVE BANACH ALGEBRAS

YNGVE DOMAR

Department of Mathematics, Uppsala University, Uppsala, Sweden

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All rings (in particular all algebras) in this paper are assumed commutative. An ideal I in a ring A is called *modular* if A/I has a unit. For a Banach algebra B, $\mathfrak{M}$ denotes its Gelfand space, the space of non-trivial complex homomorphisms of B. $b(x)$ is the map of $b \in B$ by the homomorphism $x \in \mathfrak{M}$. Thus $x \mapsto b(x)$, $x \in \mathfrak{M}$, is the Gelfand transform of b. The *hull* $h(I)$ of an ideal $I \subseteq B$ is the set of all $x \in \mathfrak{M}$ for which the kernel, i.e. the corresponding modular maximal ideal, contains I. The ideal $I \subseteq B$ is called *primary at* $x \in \mathfrak{M}$ if I is modular and $h(I) = \{x\}$, and *primary at* ∞ if $h(I) = \emptyset$.

The paper contains three independent results on the structure of the set of modular ideals in a Banach algebra. The first is a general result. It is rather a collection of observations, described in Theorems 1 and 2 and in the corollary to Theorem 2. For an arbitrary ideal A in a Banach algebra B, the theorems establish strong connections between the set of modular A-ideals and the set of modular B-ideals. Theorem 2 and its corollary give in addition results on the closely related question on the possibilities of representing an ideal in B as intersection of two ideals with disjoint hulls. Theorems 3 and 4 deal with closed primary ideals in a Banach algebra B with unit. In Theorem 3 it is assumed that the elements which are rational functions of a fixed set of elements $a_1, a_2, \ldots, a_n$ form a dense subspace. Identifying the Gelfand space with the joint spectrum of these elements, a complete description is given of the closed primary ideals at interior points. In Theorem 4 we specialize in a different way. Here we assume that $a \in B$ has the property that the closure I_n of the ideal generated by a^n has co-dimension n, $n \geqslant 1$. Under a supplementary condition on the norm, it is shown that all remaining closed primary ideals in I_1 are contained in $\bigcap_{n \geqslant 1} I_n$. Theorem 4 extends Theorem 1 in [3], and the basic idea of the proof is the same.

Elementary Banach algebra theory which can be found in [7] or [8] will be used freely without specific references.

The author acknowledges valuable discussions with S. Kaijser, C. O. Kiselman and L.-Å. Lindahl.

1

The following algebraic lemma is probably known. But the author has been unable to find a reference, and for that reason a proof is given.

LEMMA 1. *Let A be an ideal in a commutative ring B. Then $I \mapsto I \cap A$ defines a bijection from the set of modular B-ideals I, satisfying $I+A = B$, onto the set of modular A-ideals. The inverse mapping is given by $J \mapsto \{b \in B\colon bA \subseteq J\}$. For every I in the family above, the mapping $a+I \mapsto a+I\cap A$, $a \in A$, is an isomorphism of B/I onto $A/I\cap A$.*

Proof. Let I be a modular B-ideal, satisfying $I+A = B$. $I \cap A$ is a B-ideal, hence an A-ideal. A unit modulo I in B can be represented as $i+a$, where $i \in I$, $a \in A$. Then a is a unit modulo $I \cap A$ in A, proving that $I \cap A$ is a modular A-ideal.

Then let J be a modular A-ideal. Let $e \in A$ be a unit modulo J. We shall first prove that J is a B-ideal. For this it suffices to show that $j \in J$, $b \in B$ imply that $jb \in J$. Obviously $jb \in A$, and hence $jb - jbe \in J$. Thus it remains to prove that $jbe \in J$, but this is obvious since $be \in A$. Since J has been proved to be a B-ideal, we know that $\psi(J) = \{b \in B\colon bA \subseteq J\}$ defines a B-ideal. We have to show that it is modular, and that $\psi(J)+A = B$. But $(b-be)a = (a-ae)b \in J$, since J is a B-ideal, and hence

$$b - be \in \psi(J). \tag{1}$$

Thus e is a unit in B modulo $\psi(J)$. Since $be \in A$, (1) shows moreover that $\psi(J)+ +A = B$.

We know now that each of the mappings $\varphi\colon I \to I\cap A$ and ψ has its image in the set of definition of the other mapping. To show that φ is a bijection with inverse ψ it remains to show that $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity mappings, i.e. that

$$\{b \in B\colon bA \subseteq I \cap A\} = I, \tag{2}$$

$$\{b \in A\colon bA \subseteq J\} = J, \tag{3}$$

if I and J are as above. To prove (2), we use our knowledge that $I \cap A$ is modular in A, and let $e \in A$ be a unit modulo $I \cap A$. The left member of (2) equals

$$\{b \in B\colon be \subseteq I\} = \{b \in I+A\colon be \in I\} = I + \{b \in A\colon be \in I\} = I,$$

which proves (2). To prove (3) we let $e \in A$ be a unit in A modulo J and find that the left member of (3) equals

$$\{b \in A\colon be \subseteq J\} = J,$$

which proves (3). Thus the bijectivity is proved.

The last assertion of the lemma is an immediate consequence of the fact that $B/I\cap A$ is the direct sum of $A/I\cap A$ and $I/I\cap A$.

We are now prepared to prove our first theorem.

THEOREM 1. *Let A be an ideal in a Banach algebra B. Then $I \mapsto I\cap A$ defines a bijection from the set of modular B-ideals I, satisfying $h(I) \cap h(A) = \emptyset$, onto the set of modular A-ideals. The inverse mapping is given by $J \mapsto \{b \in B\colon bA \subset J\}$. For every I in the family above, the mapping $a+I \mapsto a+I\cap A$, $a \in A$, is an algebra isomorphism of B/I onto $A/I\cap A$.*

If A is closed and I is as above, then I and $I \cap A$ are closed simultaneously.

Proof. I modular implies that $I+A$ is modular. The condition $h(I)\cap h(A) = \emptyset$ is equivalent to $h(I+A) = \emptyset$. But since $I+A$ is modular, the last condition is equivalent to $I+A = B$. Hence the assumption that $h(I)\cap h(A) = \emptyset$ is equivalent to $I+A = B$, and from this we see that the first part of the theorem is nothing but a special case of Lemma 1. The second part follows from the definition of the mapping and the obtained explicit formula for its inverse.

For the next theorem we need Šilov's idempotent theorem. By taking quotients with respect to the ideal J, we find that Šilov's theorem can be formulated in the following way.

LEMMA 2. *Let B be a Banach algebra and J a closed ideal with $h(J) = K \cup F$, where K is compact, F is closed, and $K \cap F = \emptyset$. Then B contains an idempotent e_0 modulo J, with $e_0(x) = 1$, $x \in K$, $e_0(x) = 0$, $x \in F$.*

THEOREM 2. (a) *Let B be a Banach algebra, and let K and F be disjoint subsets of $\mathfrak{M}$ with K compact and F closed. Then every ideal J in B with $h(J) = K\cup F$, has at most one representation $J = I \cap A$, where I and A are ideals with $h(I) = K$, $h(A) = F$, and where I is modular. If J moreover is closed, there exists a unique representation. I and A are then closed, and the mapping $a+I \mapsto a+J$, $b \in A$, is a Banach algebra isomorphism of B/I onto A/J.*

(b) *If I and A are closed, the isomorphism in Theorem 1 is as well a Banach algebra isomorphism.*

Proof. Part (a). Let J be an A-ideal with two representations, $I\cap A$ and $I' \cap A'$. Since $h(A\cap A') = F$, $I\cap(A\cap A')$ is another representation. Both $I+A$ and $I+(A\cap A')$ are modular ideals with empty hull. Thus $I+A = I+(A\cap A') = B$. Taking quotients with respect to $I\cap A = I\cap(A\cap A')$ we obtain

$$I/I\cap A + A/I\cap A = I/I\cap A + A\cap A'/I\cap A,$$

where both members are direct sums. Hence $A = A\cap A'$. Permutating A and A', we obtain $A' = A' \cap A$, thus $A = A'$. Then $I = I'$ follows from the injectivity of the mapping in Theorem 1. Thus the uniqueness is proved.

If J is closed the existence of a representation with I and A closed follows directly from the lemma with $A/J = (e_0+J)B/J$, $I/J = (e-e_0+J)B/J$, which gives $B/J = A/J \oplus I/J$ in Banach algebra sense. This representation also gives the Banach algebra isomorphism.

Part (b) follows directly from the last assertions of Part (a).

COROLLARIES. (a) *A closed ideal J with compact hull has a unique representation $J = I\cap A$, where I is a closed modular ideal, and A is a closed primary ideal at ∞.*

(b) *A closed ideal is modular if and only if it has compact hull and is not included in any proper closed ideal at* ∞.

(c) *A closed ideal with finite hull is the intersection of a finite set of closed primary ideals.*

The proofs of the corollaries are easy and are omitted here. Corollary (b) implies that our definition of closed primary ideal agrees with the definition used by V. P. Gurariĭ [6].

Remark. L. Waelbroeck has communicated to the author that the assertion of Lemma 2 can be proved even if J is not closed. This interesting extension of Šilov's theorem has the consequence that the existence of a unique representation in Theorem 2 holds even without the closedness assumption on J. And all three corollaries are true with the prefix "closed" deleted at all places.

2

In this section we assume that the Banach algebra B has unit e and contains elements $a_1, a_2, \ldots, a_n$ such that the rational elements of $a = (a_1, a_2, \ldots, a_n)$ form a dense subalgebra. With a rational element is then meant an element $P(a)\cdot Q(a)^{-1}$, where P and Q are polynomials, and $Q(a)$ is invertible. It is then known that $\mathfrak{M}$ can be identified with a compact subset of C^n (the joint spectrum of $a_1, a_2, \ldots, a_n$) in such a way that the Gelfand transform of a_i is the ith coordinate projection $z = (z_1, z_2, \ldots, z_n) \mapsto z_i$ on $\mathfrak{M}$. The rational elements $P(a)\cdot Q(a)^{-1}$ have Gelfand transforms $z \mapsto P(z)\cdot G(z)^{-1}$, and a general element $b \in B$ has a Gelfand transform $z \mapsto b(z)$, which is continuous on $\mathfrak{M}$, and analytic in $\mathfrak{M}^0$.

Let O_n be the ring of all power series $\sum c_\alpha z^\alpha$ with complex coefficients and positive radius of convergence. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where α_i are non-negative integers. We define K as the family of all ideals in O_n which contain every monomial z^α, if $|\alpha|$ is large enough. Every such ideal has finite co-dimension, and it can be described as the subset of O_n of all $\sum c_\alpha z^\alpha$, satisfying a finite number of specified relations

$$\sum d_\alpha c_\alpha = 0, \tag{4}$$

where d_α are fixed complex numbers, vanishing for large $|\alpha|$.

Now let $z^0 = (z_1^0, z_2^0, \ldots, z_n^0)$ be a given point in $\mathfrak{M}^0$. For every ideal J in O_n we denote by $B(J)$ the ideal in B of all $b \in B$ for which the power series expansion in z of $b(z^0+z)$ belongs to J. If $J \in K$, $B(J)$ is primary at z^0, since $(a_i - z_i^0 e)^N \in B(J)$ if N is large enough. Since the coefficients in the power series expansion are bounded linear functionals on B, the relations (4) show that $J \in K$ implies as well that $B(J)$ is closed and of finite co-dimension. Thus we have proved the sufficiency part and the second assertion of the following theorem.

THEOREM 3. *In order that I is a closed primary ideal of B at $z^0 \in \mathfrak{M}^0$ it is necessary and sufficient that $I = B(J)$ where J belongs to the family K of ideals in O_n. All these ideals I have finite co-dimension.*

Proof. In proving the remaining necessity part we can, without loss of generality, assume $z^0 = 0$. First we observe that B/I has Gelfand space $h_B(I) = \{0\}$, with Gelfand transforms $(b+I)(0) = b(0)$, $b \in B$. By the spectral radius formula, $D \in B/I$ and $D(0) = 0$ imply $\|D^n\|^{1/n} \to 0$, as $n \to \infty$. In particular, this holds for the elements $A_i = a_i + I$. Thus if $\sum c_\alpha z^\alpha$ is an element in O_n, $\sum c_\alpha A^\alpha$ represents an element in B/I if $A = (A_1, A_2, \ldots, A_n)$. The mapping $\sum c_\alpha z^\alpha \mapsto \sum c_\alpha A^\alpha$ is a ring homomorphism from O_n to B/I. Defining J as its kernel, we have thus that J is an ideal in O_n. The necessity part is proved if we can show that $I = B(J)$, and that $J \in K$.

To prove that $I = B(J)$, it suffices to show that if $b \in B$, and $\sum b_\alpha z^\alpha$ is the expansion of $b(z) = 0$, then

$$\sum b_\alpha A^\alpha = b + I. \tag{5}$$

For then, for every $b \in B$, the relation $b \in I$ is equivalent to $\sum b_\alpha A^\alpha = I$, which in its turn is equivalent to $\sum b_\alpha z^\alpha \in J$. The proof of (5) is as follows. If $b = P(a)\cdot Q(a)^{-1}$, where P and Q are polynomials and $Q(a)$ is invertible,

$$\begin{aligned} Q(A)\sum b_\alpha A^\alpha &= \Big[Q(t)\sum b_\alpha t^\alpha\Big]_{t=A} = [P(t)]_{t=A} = P(A) = P(a)+I \\ &= Q(a)b+I = \big(Q(a)+I\big)(b+I) = Q(A)(b+I), \end{aligned}$$

and from this (5) follows directly. To prove (5) in the general situation, we take a sequence $(b^m)_1^\infty$ of rational elements, converging to b. Then the analytic functions $z \mapsto b^m(z)$ in $\mathfrak{M}$ converge uniformly to $z \mapsto b(z)$. The analyticity at 0 implies the existence of a positive constant C such that the coefficients b_α^m in the expansion of $b^m(z)$ satisfy

$$|b_\alpha^m| \leqslant C^{|\alpha|+1},$$

for every m and α. Thus we can pass to the limit in the relation

$$\sum b_\alpha^m A^\alpha = b^m + I,$$

obtaining (5) in the general case.

Finally we shall prove that $J \in K$. Since $I = B(J)$, J contains all power series which are expansions of Gelfand transforms of elements in I. These power series converge in a common neighborhood of $0 \in C^n$, but since I is primary, 0 is their only common zero in this neighborhood. By [5], Theorem II.D.2, the ideal J (which is finitely generated, since O_n is Noetherian) has 0 as its locus ([5], Definition II. E.8). By Hilbert's Nullstellensats ([5], II.E.20 and III.A.7) this implies that $z_i^N \in J$, if the integer N is large enough. Thus $J \in K$.

Remarks. In addition to the closed primary ideals of Theorem 3, there exist in general non-closed primary ideals at points in $\mathfrak{M}^0$, even if $n = 1$. This is shown by the following example. Let $(w_n)_0^\infty$ be a sequence of real numbers $\geqslant 1$, satisfying $w_{m+n} \leqslant w_m w_n$, $m \geqslant 0$, $n \geqslant 0$, and $w_n^{1/n} \to 1$, as $n \to \infty$. Then the Banach space l_w of sequences $b = (b_n)_0^\infty$ with $\|b\| = \sum |b_n| w_n < \infty$, is a convolution Banach algebra, and l_w is generated by the element $a = (0, 1, 0, 0, \ldots)$. Correspondingly, the Gelfand transform is $z \mapsto \sum b_n z^n$ defined on the Gelfand space $\{z \in \boldsymbol{C}: |z| \leqslant 1\}$. If $\sup_n w_n/w_{n+1} = \infty$, the ideal generated by a is primary at 0, but not closed.

Another circumstance, worth pointing out is that Theorem 3 implies that every closed primary ideal at $z^0 \in \mathfrak{M}^0$ contains the radical of B. This is in general not true if $z^0 \notin \mathfrak{M}^0$, not even if z^0 belongs to the closure of $\mathfrak{M}^0$. This is seen by the following example. Let $A(D)$ be the disc algebra, i.e. the Banach algebra of complex-valued functions on the unit disc D in $\boldsymbol{C}$, continuous on D and analytic in D^0, under the uniform norm. Then we consider the algebra of pairs (f, α), $f \in A(D)$, $\alpha \in \boldsymbol{C}$, with component-wise addition and with

$$(f_1, \alpha_1)\cdot(f_2, \alpha_2) = \big(f_1 f_2, \alpha_1 f_2(1)+\alpha_2 f_1(1)\big).$$

We obtain a commutative Banach algebra with the norm

$$\|(f, \alpha)\| = \|f\|_{A(D)}+|\alpha|.$$

If f_0 is the function $z \mapsto z$ in $A(D)$, it is easy to see that polynomials in $a = (f_0, 1)$ are dense in B. The corresponding Gelfand space is D and the Gelfand transform of an arbitrary element (f, α) is $z \mapsto f(z)$. The subspace of all elements of the form $(f, 0)$, where $f(1) = 0$, is a closed primary ideal at 1, but it is not included in the radical of B.

Moreover, it should be observed that the definition of $\mathfrak{M}^0$ depends on $(a_1, a_2, \ldots, a_n)$. Thus the theorem can be applied to those points in $\mathfrak{M}$ which are interior for some admissible choice of these elements.

A final remark is that Theorem 3 follows, in the case $n = 1$, from Theorem 4.12 in [4]. The methods in that paper do however not generalize to $n > 1$.

3

In this section, the Banach algebra B is assumed to have a unit e. Then a closed ideal m is maximal if and only if it has co-dimension 1. If m is maximal, we call an ideal in B primary at m if it is primary at the point $x \in \mathfrak{M}$ for which the corresponding kernel is m. We call an element $a \in B$ primary at m if (a), the principal ideal generated by a, is primary at m.

THEOREM 4. *Suppose that $a \in B$ has the property that $I_n = (a^n)$ has co-dimension n, for every $n \geqslant 1$. Then I_n are the only closed ideals of finite co-dimension which are primary at I_1. Suppose that there exists a sequence $(C_n)_1^\infty$ of positive constants such that*

$$\|a^n bc\| \leqslant C_n \|a^n b\|\,\|a^n c\|, \tag{6}$$

for every $b, c \in B$, $n \geqslant 1$. Then every closed ideal of infinite co-dimension, which is primary at I_1, is contained in $\bigcap I_n$.

To prove the theorem we need two lemmas.

LEMMA 3. *Let $d \in B$ be primary at the maximal ideal m, and assume that the closed ideal I is primary at m and satisfies $0 < \operatorname{Dim}(\overline{d})/I < \infty$. Then $I \subseteq \overline{dm}$.*

Proof. Let J denote the ideal of all $b \in B$ such that $db \subseteq I$. Since $(\overline{d}) \neq I$, J is proper and hence it is included in a maximal ideal m_0. It is well known and easy to prove that if Z is a dense subspace of a Banach space X, and if Y is a closed subspace of X of finite co-dimension, then $Z \cap Y$ is dense in Y. Since $\{db\colon b \in B\}$ is dense in $(\overline{d})$, and I has finite co-dimension in $(\overline{d})$, $\{db\colon b \in J\}$ is dense in I. For that reason $I \subseteq \overline{dm_0}$. In particular, this implies $I \subseteq m_0$. But I was primary at m, hence $m = m_0$, giving $I \subseteq \overline{dm}$.

LEMMA 4. *Suppose that $d \in B$ is primary at the maximal ideal m, and that*

$$\|dbc\| \leqslant \|db\|\cdot\|dc\|, \tag{7}$$

for $b, c \in B$. Let the ideal I be primary at m and satisfy $I \subseteq (\overline{d})$. Then either $I \subseteq \overline{dm}$, or $d^2 \in I$.

Proof. Put $(\overline{d}) = K$. In a unique way we can define a composition $*$ on (d) by the definition $db * dc = dbc$, $b, c \in B$. Using (7), it is easy to see that the composition can be extended to K, making it to a Banach algebra $\langle K, *\rangle$, with unit d. Again using (7), we find easily that

$$(db) * c = bc, \tag{8}$$

for $b \in B$, $c \in K$. By (8), an ideal in $\langle K, *\rangle$ is as well an ideal in B.

I generates an ideal J in $\langle K, *\rangle$. Let us first assume that J is proper. Then it is contained in a maximal ideal M of $\langle K, *\rangle$. But M is then a closed ideal in B, such that K/M has dimension 1, and it is primary at m since $M \supseteq I$. Thus Lemma 3 is applicable, giving $I \subseteq dm$. Let us then assume that $J = K$. Then $d \in J$, that is we have a representation $d = \sum a_i * b_i$, $a_i \in I$, $b_i \in K$, $1 \leqslant i \leqslant n$. By (8) we obtain

$$d^2 = d^2 * d = \sum (d^2 * a_i) * b_i = \sum da_i * b_i = \sum a_i b_i \in I.$$

Thus the lemma is proved.

Proof of Theorem 4. I_1 is a maximal ideal, and we denote by x_0 the corresponding point in $\mathfrak{M}$. Since I_1 is generated by a, the Gelfand transform of a has x_0 as its only zero. The same holds for all a^n, $n \geqslant 2$, and subsequently all I_n, $n \geqslant 1$, are primary at I_1. Let I be a closed ideal, primary at I_1, strictly contained in I_n, for fixed $n \geqslant 1$, and of finite co-dimension. Then Lemma 3 can be applied with $d = a^n$, and this gives $I \subseteq \overline{a^n I_1} = I_{n+1}$. By induction we obtain $I = I_p$, where p is the co-dimension of I. Thus the first part of the theorem is proved.

To prove the second part, let I be a closed ideal, primary at I_1. It suffices to show that, for fixed $n \geqslant 1$, the relation $I \subset I_n$ implies $I \subseteq I_{n+1}$. Put $d = C_n a^n$. (6) shows that (7) is fulfilled, and d is primary at I_1. Hence the conditions of Lemma 4 are fulfilled, and we can conclude that $I \subseteq \overline{dI_1} = I_{n+1}$, or $d^2 \in I$. In the second case we have therefore $I_{2n} \subseteq I$, showing that I is of finite co-dimension. But then we can apply Lemma 3, which gives $I \subseteq I_{n+1}$. Thus $I \subseteq I_{n+1}$ holds in either case, and this proves the second part of the theorem.

Remarks. It would be of interest to find examples which show that Theorem 4 is no longer true if the condition (6) is removed. Applications of Theorem 3 to particular Banach algebras can be found in [2] and [3]. Here is another application, which extends results in [1] and complements investigations of V. P. Gurariĭ in [6].

Let w be a positive, Borel measurable, locally bounded, and submultiplicative function on the additive semigroup $\boldsymbol{R}^+ \cup \{0\}$. M_w is the Banach space of regular Borel measures μ on $\boldsymbol{R}^+ \cup \{0\}$ with $\|\mu\| = \int w(t)\,d\mu(t) < \infty$. Then M_w is a commutative Banach algebra with convolution as operation. L_w is the closed ideal of all absolutely continuous measures in M_w, and L'_w is the closed ideal of all measures in M_w, absolutely continuous except possibly at 0. We assume that $t \mapsto t^{-1} \times \times \log w(t)$ is bounded below on $\boldsymbol{R}^+$. Then the Gelfand space $\mathfrak{M}$ of L_w is non-empty, and can be identified with a closed half-plane in $\boldsymbol{C}$, with the Gelfand transform of an element μ given by $z \to \int e^{zt}\,d\mu(t)$. Since L_w is a closed ideal in M_w, $\mathfrak{M}$ is an open subset of the Gelfand space of M_w. And the Gelfand space of L'_w is, of course, the one-point compactification of $\mathfrak{M}$.

As for closed primary ideals at points in $\mathfrak{M}$, we have by the results in Section 1 the same structure in the three algebras. Thus we can restrict our attention to L'_w. If α is real and large enough, this space contains an element a with Gelfand transform $z \mapsto z(z-i\alpha)^{-1}$, and it is easy to see that polynomials in a are dense in L'_w. The corresponding identification of the Gelfand space, in the sense of Section 2, gives a Möbius map of the compactified half plane to a circular disc in $\boldsymbol{C}$. Hence the results in Section 2 give a complete knowledge of the closed primary ideals at interior points of $\mathfrak{M}$.

Let us consider a point at the boundary of $\mathfrak{M}$. By an easy transformation we are free to assume that the point is 0, which means that w has a positive lower bound. If we assume a little more, namely that $w(t)t^{-n} \to \infty$, as $|t| \to \infty$, for every n, then the subspaces I_n of all $\mu \in L'_w$ for which

$$\int t^m\,d\mu = 0, \quad m = 0, 1, \ldots, n-1,$$

are closed primary ideals at 0, of co-dimension n, and it is easy to see that $I_n = \overline{(a^n)}$. Using arguments that are completely analogous to the discussion on pp. 363-365 in [2], it is possible to show that (6) holds for certain constants C_n, if

$$w(x)^{-1}w(y)^{-1}\Big(\int\limits_{0 \leqslant t/x \leqslant 1} + \int\limits_{0 \leqslant (t-y)/x \leqslant 1} \big(1+(x+y-t)\big)^q w(x)\,dt\Big) \tag{9}$$

is bounded, for every integer $q \geqslant 0$, independently of x and y. Thus we can apply Theorem 4 in this case.

Similar results can be obtained for weighted convolution algebras on $\boldsymbol{R}$.

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Presented to the semester
Spectral Theory
September 23–December 16, 1977