SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

DISCONTINUOUS HOMOMORPHISMS FROM $\mathscr{C}(K)$

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1. Introduction

We shall outline here a proof of the following results.

THEOREM 1. *Let K be an infinite compact space and let $\mathscr{B}$ be a commutative radical Banach algebra having bounded approximate identities. If the continuum hypothesis ($2^{\aleph_0} = \aleph_1$) is assumed, there exist a discontinuous homomorphism from $\mathscr{C}(K)$, the algebra of all continuous complex valued functions over K, into $\mathscr{B} \oplus \mathbf{C}e$.*

COROLLARY. *If the continuum hypothesis is assumed, there exists an incomplete algebra norm over $\mathscr{C}(K)$.*

Similar results were obtained independently by Dales [4], by a very different way. A very short summary of both construction can be found in [7], and some detailed comments are given in Dales's survey article about automatic continuity [6]. On the other hand, R. Solovay has constructed models of set theory including the axiom of choice in which every homomorphism from $\mathscr{C}(K)$ is continuous.

2. Properties of algebra semi-norms defined over $\mathscr{C}(K)$ [9]

Badé and Curtis [3] showed in 1960 that for every algebra homomorphism φ from $\mathscr{C}(K)$ into a Banach algebra there exists a dense subalgebra $\mathscr{D}_\varphi$ of $\mathscr{C}(K)$ (the subalgebra depending of the homomorphism) such that $\varphi|\mathscr{D}_\varphi$ is continuous. More precisely, $\mathscr{D}_\varphi$ must contain all the function which are constant in some neighbourhood of a finite family T_φ of elements of K.

The results of Badé and Curtis were strengthened by the author in [9]. (Many results of [9] were obtained before by Allan Sinclair in [23] in a different way.)

The methods of [9] are based upon a generalization of the theory of "elements of finite closed descent in Banach algebras" due to G. R. Allan [1], [2].

THEOREM 2.1 ([9]). *Let $\mathscr{A}$ be a metrizable complete commutative algebra and let (a_n) be a sequence of elements of $\mathscr{A}$. If $[\mathscr{A} a_n]^- = \mathscr{A}$ $(n \in N)$, there exists for any linear semi-norm p on $\mathscr{A}$ a positive integer m_p such that the p-closure of $Aa_1 \dots a_n$ equals the p-closure of $Aa_1 \dots a_{m_p}$ for every $n \geqslant m_p$.*

COROLLARY 2.2 [9]. *Let (f_n) be a sequence of elements of $\mathscr{C}(K)$. If $f_n^{-1}(\{0\}) = f_1^{-1}(\{0\})$ for every $n \in N$, then for every linear norm p over $\mathscr{C}(K)$ there exists a positive integer m_p such that the p-closure of $f_1 \dots f_n \cdot \mathscr{C}(K)$ equals the p-closure of $f_1 \dots f_{m_p} \circ \mathscr{C}(K)$ for every $n \geqslant m_p$.*

Applying Corollary 2.2 we can obtain more precise results for algebra semi-norms.

LEMMA 2.3 ([9]). *Let f be an element of $\mathscr{C}(K)$ taking real non negative values. Then for every positive real r the p-closure of $f^r \cdot \mathscr{C}(K)$ equals the p-closure of $f \cdot \mathscr{C}(K)$.*

This lemma implies that for every algebra norm q over $\mathscr{C}(K)$ all non invertible elements of $\mathscr{C}(K)$ are of closed descent one in Allan's sense ([2]).

COROLLARY 2.4 ([9]). *Let q be any algebra semi-norm over $\mathscr{C}(K)$. Every q-closed ideal of $\mathscr{C}(K)$ equals the intersection of the primes containing it.*

Using Corollary 2.4 and the classical methods of Badé and Curtis we can prove much more.

THEOREM 2.5 ([9]). *Let q be any algebra semi-norm over $\mathscr{C}(K)$, and let I be a proper ideal of $\mathscr{C}(K)$. The q-closure of I equals the intersection of all q-closed primes containing I.*

Using again Badé and Curtis's results, we obtain:

THEOREM 2.6 ([9]). *Let q be any algebra seminorm over $\mathscr{C}(K)$, let $\mathscr{F}_q$ be the set of all nonmaximal q-closed primes of $\mathscr{C}(K)$ and let J_q be the intersection of all elements of $\mathscr{F}_q$ (we put $J_q = \mathscr{C}(K)$ if $\mathscr{F}_q$ is empty).*

(a) *The semi-norm q is continuous if and only if $\mathscr{F}_q$ is empty.*

(b) *$q|J_q$ is continuous, and J_q contains every ideal I of $\mathscr{C}(K)$ such that $q|I$ is continuous.*

(c) *The set of maximal ideals of $\mathscr{C}(K)$ containing some elements of $\mathscr{F}_q$ is finite.*

COROLLARY 2.7 ([9]). *There exists a discontinuous homomorphism from $\mathscr{C}(K)$ into a Banach algebra if and only if there exists a nonmaximal prime I of $\mathscr{C}(K)$ such that the quotient algebra $\mathscr{C}(K)/I$ is normable with an algebra norm.*

Let q be any algebra semi-norm of $\mathscr{C}(K)$. Every q-closed prime contains a minimal q-closed prime and the set of all primes containing a given prime is fully ordered by inclusion ([15]). So the set $\mathscr{F}_q$ is a union of chains of primes. For some compact spaces it is possible to prove that $\mathscr{F}_q$ must be a union of a finite number of chains of primes ([9], Theorem 5.1). This is true in particular when K is the Alexandroff compactification of the set of integers. It is not known, as far as the author is aware, whether this result is true in general (see [9], § 5).

Using classical order properties of $\mathscr{C}_R(K)/I$ (where I is a prime ideal of $\mathscr{C}_R(K)$, the algebra of continuous real-valued functions over K) we obtain another result about chains of primes.

THEOREM 2.8 ([9]). *Let p be any linear semi-norm over $\mathscr{C}(K)$. Every chain of p-closed primes is well ordered by inclusion.*

3. Discontinuous homomorphisms from $\mathscr{C}(K)$ and an algebra of power series [10]

The problem of constructing a discontinuous homomorphism of $\mathscr{C}(K)$ is equivalent to the problem of norming a complicated integral domain: the quotient algebra $\mathscr{C}(K)/I$ for some nonmaximal prime I. This is not possible for some nonmaximal primes of $\mathscr{C}(K)$ for some complicated compact spaces K ([9], Theorem 7.1) but we shall see that it is possible for every nonmaximal prime I in the case of separable compact spaces.

In the real case the quotient algebra $\mathscr{C}_R(K)/I$, which is fully ordered under a natural order, has very special order properties ([15], Chapter XIII). For example if I is a minimal non trivial prime of l_R^∞, the algebra of all real-valued bounded sequences, the field of fractions L of l_R^∞/I is a "real closed η_1-field" and l_R^∞/I is the ring of valuation of L for the "valuation of order". (The notion of η_1-fully ordered sets" was introduced by Hausdorff in [17]. See [15], Chapter XIII for a definition and properties of η_1-sets and real closed fields.) We outline here some results obtained by the author in [10] where an algebra of formal power series B_{ω_1} is introduced. This algebra has the following property: If the continuum hypothesis is assumed, there exists a discontinuous homomorphism of $\mathscr{C}(K)$ if and only if B_{ω_1} is normable with a real algebra norm (B. E. Johnson had pointed out in [19] the relation between Kaplansky's problem and the problem of norming the algebra of infinitesimals of some η_1-real-closed fields without introducing algebras of formal power series). We denote by ω_1 the first uncountable ordinal and we denote by S_{ω_1} the set of all (0, 1) valued sequences $t = (t_\xi)_{\xi<\omega_1}$ such that the set $\Delta_t = \{\xi < \omega_1 : t_\xi = 1\}$ has a greatest member. We equip S_{ω_1} with the lexicographic order.

Now we denote by $\mathscr{G}_{\omega_1}^{(1)}$ the real linear space of all real valued functions over S_{ω_1} which vanish outside some countable well-ordered subset of S_{ω_1}, the subset depending of the function. We say that a nonzero element f of $\mathscr{G}_{\omega_1}^{(1)}$ is positive if and only if it takes a positive value on the smallest element of the set $\{t \in S_{\omega_1} : f(t) \neq 0\}$. Equipped with this order, $\mathscr{G}_{\omega_1}^{(1)}$ is a fully ordered real linear space.

Now denote by $\mathscr{F}_{\omega_1}^{(1)}$ (resp. $\mathscr{H}_{\omega_1}^{(1)}$) the set of all real valued (resp. complex valued) functions over $\mathscr{G}_{\omega_1}^{(1)}$ which vanish outside some countable well-ordered subset of $\mathscr{G}_{\omega_1}^{(1)}$, the subset depending of the function. We define as above a structure of fully ordered real linear space over $\mathscr{F}_{\omega_1}^{(1)}$ (resp. a structure of complex linear space over $\mathscr{H}_{\omega_1}^{(1)}$). Now for every pair g_1, g_2 of elements of $\mathscr{F}_{\omega_1}^{(1)}$ (resp. $\mathscr{H}_{\omega_1}^{(1)}$) denote by $g_1 g_2$ the real valued (resp. complex valued) function defined over $\mathscr{G}_{\omega_1}^{(1)}$ by the formula:

$$(g_1 g_2)(d) = \sum_{\delta+\delta'=d} g_1(\delta) g_2(\delta') \qquad (d \in \mathscr{G}_{\omega_1}^{(1)}).$$

It is easy to see and well known ([16]) that the nonzero terms in the sum of the above formula are finite and that $g_1 g_2$ belongs to $\mathscr{F}_{\omega_1}^{(1)}$ (resp. $\mathscr{H}_{\omega_1}^{(1)}$). Under this multiplication $\mathscr{F}_{\omega_1}^{(1)}$ becomes a fully ordered field (and $\mathscr{H}_{\omega_1}^{(1)}$ becomes a field), see

[16] for the original proof of Hahn (some simpler proofs can be found in [14], [18], [22]).

THEOREM 3.1 ([10]). (a) *$\mathscr{F}^{(1)}_{\omega_1}$ is a η_1-real closed field, and $\mathscr{H}^{(1)}_{\omega_1}$ is an algebraically closed field.*

(b) Card $(\mathscr{H}^{(1)}_{\omega_1}) = \mathrm{Card}(\mathscr{F}^{(1)}_{\omega_1}) = \mathrm{Card}(\mathscr{G}^{(1)}_{\omega_1}) = \mathrm{Card}(S_{\omega_1}) = 2^{\aleph_0}$.

(c) *There exists an increasing family $(G_\xi)_{\xi<\omega_1}$ of linear subspaces of $\mathscr{G}^{(1)}_{\omega_1}$ such that $\mathscr{G}^{(1)}_{\omega_1} = \bigcup_{\xi<\omega_1} G_\xi$ and such that for every $\xi < \omega_1$ each subset of G_ξ has a countable cofinal and coinitial sequence.*

(d) *For every $\xi < \omega_1$, the set F_ξ (resp. H_ξ) of all elements of $\mathscr{F}^{(1)}_{\omega_1}$ (resp. $\mathscr{H}^{(1)}_{\omega_1}$) vanishing outside G_ξ is a real-closed subfield of $\mathscr{F}^{(1)}_{\omega_1}$ (resp. an algebraically closed subfield of $\mathscr{H}^{(1)}_{\omega_1}$).*

(e) *$\mathscr{F}^{(1)}_{\omega_1} = \bigcup_{\xi<\omega_1} F_\xi$, $\mathscr{H}^{(1)}_{\omega_1} = \bigcup_{\xi<\omega_1} H_\xi$ and for every $\xi < \omega_1$, each subset of F_ξ has a countable cofinal and coinitial sequence.*

For every nonzero element x of $\mathscr{H}^{(1)}_{\omega_1}$ denote by $V(x)$ the smallest element of the set $\{d \in \mathscr{G}^{(1)}_{\omega_1}: x(d) \neq 0\}$. The map $x \to V(x)$ defines a valuation over $\mathscr{H}^{(1)}_{\omega_1}$ (we put as usual by convention $V(x) = +\infty$).

Put: $B_{\omega_1} = \{x \in \mathscr{F}^{(1)}_{\omega_1}: V(x) \geqslant 0\}$, $B'_{\omega_1} = \{x \in \mathscr{F}^{(1)}_{\omega_1}: V(x) > 0\}$, $C_{\omega_1} = \{x \in \mathscr{H}^{(1)}_{\omega_1}: V(x) \geqslant 0\}$, $C'_{\omega_1} = \{x \in \mathscr{H}^{(1)}_{\omega_1}: V(x) > 0\}$.

B_{ω_1} and C_{ω_1} are rings of valuation, and B'_{ω_1} and C'_{ω_1} are respectively the unique maximal ideals of B_{ω_1} and C_{ω_1}. In fact B_{ω_1} is the set of "finite elements" of $\mathscr{F}^{(1)}_{\omega_1}$ ($B_{\omega_1} = \{x \in \mathscr{F}^{(1)}_{\omega_1}: |x| < n\cdot 1\}$ for some $n \in N$) and B'_{ω_1} is the set of "infinitesimals" of $\mathscr{F}^{(1)}_{\omega_1}$ ($B'_{\omega_1} = \{x \in \mathscr{F}^{(1)}_{\omega_1}: |x| < 1/n\}$ for every $n \in N$).

Note also that as a field $\mathscr{H}^{(1)}_{\omega_1}$ (and also H_ξ for every $\xi < \omega_1$) is isomorphic with the field of complex numbers. This follows easily from Steinitz theory.

Using methods of ordered fields theory, we obtain the following result:

THEOREM 3.2 ([10]). (a) *Let $\mathscr{F}$ be a real closed η_1-field, equipped with a real algebra structure compatible with its order. There exists an order preserving algebra homomorphism from $\mathscr{F}^{(1)}_{\omega_1}$ into $\mathscr{F}$. Moreover, if* $\mathrm{Card}(\mathscr{F}) = 2^{\aleph_0}$ *and if the continuum hypothesis is assumed there exists an order preserving algebra isomorphism from $\mathscr{F}^{(1)}_{\omega_1}$ onto $\mathscr{F}$.*

(b) *Let F be a fully ordered field having a real algebra structure compatible with its order such that* $\mathrm{Card}(F) = 2^{\aleph_0}$. *If the continuum hypothesis is assumed there exists an order preserving algebra homomorphism from F into $\mathscr{F}^{(1)}_{\omega_1}$.*

Some other purely algebraic results are obtained in [10]. In particular, it is proved that if the continuum hypothesis is assumed to be false there exists non isomorphic η_1-real-closed fields of cardinality $2^{\aleph_0}$, which solves a problem raised by Erdös, Gillman and Henriksen in [8].

We now turn to the construction of discontinuous homomorphisms from $\mathscr{C}(K)$. Using Theorem 3.2 and order properties of the quotient algebra $\mathscr{C}_R(K)/I$, where I is a nonmaximal prime of $\mathscr{C}_R(K)$, we obtain:

THEOREM 3.3 [10]. *Let K be any infinite compact space.*

(a) *For any nonmaximal prime I of $\mathscr{C}_R(K)$ there exists a faithful algebra homomorphism from B_{ω_1} into the quotient algebra $\mathscr{C}_R(K)/I$.*

(b) *If the continuum hypothesis is assumed there exists for every nonmaximal prime I of $\mathscr{C}_R(K)$ such that* $\mathrm{Card}(\mathscr{C}_R(K)/I) = 2^{\aleph_0}$ *a faithful algebra homomorphism from $\mathscr{C}_R(K)/I$ into B_{ω_1}.*

Of course similar results are true in the complex case if we put C_{ω_1} in place of B_{ω_1}. So we obtain:

COROLLARY 3.4 [10]. *If the continuum hypothesis is assumed the existence of an algebra norm over C_{ω_1} implies the existence of a discontinuous homomorphism from $\mathscr{C}(K)$ for every infinite compact space K.*

Note that Theorem 3.3 and Corollary 2.7 show that the existence of a discontinuous homomorphism from $\mathscr{C}(K)$ for some compact space K imply the existence of an algebra norm over C_{ω_1}, and this result does not involve the continuum hypothesis. So in some sense embedding C_{ω_1} into a Banach algebra is necessary to construct discontinuous homomorphisms from $\mathscr{C}(K)$.

4. Embedding C_{ω_1} into a Banach algebra [11], [12]

Denote by $C[[X]]$ the algebra of all formal power series in one variable ($C[[X]]$ is roughly the set of formal sums $\sum_{n=0}^{\infty} a_n X^n$, where the sequence (a_n) runs over all complex sequences); $C[[X]]$ is a ring of valuation of the field of all Laurent series, and the semi-group of values of $C[[X]]$ for this valuation is the additive semi-group of all nonnegative integers. In fact the relation between Z and $C[[X]]$ is similar with the relation between $\mathscr{G}^{(1)}_{\omega_1}$ and C_{ω_1}.

G. R. Allan has constructed in [1] an embedding from $C[[X]]$ into $\mathscr{B}\oplus Ce$, where $\mathscr{B}$ is any commutative radical Banach algebra having bounded approximate identities. His proof is divided in two steps:

(1) Let $\mathscr{B}$ be a commutative radical Banach algebra. If there exists $x \in \mathscr{B}$ such that the sequence $[x^n\mathscr{B}]^-$ is ultimately constant, there exists a faithful algebra homomorphism from $C[[X]]$ into $\mathscr{B}\oplus Ce$ such that x is the image of X.

(2) For every non unital commutative separable Banach algebra $\mathscr{B}$ having bounded approximate identities there exists $x \in \mathscr{B}$ such that $[x\mathscr{B}]^- = \mathscr{B}$.

In other terms, for every commutative radical Banach algebra $\mathscr{B}$ having a b.a.i. there exists a map $\varphi: N \to \mathscr{B}$ such that $\varphi(n+n') = \varphi(n)\cdot\varphi(n')$, $[\varphi(n)\mathscr{B}]^- = [\varphi(n')\mathscr{B}]^-$ for every $n, n' \in N$, and this implies the existence of a faithful algebra homomorphism ψ from $C[[X]]$ into $\mathscr{B}\oplus Ce$ such that $\psi(X^n) = \varphi(n)$ for every $n \in N$.

Our strategy for the construction of an embedding from C_{ω_1} into a Banach algebra is similar, but the proofs of the analogous two steps are of course much more complicated.

We denote respectively by T_{ω_1} and T'_{ω_1} the set of all nonnegative elements and the set of all strictly positive elements of $\mathcal{G}^{(1)}_{\omega_1}$. Also for every $d \in T'_{\omega_1}$ we denote by X^d the element of C'_{ω_1} which takes the value 1 at d and vanishes outside $\{d\}$. The first step of our embedding of C_{ω_1} into a Banach algebra is given by the following result:

THEOREM 4.1 [11]. *Let $\mathcal{B}$ be a radical commutative Banach algebra. If there exists a faithful map φ: $T'_{\omega_1} \to \mathcal{B}$ such that $\varphi(d+d') = \varphi(d)\cdot\varphi(d')$ and $[\varphi(d)\mathcal{B}]^- = [\varphi(d')\mathcal{B}]^-$ for every $d, d' \in T'_{\omega_1}$, there exists a faithful algebra homomorphism ψ: $C'_{\omega_1} \to \mathcal{B}$ such that $\psi(X^d) = \varphi(d)$ for every $d \in T'_{\omega_1}$.*

The proof involves as in [1] a theorem of Arens and Calderón which asserts that the equation $a_0+a_1x+\ldots+a_{n-1}x^{n-1} = 0$ has a solution in $\mathcal{B}$ for every commutative radical Banach algebra $\mathcal{B}$ and for every finite family $(a_0, a_1, \ldots, a_{n-1})$ of elements of $\mathcal{B}$. The proof involves also the following lemma, where the Mittag-Leffler theorem about projective limits is used as in [1]:

LEMMA 4.2 [11]. *Let $\mathcal{B}$ be a commutative Banach algebra and let (a_n) be a sequence of elements of $\mathcal{B}$ satisfying $[a_n\mathcal{B}]^- = \mathcal{B}$ for every $n \in N$. There exists for every sequence (δ_n) of elements of $\mathcal{B}$ an element ϱ of $\mathcal{B}$ such that $\varrho - \sum_{k=1}^{n} a_1 a_2 \ldots a_k \delta_k \in a_1 a_2 \ldots a_{n+1}\mathcal{B}$ for every $n \in N$.*

We shall not outline here the whole proof, in which some results of valuation theory are used (in particular an old result by Mac Lane [21] which ensures that some maximal fields with valuation are algebraically closed). In some sense certain subrings of $\mathcal{B}$ can be considered as rings of valuation of "countably maximally complete" valued fields in which the theorem of Arens and Calderón plays the role of Hensel's lemma, and some of our methods are related with Kaplansky's thesis about maximal fields with valuations [20].

Remark 4.3. The condition of Theorem 4.1 is in fact necessary for a radical Banach algebra $\mathcal{B}$ to contain a copy of C'_{ω_1}. This follows from a result of [10] which shows that for any algebra norm p over C'_{ω_1} there exists a subalgebra of C'_{ω_1} isomorphic with C'_{ω_1} in which each nonzero ideal is p-dense.

The second step of our embedding is given by the following theorem.

THEOREM 4.4 [12]. *Let $\mathcal{B}$ be any nonunital commutative separable Banach algebra having bounded approximate identities. There exists a faithful map φ: $T'_{\omega_1} \to \mathcal{B}$ satisfying:*

$$[\varphi(d)\mathcal{B}]^- = \mathcal{B} \qquad (d \in T'_{\omega_1}),$$
$$\varphi(d+d') = \varphi(d)\cdot\varphi(d') \qquad (d, d' \in T'_{\omega_1}).$$

In the case of $C[[X]]$ the "second step" was an immediate consequence of the Johnson-Varopoulos strenghtening of Cohen's factorization theorem:

For any Banach algebra $\mathcal{B}$ with b.a.i. and for any sequence (x_n) of elements of $\mathcal{B}$ there exists a sequence (y_n) of elements of $\mathcal{B}$ and an element b of $\mathcal{B}$ such that $x_n = by_n$ for every $n \in N$. So in the separable case applying this result to a sequence (x_n) dense in $\mathcal{B}$ we obtain $[b\mathcal{B}]^- = \mathcal{B}$ and $[b^n\mathcal{B}]^- = \mathcal{B}$ $(n \in N)$, the desired result.

The proof of Theorem 4.4 is much more complicated, because the semi-group T'_{ω_1} is an η_1-set. Also T'_{ω_1} is divisible, so the elements of the range of the map ψ must have nth roots for every $n \in N$.

The proof given in [12] involves a special class of elements of $\mathcal{B}$ (the "Cohen elements"). We give here a definition of these elements which is simpler than but equivalent to the original definition given by the author in [12].

DEFINITION 4.5. Let $\mathcal{B}$ be a nonunital commutative separable Banach algebra with b.a.i. An element α of $\mathcal{B}$ shall be called a *Cohen element* if and only if there exists a sequence (a_n) of elements of $\mathcal{B}$ satisfying the following conditions:

(1) $\alpha = \lim_{n\to\infty} \exp(a_n)$.

(2) The sequence $(\exp(p^{-1}\cdot a_n))$ converges for every $p \in N$ towards an element of $\mathcal{B}$ which we denote by $a^{1/p}$.

(3) $\beta = \lim_{n\to\infty} \beta\alpha\exp(-a_n)$ for every $\beta \in \mathcal{B}$.

(4) $\sup_{n\in N} ||\alpha^{1/p}\exp(-p^{-1}a_n)|| < +\infty$ for every $p \in N$.

The idea of the proof of Theorem 4.4 is roughly that every maximal divisible semigroup of the set of Cohen elements is an "η_1-set" so contains a copy of T'_{ω_1}. This property follows from four lemmas, two of them using some refinements of classical factorization methods and two of them using only the definition of Cohen's elements and arguments related with the proof of the Mittag-Leffler's theorem about projective limits.

We thus obtain the desired embedding of C_{ω_1} into some Banach algebras:

THEOREM 4.6 [12]. *Let $\mathcal{B}$ be a commutative separable radical Banach algebra. If $\mathcal{B}$ possesses bounded approximate identities, there exists a faithful algebra homomorphism from C_{ω_1} into $\mathcal{B}\oplus Ce$.*

Using Corollary 3.4, we can now give an answer to the so-called Kaplansky's problem:

COROLLARY 4.7 [12]. *Let K be any infinite compact space and let $\mathcal{B}$ be a commutative separable radical Banach algebra having bounded approximate identities; if the continuum hypothesis is assumed there exists a discontinuous homomorphism from $\mathcal{C}(K)$ into $\mathcal{B}\oplus Ce$.*

We obtain in this way discontinuous homomorphisms from $\mathcal{C}(K)$ into $L^1_*(0,1)\oplus$ $\oplus Ce$, where $L^1_*(0,1)$ is the "Volterra algebra", and into the weighted algebras $L^1(R^+, w)\oplus Ce$, where w is a measurable positive valued function over R^+ such that $w(t+t') \leqslant w(t)w(t')$ for every $t, t' \in R^+$ and such that $\lim_{t\to\infty}(w(t))^{1/t} = 0$.

Note also that every nonmaximal prime I of $\mathcal{C}(K)$ such that $\mathrm{Card}(\mathcal{C}(K)/I) = 2^{\aleph_0}$ is the kernel of a discontinuous homomorphism from $\mathcal{C}(K)$. So if K is separable every nonmaximal prime of $\mathcal{C}(K)$ is the kernel of a discontinuous homomorphism from $\mathcal{C}(K)$.

5. Discontinuous homomorphism from commutative separable Banach algebras [13]

The above construction of discontinuous homomorphisms from $\mathscr{C}(K)$ can be summarized by the following diagram, where I is any nonmaximal prime of $\mathscr{C}(K)$ such that $\mathrm{Card}(\mathscr{C}(K)/I) = 2^{\aleph_0}$ and $\mathscr{B}$ is any commutative radical Banach algebra with bounded approximate identities.

$$\mathscr{C}(K) \xrightarrow{Q_I} \mathscr{C}(K)/I \xrightarrow{\theta} C_{\omega_1} \xrightarrow{\psi} \mathscr{B}\oplus Ce.$$

In this diagram Q_I denotes the canonical map from $\mathscr{C}(K)$ onto $\mathscr{C}(K)/I$, and the maps θ and ψ are faithful. The continuum hypothesis is used only in the construction of θ.

G. Dales constructs in [5] discontinuous homomorphisms from any commutative Banach algebra having infinitely many characters and from the classical radical algebras $L^1_*(0,1)$ and $L^1_*(R^+, w)$, where w is a "weight" satisfying the conditions mentioned above. The kernels of these homomorphisms are some particular nonmaximal primes the quotient algebras by which may be "naturally" embedded into l^∞/I for some suitable nonmaximal prime I of l^∞.

We outline here a much more general but much more complicated construction, the proof of which can be found in [13].

We first state a purely algebraic result, which involves an extension of Kaplansky's theorems about maximal fields with valuations [20] and Chevalley's theorem about extensions of places.

THEOREM 5.1 [13]. *Let A be any nonunital complex commutative algebra which is an integral domain of cardinality $2^{\aleph_0}$. If the continuum hypothesis is assumed, there exists a faithful algebra homomorphism from A into C'_{ω_1}.*

COROLLARY 5.2 [13]. *If the continuum hypothesis is assumed the weighted convolution algebra $L^1(R^+, e^{-t^2})$ contains a copy of any nonunital commutative complex algebra which is an integral domain of cardinality $2^{\aleph_0}$.*

Note that $L^1(R^+, e^{-t^2})$ is itself a nonunital complex algebra of cardinality $2^{\aleph_0}$ which is an integral domain by Titchmarch's convolution theorem.

COROLLARY 5.3 [13]. *If the continuum hypothesis is assumed every nonunital commutative complex algebra which is an integral domain of cardinality $2^{\aleph_0}$ is normable with an algebra norm.*

Using Theorems 5.3 and 4.5 we see that if the continuum hypothesis is assumed every nonmaximal prime I of a commutative unital Banach algebra $\mathscr{A}$ such that $\mathrm{Card}(\mathscr{A}/I) = 2^{\aleph_0}$ is the kernel of an homomorphism from $\mathscr{A}$ into a Banach algebra. This homomorphism may be continuous but using algebraic properties of C_{ω_1} it is possible to prove that for any algebra norm p defined over C_{ω_1} there exists an homomorphism ϱ from C_{ω_1} into itself such that the norm $p\circ\varrho$ takes arbitrary given positive values over a given base of transcendence of C'_{ω_1}.

So we obtain the following general result:

THEOREM 5.4 [13]. *Let $\mathscr{A}$ be any commutative unital Banach algebra and let $\mathscr{B}$ be a commutative radical Banach algebra having bounded approximate identities. If the continuum hypothesis is assumed every nonmaximal prime I of $\mathscr{A}$ such that $\mathrm{Card}(\mathscr{A}/I) = 2^{\aleph_0}$ is the kernel of a discontinuous homomorphism from $\mathscr{A}$ into $\mathscr{B}\oplus Ce$.*

The construction of the discontinuous homomorphism of Theorem 5.4 may be summarized by the following diagram:

$$\mathscr{A} \xrightarrow{Q_I} \mathscr{A}/I \xrightarrow{\theta} C_{\omega_1} \xrightarrow{\varrho} C_{\omega_1} \xrightarrow{\psi} \mathscr{B}\oplus Ce.$$

In this diagram Q_I denotes also the canonical map from $\mathscr{A}$ into $\mathscr{A}/I$, the maps θ, ϱ and ψ are faithful, the map ψ is the map constructed in § 4 and the continuum hypothesis is used only in the construction of θ. Note that if $\mathscr{S}$ is a base of transcendence of $\mathfrak{M}$ over the complex numbers modulo I we may arrange by a suitable construction of the map ϱ that the semi-norm induced over $\mathscr{A}$ via this homomorphism by the norm of $\mathscr{B}\oplus Ce$ takes arbitrary given positive values over $\mathscr{S}$, $\mathfrak{M}$ being any maximal ideal of $\mathscr{A}$ containing I.

It is easy to see that every unital commutative Banach algebra having infinitely many characters and every unital commutative separable Banach algebra whose nilradical is infinite dimensional possesses a nonmaximal prime I satisfying the hypothesis of Theorem 5.4. On the other hand, it is possible to prove that an infinite-dimensional unital commutative Banach algebra whose nilradical is finite codimensional has a maximal ideal $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{M}^2$ is infinite dimensional. In this case there exists an obvious method to construct a discontinuous homomorphism from $\mathscr{A}$ into any Banach algebra having nonzero nilpotent elements (and it is possible to arrange that the semi-norm induced over $\mathscr{A}$ via this isomorphism takes arbitrarily given positive values over a Hamel basis of $\mathfrak{M}$ modulo $\mathfrak{M}^2$). So we obtain.

THEOREM 5.5 [13]. *If the continuum hypothesis is assumed there exists a discontinuous homomorphism from any infinite-dimensional commutative separable Banach algebra into $L^1_*(0,1)\oplus Ce$.*

Remark 5.6. We actually proved much more. If $\mathscr{A}$ is any infinite-dimensional commutative separable Banach algebra and if p is any linear norm defined over $\mathscr{A}$ there exists an homomorphism from $\mathscr{A}$ into $L^1_*(0,1)\oplus Ce$ which is p-discontinuous.

This follows from the existence of sequences of elements of $\mathscr{A}$ over which the values taken by the semi-norm induced by the homomorphism of Theorem 5.5 may be arbitrarily fixed.

6. Further results

In this section we shall state some unpublished results of the author. We shall denote by $\nu(x)$ the spectral radius of an element x of a Banach algebra $\mathscr{A}$. We denote by $\mathrm{Spec}(x)$ the spectrum of x in $\mathscr{A}$ if $\mathscr{A}$ is unital and the spectrum of x in $\mathscr{A}\oplus Ce$ if $\mathscr{A}$ is not unital. The set $\{\lambda\in\mathrm{Spec}(x)\colon |\lambda| = \nu(x)\}$ shall be denoted by $\Delta(x)$.

When $\mathscr{A}$ is unital, we shall denote by $\mathscr{D}(x)$ the subalgebra of $\mathscr{A}$ generated by x and the unit element and we shall denote by $\mathscr{R}(x)$ the subalgebra of $\mathscr{A}$ generated by $\mathscr{D}(x)$ and the set of invertible elements of $\mathscr{D}(x)$; $\mathscr{D}(x)$ is the algebra of complex polynomials in x and $\mathscr{R}(x)$ is the algebra of rational functions in x, where we limit of course ourselves to the rational fractions whose poles lie outside the spectrum of x. It follows easily from Gelfand theory that for every homomorphism ψ from a Banach algebra $\mathscr{A}$ into another Banach algebra and for every $x \in \mathscr{A}$ we have: $\lim ||\psi(x^n)||^{1/n} \leqslant \nu(x)$. In some cases it is possible to prove more. For example we have the following result.

THEOREM 6.1. *Let $\mathscr{A}$ be a semi-simple commutative Banach algebra and let x be a nonzero element of $\mathscr{A}$. If $\Delta(x)$ contains only isolated points of* Spec(x), *then for every homomorphism ψ from $\mathscr{A}$ into a Banach algebra we have*:

$$\sup\left(||\psi(x^n)||/||x^n||\right) < +\infty .$$

On the other hand we can make the sequence $||\psi(x^n)||$ grow as fast as possible if the condition of Theorem 6.1 is not satisfied.

THEOREM 6.2. *Let $\mathscr{A}$ be a commutative Banach algebra and let x be an element of $\mathscr{A}$. Assume the continuum hypothesis. If $\Delta(x)$ contains a nonisolated point of* Spec(x) *there exist for every commutative separable radical Banach algebra $\mathscr{B}$ having bounded approximate identities and for every sequences (λ_n) of positive real numbers such that* $\lim(\lambda_n)^{1/n} = \nu(x)$ *an homomorphism ψ: $\mathscr{A} \to \mathscr{B}\oplus Ce$ satisfying*:

$$\liminf_{n\to\infty}\left(\lambda_n^{-1}||\psi(x^n)||\right) = +\infty .$$

Theorems 6.1 and 6.2 solve completely the question of the rate of growth of the sequence $\left(||\psi(x^n)||\right)$ for arbitrary homomorphisms from $\mathscr{A}$ in the semi-simple case. The situation is more complicated in the non semi-simple case. If we assume $\mathscr{A}$ to be separable (or we suppose simply Card$(\mathscr{A}) = 2^{\aleph_0}$) it is possible to obtain results similar to Theorems 6.1 and 6.2 with a more sophisticated condition. (If $x \in \mathrm{Rad}(\mathscr{A})$ Theorem 6.2 works if x is not nilpotent.) We shall not state these results here.

Now we turn to the problem of constructing an homomorphism ψ from $\mathscr{A}$ such that $\psi|\mathscr{D}(x)$ is discontinuous. Theorem 6.2 gives a solution in some cases, but it may happen even in the semi-simple case that $\mathscr{D}(x)$ is infinite-dimensional and that $\Delta(y)$ contains only isolated points of Spec(y) for every $y \in \mathscr{D}(x)$ (consider the element $x = \left(\exp(in)/n\right)$ on $c_0\oplus Ce$). Nevertheless the following result is true.

THEOREM 6.3. *Let $\mathscr{A}$ be a commutative unital Banach algebra and let x be an element of $\mathscr{A}$. Assume the continuum hypothesis. If $\mathscr{A}$ is a semi-simple or separable and if $\mathscr{D}(x)$ is infinite-dimensional there exists for every commutative separable Banach algebra with b.a.i. and for every algebra norm q defined over $\mathscr{A}$ an homomorphism ψ: $\mathscr{A} \to \mathscr{B}\oplus Ce$ such that $\psi|\mathscr{D}(x)$ is q-discontinuous.*

For the algebra $\mathscr{R}(x)$ we have a much stronger result:

THEOREM 6.4. *Under the hypothesis of Theorem 6.3 there exists for any linear norm p defined over $\mathscr{R}(x)$ an homomorphism ψ: $\mathscr{A} \to \mathscr{B}\oplus Ce$ such that $\psi|\mathscr{R}(x)$ is p-discontinuous.*

The proofs of Theorems 6.2, 6.3 and 6.4 are related with the construction of Theorem 5.4. The problem here is to find some Cohen elements α in $\mathscr{B}$ having special properties (this is done using factorization methods) and then to construct a map ψ: $C_{\omega_1} \to \mathscr{B}\oplus Ce$ such that $\psi(z) = \alpha$, where z is related with x via the map θ of the diagram following Theorem 5.4.

We state in conclusion some unpublished results related with chains of q-closed primes of $\mathscr{C}(K)$ for a discontinuous algebra norm q on $\mathscr{C}(K)$. Theorem 2.8 has a converse.

THEOREM 6.5. *Let K be any infinite compact space, let $\mathscr{F}$ be a chain of nonmaximal primes of $\mathscr{C}(K)$ which is well ordered by inclusion and let I be the smallest element of $\mathscr{F}$. If* Card$(\mathscr{C}(K)/I) = 2^{\aleph_0}$ *and if the continuum hypothesis is assumed there exists an algebra norm q over $\mathscr{C}(K)$ such that all elements of $\mathscr{F}$ are q-closed.*

I was not able to find a necessary and sufficient condition for a well-ordered chain $\mathscr{F}$ of nonmaximal primes of $\mathscr{C}(K)$ to be exactly the set of nonmaximal q-closed primes of some algebra norm q over $\mathscr{C}(K)$. An obvious necessary condition is that every union of an increasing family of q-closed primes which has not countable cofinal subset must be closed but I do not know whether this condition is sufficient in the separable case.

Nevertheless we have the following result, the proof of which we shall not outline here.

THEOREM 6.6. *Let K be any infinite metric compact space. If the continuum hypothesis is assumed there exists for every ordinal ω less than ω_2 an algebra norm q over $\mathscr{C}(K)$ such that the set of all nonmaximal q-closed primes is a well ordered chain order-isomorphic with ω.*

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Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

SOME RESULTS ON DUALITY FOR SPECTRAL DECOMPOSITIONS

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I should like to begin by saying that it is a honour for me to give lectures at the Banach Center and I should like to address my warmest thanks to Organizing Committee for invitation.

The first part of my lecture will be essentially a survey of some results published between 1971 and 1977 ([18], [20], [21]). In the second part I will discuss some new unpublished results.

Before giving formal definitions, which are probably less known, let me begin with some introductive ideas.

Introduction

1. Consider an operator T on a complex Banach space X, whose spectrum consists of two separate parts:

$$\mathrm{sp}(T) = F_1 \cup F_2, \quad F_1 \cap F_2 = \emptyset.$$

Then, by the Riesz decomposition theorem, X decomposes into the direct sum

$$X = X(F_1) \oplus X(F_2),$$

where $X(F_j)$ are closed subspaces invariant for T and

$$\mathrm{sp}\big(T, X(F_j)\big) = F_j, \quad j = 1, 2.$$

Moreover, $X(F_j)$ is the range of the projection P_j commuting with T which is defined by:

$$P_j = (2\pi i)^{-1} \int_{\Gamma_j} (z-T)^{-1} dz,$$

Γ_j being an admissible contour of integration which "surrounds" F_j and leaves outside F_k, $k \neq j$.

Consider now the dual operator T', defined on the dual space X' by:

$$(T'u)(x) = u(Tx), \quad u \in X', \ x \in X.$$