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SOME RESULTS ON DUALITY FOR SPECTRAL DECOMPOSITIONS

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I should like to begin by saying that it is a honour for me to give lectures at the Banach Center and I should like to address my warmest thanks to Organizing Committee for invitation.

The first part of my lecture will be essentially a survey of some results published between 1971 and 1977 ([18], [20], [21]). In the second part I will discuss some new unpublished results.

Before giving formal definitions, which are probably less known, let me begin with some introductive ideas.

Introduction

1. Consider an operator T on a complex Banach space X, whose spectrum consists of two separate parts:

$$\mathrm{sp}(T) = F_1 \cup F_2, \quad F_1 \cap F_2 = \emptyset.$$

Then, by the Riesz decomposition theorem, X decomposes into the direct sum

$$X = X(F_1) \oplus X(F_2),$$

where $X(F_j)$ are closed subspaces invariant for T and

$$\mathrm{sp}\big(T, X(F_j)\big) = F_j, \quad j = 1, 2.$$

Moreover, $X(F_j)$ is the range of the projection P_j commuting with T which is defined by:

$$P_j = (2\pi i)^{-1} \int_{\Gamma_j} (z-T)^{-1} dz,$$

Γ_j being an admissible contour of integration which "surrounds" F_j and leaves outside F_k, $k \neq j$.

Consider now the dual operator T', defined on the dual space X' by:

$$(T'u)(x) = u(Tx), \quad u \in X', \ x \in X.$$

Taking into account that $\mathrm{sp}(T') = \mathrm{sp}(T) = F_1 \cup F_2$, we have also a decomposition for X':

$$X' = X'(F_1) \oplus X'(F_2), \quad \mathrm{sp}\big(T', X'(F_j)\big) = F_j, \quad X'(F) = \mathscr{R}(Q_j),$$

$$Q_j = (2\pi i)^{-1} \int_{\Gamma_j} (z - T')^{-1} dz, \quad j = 1, 2.$$

It is easy to see that $Q_j = P_j'$, $j = 1, 2$. By applying the range-kernel duality we get:

$$X'(F_1) = \mathscr{R}(Q_1) = \mathscr{R}(P_1') = \mathscr{N}(P_1)^{\perp} = \mathscr{R}(P_2)^{\perp} = X(F_2)^{\perp}$$

and similarly $X'(F_2) = X(F_1)^{\perp}$.

Of course, this example is a very simple one and everybody knows that a situation like that when the spectrum consists of separate parts is a rara avis in the infinite dimensional case. What can we say when the spectrum of T has no separate parts? If we assume that T has a good spectral decomposition, then we get a similar duality formula.

2. Let T be a Dunford spectral operator ([10], [11], [12]); that means, there exists a projection-valued measure on the Borel sets of the complex plane:

$$\sigma \to E(\sigma), \quad \sigma \in \mathscr{B}(C),$$

such that $\mathrm{sp}\big(T, E(\sigma)X\big) \subset \bar{\sigma}$, $\forall \sigma \in \mathscr{B}(C)$.

It is easy to see that the map

$$\sigma \to \big(E(\sigma)\big)', \quad \sigma \in \mathscr{B}(C),$$

is a projection-valued measure and $\mathrm{sp}\big(T', \big(E(\sigma)\big)'X'\big) \subset \bar{\sigma}$, $\forall \sigma \in \mathscr{B}(C)$. Indeed, we have $\big(E(\sigma)\big)'X' = \mathscr{R}\big(E(\sigma)'\big) = \mathscr{N}\big(E(\sigma)\big)^{\perp} = \mathscr{R}\big(E(\sigma^c)\big)^{\perp}$ (we note that, by the additivity of $E(\sigma)$, $E(\sigma) + E(\sigma^c) = I$). By applying a well known isomorphism theorem, we get

$$\mathscr{R}\big(E(\sigma^c)\big)^{\perp} \cong \big(X/E(\sigma^c)X\big)' \cong \big(E(\sigma)X\big)',$$

so that $\mathrm{sp}\big(T', E(\sigma)'X'\big) = \mathrm{sp}\big(T', \big(E(\sigma)X\big)'\big) = \mathrm{sp}\big(T, E(\sigma)X\big) \subset \bar{\sigma}$, $\forall \sigma \in \mathscr{B}(C)$. Therefore, if we denote $X(\sigma) = E(\sigma)X$ and $X'(\sigma) = E(\sigma)'X'$, we get the following duality formula:

$$X'(\sigma) = X(\sigma^c)^{\perp}, \quad \forall \sigma \in \mathscr{B}(C).$$

3. In order to make another step towards generality let us make the following remark. Consider again a Dunford spectral operator T and $\sigma \to E(\sigma)$, its spectral measure. Then we can integrate, with respect to $E(\sigma)$, each bounded Borel measurable complex function f and we obtain an operator:

$$\mathscr{U}(f) = \int f(z) E(dz).$$

The mapping $\mathscr{U}$ from complex functions to linear bounded operators is linear, multiplicative and $\mathscr{U}(1) = I$, $\mathscr{U}(\mathrm{id}) = \int z E(dz) = S$ (the scalar part of T). Moreover, T has a unique Jordan decomposition $T = S + Q$ where Q is a quasinilpotent operator commuting with T. Thus, the scalar part of a Dunford spectral operator admits a functional calculus defined on a very large class of functions; furthermore, the spectral measure may be recaptured from the functional calculus. Now, what can we say when we only know that an operator admits a functional calculus? If the functional calculus is given on the largest algebra (in some sense) of complex functions, then the operator has a spectral decomposition of Dunford type. If the functional calculus is given on the smallest algebra — the algebra of all analytic functions in some neighbourhood of the spectrum — the best thing that we can get is a decomposition of Riesz type, i.e. a decomposition of the spectrum into separate parts. There are many interesting intermediate algebras between the smallest one and the largest one; the algebra $C^{\infty}(C)$ of all complex infinitely differentiable functions on the complex plane is an example. The operators having a C^{∞}-functional calculus are called scalar generalized operators and have been introduced and studied by C. Foiaş in 1960 ([13]).

And thus, S is called a *scalar generalized operator* if there exists a mapping

$$\mathscr{U}\colon C^{\infty}(C) \to L(X),$$

which is linear, multiplicative and $\mathscr{U}(1) = I$, $\mathscr{U}(\mathrm{id}) = S$. Such a mapping is called a *spectral distribution of S*.

Any scalar generalized operator has a spectral decomposition of a generalized type. In order to describe such a decomposition, denote:

$$X(F) := \{x \in X,\ \mathrm{supp}\,\mathscr{U}(\cdot)x \subset F\}, \tag{1}$$

where F is a closed subset of C. Then $X(F)$ is a closed linear subspace, invariant for S. A very useful and important fact is that $X(F)$ has the following intrinsic characterization:

$$X(F) = \{x \in X,\ \text{the resolvent equation } (z - T)f(z) \equiv x, \text{ has an analytic solution } f \text{ outside } F\}. \tag{2}$$

It might be worth mentioning that just by using such a characterization, N. Dunford was able to prove the uniqueness of the spectral measure. The function f appearing in (2) is unique for any given x and $f(z) = R(z; T)x$ if $z \in F^c \cap r(T)$; this means that f is, in some sense, a local resolvent of T in x. By using the characterization (2) of the space $X(F)$ one can see that $\mathrm{sp}\big(S, X(F)\big) \subset F$ and $X(F)$ contains any subspace Y invariant for S, whose spectrum is contained in F; $X(F)$ is called the *maximal spectral space corresponding to F*.

Now take an arbitrary finite open covering $\{G_j\}_1^n$ of $\mathrm{sp}(S)$. Then there exists a corresponding partition of unity $\{\varphi_j\}_1^n$ subordinated to this covering; that means:

$$\sum_{j=1}^{n} \varphi_j = 1, \quad \mathrm{supp}\,\varphi_j \subset G_j, \quad j = 1, \ldots, n.$$

Consequently, we get the following decomposition of the space X:

$$X = \sum_{j=1}^{n} X(F_j), \quad F_j \subset G_j, \quad j = 1, \ldots, n. \tag{3}$$

It is known that, unlike a Dunford scalar operator whose spectral measure is unique, a scalar generalized operator may have several spectral distributions and it is even possible that no one of them has values in the bicommutant of the operator ([1]). Nevertheless, on account of (2), the spaces $X(F)$ are the same for all possible functional calculi associated to the operator. If S is a scalar Dunford operator then $X(F)$ is the range of the projection $E(F)$. In the more general case we cannot say that $X(F)$ is associated to a projection but the family $\{X(F), F \subset \boldsymbol{C}\}$ acts as an effective partial substitute of a spectral measure. The generalized decomposition discussed before was formalized by C. Foiaş ([14]; see also [28]) in the following manner.

Definition 1. A bounded linear operator T on a complex Banach space X is called *decomposable* (or *spectral in the sense of Foiaş*) if the following conditions are fulfilled:

(i) for each closed set $F \subset \boldsymbol{C}$, there exists a maximal spectral space $X(F)$ of T, whose spectrum is contained in F,

(ii) for each finite open covering $\{G_j\}_{j=1}^n$ of $\operatorname{sp}(T)$, we have:

$$X = \sum_{j=1}^{n} X(F_j) \quad \text{for some } F_j \subset G_j, \quad j = 1, \ldots, n.$$

The maximality property of $X(F)$ stated in (i) means that $Y \subset X(F)$ whenever Y is invariant for T and $\operatorname{sp}(T, Y) \subset F$. A natural candidate for $X(F)$ would be the following:

$$M(F) = \{x \in X, \text{ the resolvent equation } (z-T)f(z) \equiv x \text{ has an analytic solution } f \text{ outside } F\}.$$

It is easy to see, looking at the definition, that $M(F)$ is a linear subspace, invariant for T and that $Y \subset M(F)$ whenever Y is invariant for T and $\operatorname{sp}(T, Y) \subset F$ (indeed, for each $x \in Y$, the resolvent equation has the obvious solution $f(z) = R(z, T/Y)x$, $z \in r(T, Y) \supset F^c$). Therefore $M(F)$ has the maximality property. The problems that arise in connection with $M(F)$ are the following:

1. Is $M(F)$ closed?

2. If $M(F)$ is closed, is it true that $\operatorname{sp}(T, M(F)) \subset F$?

If $M(F)$ is colsed and $\operatorname{sp}(T, M(F)) \subset F$, then obviously $M(F)$ is the maximal spectral space whose spectrum is contained in F. Generally speaking, the answer to problem 1 is negative ([9]). As regards problem 2, the answer is almost affirmative as is shown by the following basic result of Foiaş: If the resolvent equation has a unique analytic solution on F^c for each $x \in M(F)$ and $M(F)$ is closed (for a certain set F), then $\operatorname{sp}(T, M(F)) \subset F$ (application of the closed graph theorem). Another result of Foiaş is that if T is decomposable then, for each closed set F, the resolvent equation has a unique analytic solution on F^c for any $x \in M(F)$. In other words, a decomposable operator has the single-valued extension property (condition A of

Dunford). Moreover, $M(F)$ is closed for each closed subset F (condition C of Dunford). Therefore the conditions A and C, which are necessary for spectrality in the sense of Dunford, are necessary for spectrality in the sense of Foiaş, too.

If an operator T has the single-valued extension property, then, for each $x \in X$, there exists a smallest closed set $\sigma(x)$ outside which the resolvent equation has an analytic solution. This set $\sigma(x)$ is called the *spectrum* of x with respect to T and its complement $\sigma(x)^c = \varrho(x)$ the *resolvent set* of x with respect to T. Then we may formulate Definition 1 in the following equivalent way:

Definition 1′. T is *decomposable* iff the following conditions are fulfilled:

(i) T has the single-valued extension property,

(ii) for each closed set F the space $M(F)$ is closed,

(iii) for each finite open covering $\{G_j\}_1^n$ of $\operatorname{sp}(T)$ and any element $x \in X$, we have a decomposition:

$$x = \sum_{j=1}^{n} x_j, \quad \sigma(x_j) \subset G_j, \quad j = 1, \ldots, n. \tag{4}$$

The decomposition (4) may be considered as an abstract non-classical partition of unity. It should be observed that there exist spectral decompositions of this type which are not obtained via a functional calculus ([4]). If the property (iii) from Definition 1′ holds for coverings consisting of a fixed number k of open sets, then T is called *k-decomposable* ([30]). If the decomposition (4) holds for any finite open covering $\{G_j\}_1^n$ of $\sigma(x)$, then T is called *decomposable with almost localized spectrum* ([36]). Finally, if any element $x \in X$ has a decomposition:

$$x = \sum_{j=1}^{n} x_j, \quad \sigma(x_j) \subset \sigma(x) \cap G_j, \quad j = 1, \ldots, n, \tag{4′}$$

then T is called *strongly decomposable* ([5]). This last decomposition property is the closest to the classical partition of unity.

The duality for generalized spectral decompositions

Let T be a scalar generalized operator and $\mathscr{U}\colon C^\infty(\boldsymbol{C}) \to L(X)$ be a spectral distribution of T. Then the mapping $\mathscr{U}\colon C^\infty(\boldsymbol{C}) \to L(X')$ defined by:

$$\mathscr{U}'(f) = (\mathscr{U}(f))', \quad f \in C^\infty(\boldsymbol{C}),$$

is a spectral distribution for T' (the dual of T). By using simple arguments of partition of unity and the range-kernel duality, it is easy to see that the duality of spectral spaces that we have got in more particular cases, holds in this case, as well. Namely, we have:

$$X'(F) = X(F^c)^\perp$$

for each closed set $F \subset \boldsymbol{C}$. Indeed, we can write successively:

$$\begin{aligned} X'(F) &= \{u \in X', \operatorname{supp} \mathscr{U}'(\cdot)u \subset F\} = \bigcap \{\mathscr{N}(\mathscr{U}'(f)), \operatorname{supp} f \subset F^c\} \\ &= \bigcap \{\mathscr{R}(\mathscr{U}(f))^{\perp}, \operatorname{supp} f \subset F^c\} \\ &= [\bigvee \{\mathscr{R}(\mathscr{U}(f)), \operatorname{supp} f \subset F^c\}]^{\perp} \\ &= [\bigcup_{G \supset F} \bigcap \{\mathscr{N}(\mathscr{U}(g)), \operatorname{supp} g \subset G\}]^{\perp} = X(F^c)^{\perp}. \end{aligned}$$

Therefore, if T is a scalar generalized operator then T' is a scalar generalized operator, too, and the duality formula for spectral spaces holds.

We are now asking, up to what extent can this statement be generalized to decomposable operators. The basic question to be answered is the following.

QUESTION. Is the dual of a decomposable operator decomposable and does it satisfy the duality formula for spectral spaces?

As we will see in what follows, the answer is positive. Even more, it will be shown that the dual of a 2-decomposable operator is decomposable with almost localized spectrum. The first main result in this direction is the following.

THEOREM 1 ([18]). *If T is a 2-decomposable operator, then*

$$\operatorname{sp}(T', X(F^c)^{\perp}) \subset F$$

for any closed subset F of $\boldsymbol{C}$. Moreover, $X(F^c)^{\perp}$ is the maximal invariant subspace for T' whose spectrum is contained in F.

We send to [18] for the proof. By using Theorem 1 it is relatively easy to get the following duality result.

THEOREM 2 ([18]). *If T is a 2-decomposable operator, then T' is also a 2-decomposable operator.*

Proof. First of all, condition (i) in Definition 1 is satisfied, by Theorem 1. In order to verify condition (ii), take an open covering $\{G_1, G_2\}$ of $\operatorname{sp}(T')$ and let $\{U_1, U_2\}$ be another covering of $\operatorname{sp}(T')$ such that $\overline{U}_j \subset G_j$, $j = 1, 2$. Denote $H_j = U_j^c \cap \operatorname{sp}(T) = U_j^c \cap \operatorname{sp}(T')$, $j = 1, 2$. Then $H_1 \cap H_2 = \emptyset$, so that by applying the Riesz decomposition theorem ([6]), we get the following direct decomposition:

$$X(H_1 \cup H_2) = X(H_1) \oplus X(H_2).$$

Let now $u \in X'$ be an arbitrary element of the dual space. Define $\tilde{u}_1 \colon X(H_1 \cup H_2) \to \boldsymbol{C}$ by $u_1(x) = u(x_1)$ if $x = x_1 + x_2$, $x_1 \in X(H_1)$, $x_2 \in X(H_2)$. We obtain a continuous linear functional $\tilde{u}_1$ on $X(H_1 \cup H_2)$. By the Hahn–Banach theorem, $\tilde{u}_1$ may be extended to the whole space X by preserving the linearity and continuity. Denote by u_1 such an extension and by $u_2 = u - u_1$. From the definitions of u_1 and u_2, it follows that $u_1 \in X(H_2)^{\perp}$ and $u_2 \in X(H_1)^{\perp}$; moreover, $u = u_1 + u_2$. Now we have: $X(H_j) = M(H_j) = M(U_j^c) = X(U_j^c)$ and $X(H_j) \supset X(\overline{U}_j^c)$, $j = 1, 2$. Therefore we have obtained the decomposition:

$$X' = X'(\overline{U}_1) + X'(\overline{U}_2), \quad \overline{U}_j \subset G_j, \quad j = 1, 2.$$

The proof is finished.

COROLLARY 1. *If T is 2-decomposable, then for any finite open covering $\{G_j\}_1^n$ of $\operatorname{sp}(T)$ there exist maximal spectral spaces $X(F_j)$ of T such that:*

$$F_j \subset G_j, \ j = 1, 2 \quad \textit{and} \quad X = \textit{the closure of } \sum_{j=1}^{n} X(F_j).$$

In other words, if T is 2-decomposable then T is weakly decomposable ([9], [28]).

COROLLARY 2. *If T is 2-decomposable, then the adjoint T^* of T is also 2-decomposable.*

The adjoint T^* of T is defined on the space X^* of all continuous conjugate linear functionals defined on X, in the usual way: $(T^*v)(x) = v(Tx)$, $v \in X^*$, $x \in X$. If we denote by $\tilde{X}$ the space having X as basic set, with the same addition and with the new multiplication by scalars defined by $\lambda \cdot x = \bar{\lambda} x$, $\lambda \in \boldsymbol{C}$, $x \in X$, then $(\tilde{X})' = X^*$ and the dual of T considered as an operator on $\tilde{X}$ is T^*.

In particular, the adjoint of a 2-decomposable operator on a Hilbert space is 2-decomposable.

COROLLARY 3. *If T is 2-decomposable, then $X'(F) = X(F^c)^{\perp}$ for any closed subset F of the complex plane.*

COROLLARY 4. *If T is 2-decomposable, then $\operatorname{sp}(T, X/\overline{X(G)}) \subset G^c$ for any open subset G of the complex plane.*

A similar result for $X(F)$ instead of $\overline{X(G)}$ was obtained by C. Apostol: if T is 2-decomposable, then $\operatorname{sp}(T, X/X(F)) \subset \overline{\operatorname{sp}(T) \setminus F}$, for any closed subset F of the complex plane ([5]).

COROLLARY 5. *If T is 2-decomposable, then $X(F) = {}^{\perp}X'(F^c)$ for any closed subset F of the complex plane.*

Indeed, we have:

$$\begin{aligned} X(F)^{\perp} &= [\bigcap \{X(G), F \subset G\}]^{\perp} = \text{w.cl.} \bigcup \{X(G)^{\perp}, F \subset G\} \\ &= \text{w.cl.} \bigcup \{X'(G^c), F \subset G\} = \text{w.cl.}\, X'(F^c). \end{aligned}$$

Consequently, by applying the bipolar theorem, we get $X(F) = {}^{\perp}X'(F^c)$.

Remark. If T is 2-decomposable and X is considered as a subset of X'', then we have:

$$X(F) = X \cap X''(F),$$

for each closed subset F of the complex plane.

COROLLARY 6. *If X is reflexive, then an operator T on X is 2-decomposable iff T' is 2-decomposable.*

By following the same lines of reasoning as in the proof of Theorem 2 we can obtain the following result of F.-H. Vasilescu.

THEOREM 3 ([36]). *If T is a decomposable operator with almost localized spectrum, then T' has the same property.*

Proof ([19]). A simple inductive argument shows that an operator T is decomposable with almost localized spectrum if T is 2-decomposable and

$$X(F) \subset X(G_1)+X(G_2)$$

for any closed set F and each open covering $\{G_1, G_2\}$ of F.

We will prove that the dual operator T' has that property. First of all, by applying Theorem 2, it follows that T' is 2-decomposable and $X'(F) = X(F^c)^\perp$ for any closed set F. In view of the duality between the spectral spaces of T and T', it will be sufficient to show that

$$X(G)^\perp \subset X(F_1)^\perp + X(F_2)^\perp,$$

for any open set G and any closed sets F_1, F_2 such that $F_1 \cap F_2 \subset G$. Take two open sets D_1, D_2 such that $F_j \subset D_j$ and $\bar{D}_1 \cap \bar{D}_2 \subset G$. Since T is decomposable with almost localized spectrum, we have:

$$X(F_1 \cup F_2) \subset X(\bar{D}_1)+X(\bar{D}_2).$$

Consider an element $u \in X(G)^\perp$. Define $\tilde{u}_1\colon X(F_1 \cup F_2) \to \boldsymbol{C}$ by:

$$\tilde{u}_1(x) = u(x_2) \quad \text{if} \quad x = x_1+x_2,\ x_j \in X(\bar{D}_j),\ j = 1, 2.$$

It is easy to see that the definition of $\tilde{u}_1$ is consistent and that $\tilde{u}_1$ is a linear functional. In order to show that $\tilde{u}_1$ is continuous we shall apply the open mapping theorem ([36]). Consider the mapping $S\colon X(\bar{D}_1) \oplus X(\bar{D}_2) \to X$, defined by $S(x_1 \oplus x_2) = x_1+x_2$. It is obvious that S is linear and continuous and that the range of S contains $X(F_1 \cup F_2)$. Consequently, if we restrict S to $S^{-1}(X(F_1 \cup F_2))$, we obtain a continuous linear surjection between two Banach spaces; by the open mapping theorem, this surjection must be open, so that there exists a constant $K > 0$ such that for any element $x \in X(F_1 \cup F_2)$ we have a representation $x = x_1+x_2$, $x_j \in X(\bar{D}_j)$, $j = 1, 2$, $\|x_1\|+\|x_2\| \leqslant K\|x\|$. By using such a representation, it is easy to show that $\tilde{u}_1$ is bounded. By applying the Hahn–Banach theorem, $\tilde{u}_1$ may be extended to a continuous linear functional u_1 on X. If we define $u_2 = u-u_1$, then we get $u = u_1+u_2$, $u_j \in X(F_j)^\perp$, $j = 1, 2$, and the proof is complete.

The arguments used before suggest that it would not be possible to obtain more than 2-decomposability of the dual operator. However, as we shall see in what follows, the result of Theorem 2 can be considerably improved.

THEOREM 4 ([20]). *If T is 2-decomposable then T' is decomposable.*

Proof. By applying the duality of spectral spaces, it is easy to see that T' is decomposable if and only if the mapping

$$\mathscr{S}\colon X(F_1)^\perp \oplus X(F_2)^\perp \oplus \ldots \oplus X(F_k)^\perp \to X',$$

defined by $\mathscr{S}(u_1 \oplus u_2 \oplus \ldots \oplus u_k) = u_1+u_2+\ldots+u_k$, is surjective for any family $\{F_j\}_1^k$ of closed sets such that $\bigcap_{j=1}^k F_j = \emptyset$. It is quite natural to look for an equivalent property regarding the space X. Since $X(F_j)^\perp$ is isometrically isomorphic to the dual of the quotient space $X/X(F_j)$, $j = 1, \ldots, k$, the mapping $\mathscr{S}$ is implemented (by the isomorphisms above) by the dual of the mapping:

$$\mathscr{I}\colon X \to X/X(F_1) \oplus X/X(F_2) \oplus \ldots \oplus X/X(F_k),$$

defined by $\mathscr{I}(x) = (x+X(F_1)) \oplus (x+X(F_2)) \oplus \ldots \oplus (x+X(F_k))$. Consequently, it will be enough to prove that the mapping $\mathscr{I}$ has null kernel and closed range. The condition $\bigcap_{j=1}^k F_j = \emptyset$ implies that $\mathscr{I}$ has null kernel. It is more difficult to prove that $\mathscr{I}$ has closed range. In order to do this we shall proceed by a *reductio ad absurdum*. Suppose there exists a sequence $(x_n^0) \subset X$ such that $d(x_n^0, X(F_j)) \to 0$ for $n \to \infty$, $1 \leqslant j \leqslant k$, and $\|x_n^0\| = 1$ for any natural number n; that implies that there exist sequences $(x_{n,j}^0) \subset X(F_j)$ such that $\|x_n^0 - x_{n,j}^0\| \to 0$ for $n \to \infty$. Since $\|x_n^0\| = 1$, $n \in \boldsymbol{N}$, it follows that $(x_{n,j}^0)$ are bounded sequences. Taking into account that $x_{n,j}^0 \in X(F_j)$ and $\operatorname{sp}(T, X(F_j)) \subset F_j$, we obtain:

$$x_{n,j}^0 = (z-T)f_{n,j}(z), \quad z \in F_j^c,$$

where $f_{n,j}(z) = R(z; T|X(F_j))x_{n,j}^0$, $z \in F_j^c$. It is obvious that the sequences $f_{n,j}(z)$ are uniformly bounded on compact sets. Let us now put this situation in a more adequate framework. Consider the space $l_\infty(X)/c_0(X)$ and denote by $\tilde{x}^0$ the class defined by the sequence (x_n^0) and by $\tilde{f}_j$ the function defined by:

$$\tilde{f}_j(z) = (f_{n,j}(z))+c_0(X), \quad z \in F_j^c.$$

Then we have $\tilde{f}_j(z) = \tilde{f}_h(z)$, $z \in F_j^c \cap F_h^c$, and consequently we can define an analytic function on $\bigcup_{j=1}^k F_j^c = (\bigcap_{j=1}^k F_j)^c = \boldsymbol{C}$ by $\tilde{f}(z) = \tilde{f}_j(z)$ if $z \in F_j^c$. If $\tilde{T}$ is the operator defined by T on $l_\infty(X)/c_0(X)$, then we have $\tilde{x}^0 = (z-\tilde{T})\tilde{f}(z)$, $z \in \boldsymbol{C}$, and consequently, by taking a circumference Γ contained in the resolvent set of $\tilde{T}$ and surrounding the spectrum of $\tilde{T}$, we deduce:

$$\tilde{x}^0 = (2\pi i)^{-1} \int_\Gamma R(z; \tilde{T})\tilde{x}^0 dz = (2\pi i)^{-1} \int_\Gamma \tilde{f}(z)dz = 0.$$

We are thus led to a contradiction, since $\tilde{x}^0$ is the class defined by the sequence (x_n^0) and $\|x_n^0\| = 1$, $n \in \boldsymbol{N}$.

COROLLARY 1. *If T is decomposable then T' is also decomposable.*

COROLLARY 2. *On a reflexive Banach space, any 2-decomposable operator is decomposable.*

By using arguments of topological dimension theory, E. Albrecht and F.-H. Vasilescu have proved in [3] that any 3-decomposable operator is decomposable (on any Banach space). Recently, I have read in Notices of the American Math. Soc. that M. Radjabalipour has proved that any 2-decomposable operator is decomposable ([33]), thus improving both the result in Corollary 2 and the result of Albrecht and Vasilescu. Such a result justifies the analogy between the spectral decomposition in the sense of Foiaş and the partition of unity.

The result in Theorem 4 is yet not the best possible. By using a Cousin type theorem, it turned out possible to prove that the dual of a 2-decomposable operator is a decomposable operator with almost localized spectrum.

THEOREM 5 ([21]). *If T is a 2-decomposable operator, then T' is a decomposable operator with almost localized spectrum.*

Proof. By taking into account the duality between spectral spaces of T and T', we see that the statement of Theorem 5 is equivalent to the following: for any two closed sets F_1, F_2 and any open set G such that $F_1 \cap F_2 \subset G$, we have:

$$X(G)^{\perp} \subset X(F_1)^{\perp} + X(F_2)^{\perp}.$$

This, in its turn, is equivalent to say that for any sequence $(x_n) \subset X$ such that $d(x_n, X(F_j)) \to 0$ for $n \to \infty$, $j = 1, 2$, we have $d(x_n, X(G)) \to 0$ for $n \to \infty$. The first idea was to follow the same way as in the proof of the preceding theorem, by taking $X/\overline{X(G)}$ instead of X. From Corollary 4 of Theorem 2 we know that $\operatorname{sp}(T, X/\overline{X(G)}) \subset G^c$ and from hypothesis it follows that G^c is covered by F_1^c and F_2^c. This argument does not work because we need a decomposition property for the space $X/\overline{X(G)}$ and we do not know such a property. In order to overtake this difficulty, the following theorem of Cousin type can be used:

THEOREM. *Let G_1 and G_2 be two open sets in the complex plane such that $G_1 \cap G_2 \neq \emptyset$ and let $(f_{n,j}) \subset A(G_j, X)$, $j = 1, 2$, be two sequences of analytic functions such that $f_{n,1} - f_{n,2} \to 0$ in $A(G_1 \cap G_2, X)$ for $n \to \infty$. Then there exists a sequence $(f_n) \subset A(G_1 \cup G_2, X)$ such that $f_n - f_{n,j} \to 0$ in $A(G_j, X)$ for $n \to \infty$, $j = 1, 2$.*

The idea of the proof is to apply the open mapping theorem (for Fréchet spaces) for the mapping:

$$\mathscr{D}\colon A(G_1, X) \oplus A(G_2, X) \to A(G_1 \cap G_2, X)$$

defined by $\mathscr{D}(f_1 \oplus f_2) = f_1 - f_2$. This mapping is continuous, linear and, by the vectorial variant of the classical theorem of Cousin, is also surjective ([24]). To go on with the proof of Theorem 5, let $(x_n) \subset X$ be a sequence such that $d(x_n, X(F_j)) \to 0$ for $n \to \infty$, $j = 1, 2$. We must prove that $d(x_n, X(G)) \to 0$ for $n \to \infty$. Let $(x_{n,j}) \subset X(F_j)$ be sequences such that $d(x_n, x_{n,j}) = \|x_n - x_{n,j}\| \to 0$ for $n \to \infty$, $j = 1, 2$. Since $\operatorname{sp}(T, X(F_j)) \subset F_j$, we may write:

$$x_{n,j} = (z - T) R(z; T|X(F_j)) x_{n,j}, \quad z \in F_j^c,\ j = 1, 2,\ n \in N.$$

Let us denote, for simplicity, $f_{n,j}(z) = R(z; T|X(F_j)) x_{n,j}$, $z \in F_j^c$, $j = 1, 2$, $n \in N$. Then we have:

$$f_{n,1}(z) - f_{n,2}(z) = R(z; T|X(F_1 \cup F_2))(x_{n,1} - x_{n,2}) \to 0$$

for $n \to \infty$, uniformly on each compact subset of $F_1^c \cap F_2^c$. Let us now transpose our situation to the quotient space $Y = X/\overline{X(G)}$. Denote by S the operator coinduced by T:

$$S\xi = (\widehat{Tx}), \quad \text{if} \quad x \in \xi,\ \xi \in X/\overline{X(G)}.$$

Since $\|\hat{x}_n - \hat{x}_{n,j}\| \leqslant \|x_n - x_{n,j}\|$, it follows that $\|\hat{x}_n - \hat{x}_{n,j}\| \to 0$ for $n \to \infty$, $j = 1, 2$. By the remark above, $\hat{f}_{n,1}(z) - \hat{f}_{n,2}(z) \to 0$ in $A(F_1^c \cap F_2^c, Y)$ for $n \to \infty$. Consequently, by applying our variant of Cousin theorem, it follows that there exists a sequence $(\varphi_n) \subset A(F_1^c \cup F_2^c, Y)$ such that $\varphi_n - f_{n,j} \to 0$ for $n \to \infty$, uniformly on each compact subset of F_j^c, $j = 1, 2$. By combining this with the fact that $\hat{x}_n - (z - S)\hat{f}_{n,j}(z) = \hat{x}_n - \hat{x}_{n,j} \to 0$, for $n \to \infty$, $j = 1, 2$, we get:

$$\hat{x}_n - (z - S)\varphi_n(z) \to 0$$

for $n \to \infty$, uniformly on compact subsets of $F_1^c \cup F_2^c$. We are thus led to the following situation:

Given a complex Banach space $Y = X/\overline{X(G)}$ and a sequence $(y_n) = (\hat{x}_n) \subset Y$ such that there exists a sequence (φ_n) of analytic functions in a fixed open neighbourhood of the spectrum $\operatorname{sp}(S, Y)$ with the property $y_n - (z - S)\varphi_n(z) \to 0$ for $n \to \infty$, uniformly on compact sets. Then it is easy to deduce from these conditions that $y_n \to 0$ for $n \to \infty$. Indeed, write $y_n - (z - S)\varphi_n(z) = \chi_n(z)$ and let us choose, in the resolvent set of S, a contour of integration Γ "surrounding" $\operatorname{sp}(S, Y)$. Then we may write:

$$\begin{aligned} y_n &= (2\pi i)^{-1} \int_{\Gamma} R(z; S) y_n \, dz = (2\pi i)^{-1} \int_{\Gamma} \varphi_n(z)\, dz + (2\pi i)^{-1} \int_{\Gamma} R(z; S)\chi_n(z)\, dz \\ &= (2\pi i)^{-1} \int_{\Gamma} R(z; S)\chi_n(z)\, dz, \end{aligned}$$

whence $\|y_n\| \leqslant (2\pi)^{-1}(M|\chi_n|_{\Gamma}) l(\Gamma) \to 0$ for $n \to \infty$. The proof of Theorem 5 is finished.

COROLLARY 1. *If T is decomposable with almost localized spectrum, then T' has the same property* ([36]).

COROLLARY 2. *On a reflexive Banach space, any 2-decomposable operator is a decomposable operator with almost localized spectrum.*

A characterization of orthogonally decomposable operators

In a very interesting paper ([25]) J. W. Helton suggested that "it might be interesting to study *orthogonally decomposable operators*, namely orthogonally separated operators which are decomposable in the sense of Foiaş". An operator T is called orthogonally separated if any two invariant subspaces Y_1, Y_2 of T which have disjoint spectra are orthogonal. By using duality arguments it is possible to obtain the following characterization of orthogonally decomposable operators.

THEOREM 6. *An operator T on a complex Hilbert space is orthogonally decomposable if and only if it is a Dunford spectral operator with a normal scalar part.*

It is almost obvious that any Dunford spectral operator whose scalar part is normal is orthogonally decomposable. Indeed, if $T = N + Q$ where N is normal and Q is quasinilpotent commuting with N, then $H_T(F) = H_N(F)$ for any closed set F. Let Y_1, Y_2 be two invariant subspaces for T such that $\operatorname{sp}(T, Y_1) \cap \operatorname{sp}(T, Y_2) = \emptyset$. Then $Y_j \subset H_T(F_j)$, where $F_j = \operatorname{sp}(T, Y_j)$, $j = 1, 2$. Since a normal operator has an orthogonal resolution of the identity, we have $H_N(F_1) \perp H_N(F_2)$ and consequently $Y_1 \perp Y_2$. Therefore it remains to prove that if T is an orthogonally decomposable operator, then $T = N+Q$ where N is normal and Q is quasinilpotent commuting with N. It will be enough to show that T is a Dunford spectral operator. Once this is proved, we have the canonical decomposition $T = S+Q$ where S is a scalar operator and Q is a quasinilpotent commuting with S ([10], [11], [12]). Since T is orthogonally separate, S has an orthogonal resolution of the identity and consequently is a normal operator. In order to prove that T is a Dunford spectral operator we shall use the characterization of Dunford spectral operators on weakly (sequentially) complete Banach spaces ([12]): an operator T on a weakly complete Banach space is a Dunford spectral operator if and only if it satisfies the conditions (A), (B), (C) and (D) of Dunford. Therefore we shall prove that an orthogonally decomposable operator satisfies the conditions (A), (B), (C) and (D). Condition (A) means the single-valued extension property and is satisfied since T is decomposable ([9]). Another direct consequence of the decomposability of T is condition (C), which means that the maximal space $M(F)$ is closed for any closed set F. Condition (B) requires that $\|x\| \leqslant K\|x+y\|$ for any elements x and y having disjoint spectra (K being a positive constant). Since T is orthogonally separate and decomposable, it is easy to see that any two elements whose spectra are disjoint are orthogonal and consequently condition (B) is also satisfied. As regards condition (D), the verification is more difficult. Let us recall in what consists condition (D). Denote by $\mathscr{S}_1(T)$ the family of all subsets σ of C such that the elements of the form $x+y$ where $\sigma(x) \subset \sigma$ and $\sigma(y) \subset \sigma^c$ constitute a dense set in H. Denote by $\mathscr{S}_2(T)$ the family of all subsets σ of C such that for any $x \in H$ and any $\varepsilon > 0$ there exist $x_1, x_1' \in H$ with the properties: $\sigma(x_1) \subset \sigma(x) \cap \sigma$, $\sigma(x_1') \subset \sigma(x) \cap \cap \sigma^c$ and $\|x_1 + x_1' - x\| < \varepsilon$. If σ belongs to $\mathscr{S}_1(T)$, then there exists a projection $E(\sigma)$ such that $E(\sigma)x = x$ if $\sigma(x) \subset \sigma$ and $E(\sigma)x = 0$ if $\sigma(x) \subset \sigma^c$. Denote by $\mathscr{S}(T)$ the family of all subsets σ of C such that there exist two sequences of sets $\mu_n \subset \sigma$, $\nu_n \subset \sigma^c$, $\mu_n, \nu_n \in \mathscr{S}_2(T)$, $n \in N$, with the property:

$$x = \lim_{n\to\infty} [E(\mu_n) + E(\nu_n)]x,$$

for any $x \in H$. The condition (D) requires that any complex number have a neighbourhood of arbitrarily small diameter which belongs to $\mathscr{S}(T)$. We shall prove that $\mathscr{S}(T)$ contains any closed subset of C (if T is orthogonally decomposable). The main steps in proving this are the following.

LEMMA 1. *If T is an orthogonally decomposable operator on a Hilbert space H, then the adjoint T^* of T is orthogonally decomposable and $\sigma_{T^*}(x) = \sigma_T(x)^*$ for any element $x \in H$.*

Proof. By applying Corollary 2 it follows that T^* is 2-decomposable and $H_{T^*}(F) = H_T[(F^c)^*]^\perp$ for any closed subset F of C ($(F^c)^*$ denotes the complex conjugate set of F^c). In order to prove the equality $\sigma_{T^*}(x) = \sigma_T(x)^*$, write $F = \sigma_T(x)$ and denote by T_F the restriction of T to $H_T(F)$. It is easy to see that the function $x^*(z) = R(z^*; T_F)x$, $z \in (F^c)^*$, is analytic on $(F^c)^*$ and $(z-T^*)x^*(z) \equiv x$, $z \in (F^c)^*$. Consequently, $\sigma_{T^*}(x) \subset \sigma_T(x)^*$, for any element $x \in H$. Since T and T^* may be interchanged, we have also the opposite inclusion and so the proof is finished.

LEMMA 2. *If T is orthogonally decomposable, then $H_T(\sigma) \oplus \overline{H_T(\sigma^c)} = H$ for any closed set σ.*

Proof. Taking into account that $H_T(\sigma) \perp \overline{H_T(\sigma^c)}$, it will be enough to prove that $x \perp H_T(\sigma) \oplus \overline{H_T(\sigma^c)}$ implies $x = 0$. By using the reflexivity of H and the duality of spectral spaces, we see that the last statement is equivalent to $H_T(\sigma^c)^\perp \cap H_T(\sigma)^\perp = \{0\}$. This statement, in its turn, is a consequence of the fact that T is orthogonally decomposable.

LEMMA 3. *If T is orthogonally decomposable, then*

$$H_T(\sigma_2) = H_T(\sigma_1 \cap \sigma_2) \oplus \overline{H_T(\sigma_1^c \cap \sigma_2)}$$

for any two closed sets σ_1, σ_2.

Proof. By Lemma 2, we have: $H = H_T(\sigma_1) \oplus \overline{H_T(\sigma_1^c)}$. Denote by P_1 the orthogonal projection of H on $H_T(\sigma_1)$. Since both $H_T(\sigma_1)$ and $\overline{H_T(\sigma_1^c)} = H_T(\sigma_1)^\perp$ are invariant for T, P_1 commutes with T. Consequently, we have $\sigma_T(P_1 x) \subset \sigma_T(x)$ and $\sigma_T\big((I-P_1)x\big) \subset \sigma_T(x)$, whence $H_T(\sigma_2) = H_T(\sigma_1 \cap \sigma_2) \oplus H_T(\sigma_2) \cap \overline{H_T(\sigma_1^c)}$. It is easy to see that $H_T(\sigma_2) \cap \overline{H_T(\sigma_1^c)} = \overline{H_T(\sigma_2 \cap \sigma_1^c)}$ and so Lemma 3 is proved.

In order to close the proof of Theorem 6, it suffices to remark that, by Lemma 3, the family $\mathscr{S}(T)$ contains any closed subset of the complex plane.

The operators with strongly decomposable dual

We have already seen (Theorem 5) that the dual of a 2-decomposable operator is decomposable with almost localized spectrum. So it is quite natural to conjecture the dual might even be strongly decomposable. We have not succeeded in solving this difficult problem but, by trying to reduce it to more transparent equivalent questions, we have obtained several characterizations of the operators with strongly decomposable dual, as well as a new characterization of strongly decomposable operators. In this way, we have found a new class of decomposable operators which is, in some sense, dual to the class of strongly decomposable operators. Moreover, the results of this type reveal some more complete aspects of duality. We shall begin with a characterization of strongly decomposable operators by a spectral condition.

THEOREM 7. *Let T be a 2-decomposable operator. Then the following statements are equivalent*:

(i) *T is strongly decomposable*,

(ii) $\operatorname{sp}(T, X(F)/X(H)) \subset \overline{F \setminus H}$ *for any two closed sets* F, H *such that* $H \subset F$.

The implication (i) $\Rightarrow$ (ii) is essentially a result of C. Apostol ([5], [6]). The implication (ii) $\Rightarrow$ (i) may be proved by using the Riesz decomposition theorem in a way similar to that used in [32] and [26].

By applying Theorem 7 and the duality of spectral spaces we get the following characterization of the operators having strongly decomposable dual, by a spectral condition.

THEOREM 8. *Let T be a 2-decomposable operator. Then the following statements are equivalent:*

(i) *T' is strongly decomposable,*

(ii) $\operatorname{sp}(T, \overline{X(V)}/\overline{X(U)}) \subset \overline{V \setminus U}$ *for any two open sets* U, V *such that* $U \subset V$.

Before stating the next result let us reformulate a basic result for strongly decomposable operators ([5]).

THEOREM 9. *A 2-decomposable operator T is strongly decomposable if and only if T is decomposable on any spectral space $X(F)$. If this condition holds, then T is decomposable on any quotient space $X/X(F)$.*

The following result is a dual variant of Theorem 9.

THEOREM 9′. *The dual of a 2-decomposable operator is strongly decomposable if and only if T is decomposable on any quotient space $X/\overline{X(G)}$. If this condition holds, then T is decomposable on any space $\overline{X(G)}$.*

Another characterization of the operators with strongly decomposable dual may be given in the terms of geometry of spectral spaces.

THEOREM 10′. *Let T be a 2-decomposable operator. The following statements are equivalent:*

(i) *T' is strongly decomposable,*

(ii) $\overline{X(G_1 \cup G_2)} = \overline{X(G_1)} + \overline{X(G_2)}$ *for any two open sets* G_1, G_2 *with the property* $\overline{G_1 \setminus G_2} \cap \overline{G_2 \setminus G_1} = \emptyset$,

(iii) *The subspaces* $\overline{X(G_1)}$ *and* $\overline{X(G_2)}$ *are gaping for any two open sets* G_1, G_2 *with the property* $\overline{G_1 \setminus G_2} \cap \overline{G_2 \setminus G_1} = \emptyset$.

Theorem 10′ is a result dual to the following new characterization of strongly decomposable operators.

THEOREM 10. *Let T be a 2-decomposable operator. The following statements are equivalent:*

(i) *T is strongly decomposable,*

(ii) $X(F_1 \cup F_2) = X(F_1) + X(F_2)$ *for any two closed sets* F_1, F_2 *with the property* $\overline{F_1 \setminus F_2} \cap \overline{F_2 \setminus F_1} = \emptyset$,

(iii) $X(F_1)$ *and* $X(F_2)$ *are gaping for any two closed sets* F_1, F_2 *such that* $\overline{F_1 \setminus F_2} \cap \overline{F_2 \setminus F_1} = \emptyset$.

The preceding results point out the importance of the spectral spaces $\overline{X(G)}$, as well as their dual position with respect to spectral spaces $X(F)$. A comparison with the Dunford spectral operators would be useful. In that case we have $X(F) = E(F)X$ and $\overline{X(G)} = E(G)X$ where E denotes the spectral measure of T. Therefore both $X(F)$ and $X(G)$ are complemented in X and $X/X(F) \cong \overline{X(F^c)}$, $X/\overline{X(G)} \cong X(G^c)$. In the more general case of decomposable operators the spaces $X(F)$ and $\overline{X(G)}$ are not necessarily complemented so that we must consider four different types of spectral spaces $X(F)$, $X/X(F)$, $\overline{X(G)}$ and $X/\overline{X(G)}$. It would be interesting to study in great detail the operators which are strongly decomposable and have strongly decomposable dual.

Duality for general operators

A very deep result due to E. Bishop ([8]) shows that the essence of the duality relations between spectral spaces remains valid for general operators on reflexive Banach spaces. Furthermore, in a more recent paper ([29]), V. Lomonosov, Ju. Liubiči and V. Matzaev extend Bishop's results to nonreflexive Banach spaces. In order to formulate his duality theorem, Bishop introduced two types of spectral manifolds associated to a closed set F of the complex plane.

The strong spectral manifold $M(F, T)$ is the closure of the set of all vectors x in X which have the property that the resolvent equation $(z-T)f(z) \equiv x$ has an analytic solution f outside F.

The weak spectral manifold $N(F, T)$ is the set of all vectors x in X with the property that for any $\varepsilon > 0$ there exists an X-valued analytic function f_ε defined outside F such that $\|(z-T)f_\varepsilon(z)-x\| < \varepsilon, z \in F^c$.

THEOREM B_1 ([8]). *For any (continuous linear) operator T on a reflexive Banach space X, the following duality relations hold:*

$$M(F_2, T') \subset N(F_1, T)^\perp, \quad N(F_2, T') \subset M(F_1, T)^\perp$$

for any two disjoint closed sets F_1, F_2 and

$$M(\overline{G}_1, T)^\perp \subset N(\overline{G}_2, T'), \quad N(\overline{G}_1, T)^\perp \subset M(\overline{G}_2, T')$$

for any two open sets G_1, G_2 which cover the complex plane.

A very interesting consequence of Theorem B_1 is a general decomposition theorem. By following [8], we say that an operator T satisfies condition β if, given any open set G and any sequence (f_n) of analytic functions on G such that $(z-T)f_n(z) \to 0$ for $n \to \infty$ uniformly on G, it follows that (f_n) is uniformly bounded on compact subsets of G. There exist several equivalent reformulations of the condition β. One of them is the following. T satisfies the condition β if, given any open set G and any sequence (f_n) of analytic functions on G such that $(z-T)f_n(z) \to 0$ uniformly on compact sets, it follows that $f_n(z) \to 0$ uniformly on compact sets.

THEOREM B_2 ([8]). *Let X be a reflexive Banach space and T be an operator on X. If both T and T' satisfy the condition β, then to any open covering $\{G_j\}_1^n$ of the complex plane corresponds a family $\{M_j\}_1^n$ of invariant subspaces of T such that:*

$$\operatorname{sp}(T, M_j) \subset G_j,\ j = 1, \dots, n \quad \text{and} \quad X = \text{c.l.m.}\{M_j, j = 1, \dots, n\}.$$

By using some ideas similar to those occurring in the proof of Theorem 2 and Theorem B_1, it is possible to improve the result of Theorem B_2.

THEOREM 11. *Let X be a reflexive Banach space and T be an operator on X. If both T and T' satisfy condition β, then T is decomposable.*

Actually, the statement converse to that of Theorem 11 is also true. Indeed, by a result proved in [17] and [16], any decomposable operator satisfies condition β. Consequently, on account of Corollary 1, Theorem 4, it follows that T' also satisfies condition β. In this way we get the following interesting characterization of decomposable operators on reflexive Banach spaces.

THEOREM 12. *An operator T on a reflexive Banach space is decomposable if and only if both T and T' satisfy the condition β.*

It is very probable that a similar characterization holds on nonreflexive Banach spaces.

It can be expected that all operators satisfying a reasonable growth condition of the resolvent satisfy the condition β, too, and consequently, by Theorem 11, are decomposable.

I should like to close my lecture with a remark concerning Theorem B_1. The key to the proof of Theorem B_1 was a duality theorem for spaces of analytic functions. Recently I discovered in a paper of A. Grothendieck ([23]) a definitive result concerning the duality of spaces of analytic functions due to J. Silva, G. Köthe ([35], [27]) and A. Grothendieck. Although Grothendieck's paper was published much before Bishop's paper, the duality result contained there was probably not known neither by Bishop nor by the authors of the more recent paper [29].

It seems to me that the result of Silva–Köthe–Grothendieck will be the key to obtaining a definitive duality result for general operators on Banach spaces.

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