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GENERALIZED WEIGHTS FOR OPERATOR LIE ALGEBRAS

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The purpose of this paper is to generalize a classical theorem of Lie concerning the existence of weights for solvable Lie algebras of operators on a finite-dimensional space. Such a generalization has been suggested to the author by [1] and [5].

Let us firstly recall the classical Lie's theorem. Let V be a linear space and let $\mathscr{L} \subset \mathscr{L}(V)$ be a Lie algebra of operators on V. We mean thereby that $\mathscr{L}$ is a linear subspace of $\mathscr{L}(V)$, which is closed with respect to the cross-product defined by: $[T, S] := TS-ST$, $T, S \in \mathscr{L}(V)$; i.e. we have $[T, S] \in \mathscr{L}$ whenever $T, S \in \mathscr{L}$. Given a Lie algebra $\mathscr{L}$ of operators on V, consider the following decreasing sequence of commutator algebras: $\mathscr{L}_0 = \mathscr{L}, \mathscr{L}_1 = [\mathscr{L}_0, \mathscr{L}_0], \ldots, \mathscr{L}_n = [\mathscr{L}_{n-1}, \mathscr{L}_{n-1}], \ldots$

DEFINITION 1. A Lie algebra $\mathscr{L} \subset \mathscr{L}(V)$ is called *solvable* if there exists a natural number n such that $\mathscr{L}_n = \{0\}$.

Remark. The condition in Definition 1 is a natural generalization of the commutativity property of $\mathscr{L}$. In fact, $\mathscr{L}$ is commutative if and only if $\mathscr{L}_1 = \{0\}$.

LIE'S THEOREM ([2]). *If V is a complex finite-dimensional space ($V \neq \{0\}$) and $\mathscr{L} \subset \mathscr{L}(V)$ is a solvable Lie algebra of operators on V, then there exists a vector $v \neq 0$, $v \in V$, which is a joint eigenvector for all operators of $\mathscr{L}$.*

In other words, there exists $v \neq 0$, $v \in V$, such that $Tv = \lambda_T v$, for any $T \in \mathscr{L}$.

In such a setting, the corresponding eigenvalue λ_T is uniquely defined by T (i.e. we cannot have $Tv = \lambda v$, $Tv = \mu v$ unless $\lambda = \mu$) and the scalar function $T \to \varphi(T) = \lambda_T$ is called a *weight* for $\mathscr{L}$.

It is well known that an operator on an infinite-dimensional space may have no eigenvector and consequently Lie's theorem cannot be true, in its classical form, on infinite-dimensional spaces.

However, it is also well known that any bounded linear operator on a complex Banach space, $T \in \mathscr{L}(X)$, has at least an approximate eigenvalue, i.e. a complex number λ such that $\lim_{n\to\infty} (Tx_n - \lambda x_n) = 0$ for a certain sequence (x_n) of unit vectors, $||x_n|| = 1$, $\forall n \in N$. In fact, any point of the topological boundary of the spectrum of T is an approximate eigenvalue. Gurariĭ and Liubiči deal in their paper [1] with the group-theoretical variant of Lie's theorem.

DEFINITION 2 ([1]). Let X be a complex Banach space and let $\mathscr{G} \subset \mathscr{G}\mathscr{L}(X)$ be a separable group of operators on X. A scalar function $T \to \chi(T)$ defined on $\mathscr{G}$ is called a *weight for* $\mathscr{G}$ if there exists a sequence $(x_n) \subset X$, $||x_n|| = 1$, $\forall n \in N$, such that

$$\lim_{n\to\infty}(Tx_n - \chi(T)x_n) = 0 \quad \text{for any } T \in \mathscr{G}.$$

THEOREM OF GURARIĬ AND LIUBIČI. *If $\mathscr{G}$ is a solvable, separable, locally compact and connected group of operators on X, then $\mathscr{G}$ has a weight.*

In their proof Gurariĭ and Liubiči use essentially the assumption of local compactness of $\mathscr{G}$, which is a very strong property.

Definition 2 is natural only for separable groups. For a more general setting it has to be modified as follows.

DEFINITION 3. Let $\mathscr{L} \subset \mathscr{L}(X)$ be a Lie algebra of operators on X. A scalar function $T \to \varphi(T)$ defined on $\mathscr{L}$ is called a *generalized weight for* $\mathscr{L}$ if there exists a net $(x_\alpha) \subset X$ of unit vectors, $||x_\alpha|| = 1$, $\forall \alpha \in D$, such that $\lim_\alpha (Tx_\alpha - \varphi(T)x_\alpha) = 0$, for any $T \in \mathscr{L}$.

The main result of this paper is the following.

THEOREM 1. *Any solvable Lie algebra of operators on a complex Banach space $X \neq \{0\}$ has a generalized weight.*

As a corollary of this theorem, we obtain that any solvable Lie group of operators on a complex Banach space has a generalized weight, thus improving the result of Gurarii and Liubiči.

The main step in the proof of Theorem 1 is an extension theorem for generalized weights, which is interesting also by itself.

THEOREM 2. *Let $\mathscr{L} \subset \mathscr{L}(X)$ be a Lie algebra of operators on X. Then any generalized weight for $\mathscr{L}_1 = [\mathscr{L}, \mathscr{L}]$ can be extended to a generalized weight for $\mathscr{L}$.*

As soon as this is settled, Theorem 1 may be proved by using the following theorem due to W. Żelazko:

ŻELAZKO'S THEOREM ([5]). *Let A be a commutative complex Banach algebra with unity. Then any maximal ideal of A that belongs to the Šilov boundary $\Gamma(A)$ of A consists of joint topological divisors of zero. In other words, if $M \in \Gamma(A)$ then there exists a net $(z_\alpha) \subset A$, $||z_\alpha|| = 1$, $\forall \alpha$, such that $\lim_\alpha z_\alpha x = 0$ for any $x \in M$.*

With Żelazko's theorem and Theorem 2 at hand, we can derive Theorem 1 as follows. First, consider the algebra $\mathscr{L}_{n-1}$ which is a commutative Lie algebra of operators (since $\mathscr{L}_n = [\mathscr{L}_{n-1}, \mathscr{L}_{n-1}] = \{0\}$). By using Żelazko's theorem, it is easy to prove that $\mathscr{L}_{n-1}$ has a generalized weight. Then, by applying Theorem 2, we can extend this weight step by step, until we get a generalized weight for the whole algebra $\mathscr{L}$. Therefore, it will be sufficient to prove that any commutative Lie algebra of operators has a generalized weight.

PROPOSITION 1. *Any commutative Lie algebra $\mathscr{L}$ of operators on X has a generalized weight.*

Proof. Consider the smallest commutative Banach algebra $\mathscr{A}$ of operators which contains $\mathscr{L}$ and the identity operator I.

By applying Żelazko's theorem to $\mathscr{A}$, we see that any maximal ideal $\mathscr{M}$ that belongs to the Šilov boundary $\Gamma(\mathscr{A})$ of $\mathscr{A}$ consists of joint topological divisors of zero. Therefore, there exists a net $\{T_\alpha\} \subset \mathscr{A}$, $||T_\alpha|| = 1$ for any index α, such that $\lim_\alpha T_\alpha T = 0$ for any $T \in \mathscr{M}$. We will prove that the character φ of $\mathscr{A}$ corresponding to $\mathscr{M}$ (more precisely, the restriction of this character to $\mathscr{L}$) is a generalized weight for $\mathscr{L}$. Indeed, let us remark that $T - \varphi(T)I \in \ker\varphi = \mathscr{M}$ for any $T \in \mathscr{A}$ and consequently, $\lim_\alpha [T - \varphi(T)I]T_\alpha = 0$ for any $T \in \mathscr{A}$. (The convergence is taken in the norm operator topology.) Taking into account that $||T_\alpha|| = 1$ for each index α, it is easy to construct a sequence $(y_\alpha) \subset X$, $||y_\alpha|| = 1$, $\forall \alpha$, such that $\lim_\alpha [T - \varphi(T)I]y_\alpha = 0$ for any $T \in \mathscr{A}$ and, in particular, for any $T \in \mathscr{L}$.

($||T_\alpha|| = 1 \Rightarrow \exists x_\alpha \in X$, $||x_\alpha|| = 1$, $||T_\alpha x_\alpha|| \geqslant 1/2$; write $z_\alpha := T_\alpha x_\alpha \Rightarrow ||z_\alpha|| \geqslant 1/2$,

$$y_\alpha := \frac{1}{||z_\alpha||} z_\alpha, \ ||y_\alpha|| = 1, \ ||[T - \varphi(T)]y_\alpha|| = \frac{1}{||z_\alpha||} ||[T - \varphi(T)]\, T_\alpha x_\alpha||$$
$$\leqslant 2||[T - \varphi(T)]\, T_\alpha|| \to 0 \text{ with respect to } \alpha.)$$

Thus the proof of Proposition 1 is concluded.

The proof of Theorem 2 is more complicated and requires some auxiliary facts.

First of all, we need the following characterization of generalized weights, which is similar to the characterization given by Żelazko for sets consisting of joint topological divisors of zero in Banach algebras (cf. [5]).

LEMMA 1. *Let $\mathscr{L} \subset \mathscr{L}(X)$ be a Lie algebra of operators on a complex Banach space X. A scalar function $\varphi\colon \mathscr{L} \to \mathbf{C}$ is a generalized weight for $\mathscr{L}$ if and only if for any finite subset $\mathscr{F} \subset \mathscr{L}$, we have:*

$$\inf_{||x||=1} \{\max_{S\in\mathscr{F}} ||Sx - \varphi(S)x||\} = 0. \tag{1}$$

Proof. If φ is a generalized weight for $\mathscr{L}$ then there exists a net (x_α), $||x_\alpha|| = 1$, such that $\lim_\alpha [Sx_\alpha - \varphi(S)x_\alpha] = 0$.

Consequently, for each $S \in \mathscr{L}$ and any $\varepsilon > 0$, there exists an index $\alpha(S)$ such that $||Sx_\alpha - \varphi(S)x_\alpha|| < \varepsilon$ whenever $\alpha > \alpha(S)$. If we now take a finite subset $\mathscr{F} \subset \mathscr{L}$, there exists an index α_0 such that $\alpha_0 > \alpha(S)$ for any $S \in \mathscr{F}$ (since the set of indices is directed). Hence, for $\alpha > \alpha_0$, we get $||Sx_\alpha - \varphi(S)x_\alpha|| < \varepsilon$, for any $S \in \mathscr{F}$. Therefore:

$$\inf_{||x||=1} \{\max_{S\in\mathscr{F}} ||Sx - \varphi(S)x||\} = 0.$$

Conversely, if condition (1) is satisfied, then for any finite subset $\mathcal{F} \subset \mathcal{L}$ and each natural number n, we can find an element x, depending of $\mathcal{F}$ and n, with $\|x\| = 1$ and $\max\limits_{S\in\mathcal{F}} \|Sx-\varphi(S)x\| < 1/n$. By applying the choice axiom, we obtain a net (x_α) of unit vectors, having as index set the set of all pairs $\alpha = (\mathcal{F}, n)$ where $\mathcal{F}$ is a finite subset of $\mathcal{L}$ and n is a natural number, ordered increasingly with respect to both $\mathcal{F}$ and n. Furthermore, for each $x_\alpha = x_{\mathcal{F},n}$, we have:

$$\|x_\alpha\| = 1, \quad \max_{S\in\mathcal{F}} \|Sx_\alpha-\varphi(S)x_\alpha\| < 1/n.$$

Now, it is easy to see that $\lim\limits_{\alpha} [Sx_\alpha-\varphi(S)x_\alpha] = 0$, for every $S \in \mathcal{L}$.

In order to verify this last statement, let us consider an arbitrary positive number $\varepsilon > 0$ and an arbitrary operator $S \in \mathcal{L}$. Pick a natural number n_0 such that $1/n_0 < \varepsilon$ and consider the index $\alpha_0 = (\{S\}, n_0)$.

Then, for each $\alpha = (\mathcal{F}, n) > \alpha_0 = (\{S\}, n_0)$, we have $S \in \mathcal{F}$ and $n > n_0$, so that, by the definition of $x_\alpha = x_{\mathcal{F},n}$, we get:

$$\max_{S'\in\mathcal{F}} \|S'x_\alpha-\varphi(S')x_\alpha\| < 1/n < 1/n_0 < \varepsilon.$$

In particular, we obtain

$$\|Sx_\alpha-\varphi(S)x_\alpha\| < \varepsilon$$

for any $\alpha > \alpha_0$.

Consequently, we have proved that $\lim\limits_{\alpha} [Sx_\alpha-\varphi(S)x_\alpha] = 0$ for any operator $S \in \mathcal{L}$, and the proof of Lemma 1 is finished.

The proof of Theorem 2 will be reduced to the particular case when we have an ordinary weight on $\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}]$.

THEOREM 2′. *Any ordinary weight for $\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}]$ can be extended to a generalized weight for $\mathcal{L}$.*

Proof. Let φ_0 be an ordinary weight for $\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}]$. That means, there exists an element $x_0 \in X$, $x_0 \neq 0$, such that:

$$Tx_0 = \varphi_0(T)x_0 \quad \text{for any } T \in \mathcal{L}_1.$$

Consider the following subspace X_0 of X:

$$X_0 = \{x \in X;\ Tx = \varphi_0(T)x \text{ for any } T \in \mathcal{L}_1\}.$$

Since $x_0 \in X_0$, $x_0 \neq 0$, it follows that $X_0 \neq \{0\}$ and we will prove that X_0 is invariant for all operators $S \in \mathcal{L}$.

Let S be an arbitrary operator from $\mathcal{L}$ and consider the mapping:

$$\operatorname{Ad} S\colon \mathcal{L}(X) \to \mathcal{L}(X)$$

defined by:

$$(\operatorname{Ad} S)\, T = [S, T], \quad T \in \mathcal{L}(X).$$

We have the following well known formula:

$$\exp(-zS)\, T\exp(zS) = \exp[z\operatorname{Ad} S]\, T, \quad \forall z \in \boldsymbol{C}.$$

If $S \in \mathcal{L}$ and $T \in \mathcal{L}_1$, then $(\operatorname{Ad} S)T \in \mathcal{L}_1$ and $\exp(z\operatorname{Ad} S)T \in \bar{\mathcal{L}}_1$ (the closure of $\mathcal{L}_1$ in the norm operator topology). It is easy to see that $T \to \varphi_0(T)$ is a linear continuous scalar function on $\mathcal{L}_1$ and, consequently, it has a unique continuous extension $\tilde{\varphi}_0$ to $\bar{\mathcal{L}}_1$, which is an ordinary weight for $\bar{\mathcal{L}}_1$ with the same eigenvector x_0. Therefore, if $x \in X_0$ then:

$$\exp(-zS)\, T\exp(zS)x = [\exp(z\operatorname{Ad} S)\, T]x = \tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T])x, \quad \forall z \in \boldsymbol{C},$$

whence:

$$T\exp(zS)x = \tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T])\exp(zS)x, \quad \forall z \in \boldsymbol{C}. \tag{2}$$

In particular, the equality (2) holds for $x = x_0$; hence the number $\tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T])$ is an eigenvalue of T for each $z \in \boldsymbol{C}$, and consequently it belongs to the spectrum of T for each $z \in \boldsymbol{C}$. Since the spectrum of T is a bounded set, the function $z \to \tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T])$ is a bounded analytic function on the whole complex plane. By Liouville's theorem, this function must be constant, namely:

$$\tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T]) = \tilde{\varphi}_0([\exp(z\operatorname{Ad} S)\, T])|_{z=0} = \varphi_0(T), \quad \forall z \in \boldsymbol{C}.$$

Consequently, we get from (2):

$$T\exp(zS)x = \varphi_0(T)\exp(zS)x, \quad \forall z \in \boldsymbol{C},\ \forall x \in X_0. \tag{2′}$$

By derivating (2′) with respect to z and putting $z = 0$, we get:

$$TSx = \varphi_0(T)Sx, \quad \forall x \in X_0. \tag{3}$$

The relation (3) holds for each $T \in \mathcal{L}_1$, $S \in \mathcal{L}$ and $x \in X_0$. Therefore we have proved in fact that the subspace X_0 is invariant for all operators $S \in \mathcal{L}$. Let us now restrict all operators $S \in \mathcal{L}$ to the subspace X_0. Note that X_0 is a closed linear subspace of X, $X_0 \neq \{0\}$.

We will prove that the algebra of all restrictions $S|X_0$, $S \in \mathcal{L}$ is commutative. Indeed, for any $x \in X_0$ we have:

$$[S_1|X_0, S_2|X_0]x = \varphi_0([S_1, S_2])x, \quad S_1, S_2 \in \mathcal{L}.$$

Therefore, $[S_1|X_0, S_2|X_0]$ is a scalar multiple of the identity operator on X_0. By the theorem of Wintner–Wielandt (or Kleinecke–Širokov), we must have: $\varphi_0([S_1, S_2]) = 0$, and consequently $[S_1|X_0, S_2|X_0] = 0$ for every pair $S_1, S_2 \in \mathcal{L}$.

Thus, the algebra of all restrictions $S|X_0$, $S \in \mathcal{L}$, is a commutative Lie algebra of operators on the (Banach) space $X_0 \neq \{0\}$, and, by Lemma 1, it has a generalized weight. Therefore, there exists a scalar function $S \to \varphi(S)$ and a net $(x_\alpha) \subset X_0$, $\|x_\alpha\| = 1$, such that $\lim\limits_{\alpha}[Sx_\alpha-\varphi(S)x_\alpha] = 0$ for each $S \in \mathcal{L}$.

If $T \in \mathcal{L}_1$, then $Tx_\alpha = \varphi_0(T)x_\alpha$ for each α and consequently $\varphi_0(T) = \varphi(T)$. The proof of Theorem 2′ is concluded.

It remains only to prove Theorem 2.

The proof of Theorem 2. Let φ_0 be a generalized weight for the algebra $\mathcal{L}_1 = [\mathcal{L}, \mathcal{L}]$. That means, there exists a net $(x_\alpha) \subset X$, $\|x_\alpha\| = 1$, such that $\lim\limits_{\alpha}[Tx_\alpha - \varphi_0(T)x_\alpha] = 0$ for any $T \in \mathcal{L}_1$. Let D be the set of indices of (x_α) and consider the quotient space $\tilde{X} = l_\infty(D, X)/c_0(D, X)$ of the space of all bounded functions from D to X, by its subspace consisting of all bounded functions with the limit equal to zero. On this new space the weight induces an ordinary weight.

To any operator $S \in \mathscr{L}$ there corresponds a linear bounded operator $\tilde{S}$ on the space $\tilde{X}$: $\tilde{S}[(y_\alpha)+c_0(D, X)] = (Sy_\alpha)+c_0(D, X)$. It is easy to see that the set $\tilde{\mathscr{L}} = [\tilde{S}, S \in \mathscr{L}\}$ is a Lie algebra of operators on $\tilde{X}$ and that the mapping

$$S \to \tilde{S}$$

from $\mathscr{L}$ to $\tilde{\mathscr{L}}$ is a homomorphism of Lie algebras. Since $||x_\alpha|| = 1$ for any α, the net (x_α) defines an element ξ of $\tilde{X}$ different from zero, $\xi = (x_\alpha)+c_0(D, X)$. Taking into account that $\lim_\alpha [Tx_\alpha - \varphi_0(T)x_\alpha] = 0$, for any $T \in \mathscr{L}_1$, we get $\tilde{T}\xi = \varphi_0(T)\xi$, for any $T \in \mathscr{L}_1$.

Consequently, the scalar function $\tilde{T} \to \varphi_0(T)$, $\tilde{T} \in \tilde{\mathscr{L}}_1 = [\tilde{\mathscr{L}}, \tilde{\mathscr{L}}]$, becomes an ordinary weight for $\tilde{\mathscr{L}}_1 = [\tilde{\mathscr{L}}, \tilde{\mathscr{L}}]$ on the space $\tilde{X}$. By applying Theorem 2′, we infer that this weight can be extended to a generalized weight $\tilde{\varphi}$ for $\tilde{\mathscr{L}}$. It remains to prove that the scalar function $\varphi(S) = \tilde{\varphi}(\tilde{S})$, $S \in \mathscr{L}$, is a generalized weight for $\mathscr{L}$ (it is obvious that φ extends φ_0). To this purpose we use the characterization given in Lemma 1. We know that for each finite subset $\{S_1, \ldots, S_k\} \subset \mathscr{L}$ and any $\varepsilon > 0$, there exists an element $\eta \in \tilde{X}$, $||\eta|| = 1$, such that $||\tilde{S}_j\eta - \tilde{\varphi}(\tilde{S}_j)\eta|| < \varepsilon/4$, $j = 1, \ldots, k$. If $(y_\alpha) \in \eta$, then, by the definition of the norm on a quotient space, it follows that there exists a bounded net $(z_\alpha^j) \in c_0(D, X)$ such that:

$$\sup_\alpha ||S_j y_\alpha - \tilde{\varphi}(\tilde{S}_j)y_\alpha - z_\alpha^j|| < \varepsilon/4, \quad j = 1, \ldots, k.$$

Now, since $(z_\alpha^j) \in c_0(D, X)$, it is possible to find an index α_0 such that $||z_\alpha^j|| < \varepsilon/4$ for each $\alpha > \alpha_0$ and any $j = 1, \ldots, k$. Consequently, we get

$$||S_j y_\alpha - \tilde{\varphi}(\tilde{S}_j)y_\alpha|| < \varepsilon/2 \quad \text{for any } \alpha > \alpha_0,\ j = 1, \ldots, k.$$

Since $(y_\alpha) \in \eta$ and $||\eta|| = 1$ in the quotient space $l_\infty(D, X)/c_0(D, X)$, it follows that $\sup\{||y_\alpha||, \alpha > \alpha_0\} \geqslant 1$ (if $\{\sup||y_\alpha||, \alpha > \alpha_0\} < 1$, then the net: $y'_\alpha = 0$ for $\alpha > \alpha_0$ and $y'_\alpha = y_\alpha$ for $\alpha > \alpha_0$, belongs to $c_0(D, X)$ and

$$||(y_\alpha) - (y'_\alpha)|| = l_\infty(D, X) = \sup\{||y_\alpha||, \alpha > \alpha_0\} < 1,$$

whence $||\eta||_{\tilde{X}} < 1$, a contradiction).

Therefore, $\sup\{||y_\alpha||, \alpha > \alpha_0\} \geqslant 1$ and $||S_j y_\alpha - \tilde{\varphi}(\tilde{S}_j)y_\alpha|| < \varepsilon/2$ for any $\alpha > \alpha_0$, $j = 1, \ldots, k$.

Take an index $\beta > \alpha_0$ such that $||y_\beta|| \geqslant 1/2$; then

$$||S_j y_\beta - \tilde{\varphi}(\tilde{S}_j)y_\beta|| < \varepsilon/2, \quad j = 1, \ldots, k.$$

Put $x_0 := \frac{1}{||y_\beta||} y_\beta$; then $||x_0|| = 1$ and

$$||S_j x_0 - \tilde{\varphi}(\tilde{S}_j)x_0|| < 2||S_j y_\beta - \tilde{\varphi}(\tilde{S}_j)y_\beta|| < \varepsilon.$$

The proof of Theorem 2 is finished.

Other generalizations of Lie's theorem have been obtained by M. Şabac ([3], [4]).

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