SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

FIXED POINTS OF SEMIGROUPS OF POSITIVE OPERATORS IN KB-SPACES

R. KÜHNE

Sektion Mathematik der Technischen Universität, Dresden, DDR

1. Introduction

Let $[\Omega, \mathfrak{B}, m]$ be a measure space, and let Γ be a semigroup of non-singular measurable transformations defined on $[\Omega, \mathfrak{B}, m]$. The classical invariant measure problem can be formulated as follows: find conditions for the existence of a finite invariant measure μ (i.e. $\mu(\tau^{-1}B) = \mu(B)$, $\tau \in \Gamma$, $B \in \mathfrak{B}$) with the same zero sets as m. With the semigroup Γ a semigroup G of linear operators T_τ:

$$T_\tau f := \frac{d}{dm}\Big[\int_{\tau^{-1}(\cdot)} f\,dm\Big] \quad (\tau \in \Gamma) \quad \text{in} \quad L^1(\Omega, \mathfrak{B}, m)$$

can be associated. Hence the more general problem arises of finding conditions for the existence of an m-almost everywhere positive fixed point of a bounded semigroup G of positive linear operators in $L^1(\Omega, \mathfrak{B}, m)$. This problem was discussed by many authors under various special conditions ([1], [2], [4], [6], [7], [8], [9], [13], [14], [16], [17], [18], and others). As a generalization P. C. Shields [16] (see also [10]) treated the special case of a cyclic semigroup as a fixed point problem in a special Banach lattice, a so-called KB-space. In the present paper we intend to investigate corresponding questions for more general semigroups and resolvents in KB-spaces. The results are given without proofs, some proofs and similar results are published in [11], [12].

We use the following definitions (see [19], [15]). A (real) Banach lattice X is called a *KB-space*, if

(i) $x_1 \geqslant x_2 \geqslant \ldots$, $\inf\limits_n x_n = 0$ implies $\lim\limits_{n\to\infty} x_n = 0$,

(ii) $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots$, $\sup\limits_n \|x_n\| < \infty$ implies the existence of $x := \sup\limits_n x_n \in X$.

An element $e \in X$ is called a *weak order unit* (*wou*) if $\inf\{e, x\} = 0$ implies $x = 0$ for every $x \in X$ with $x \geqslant 0$, $x \neq 0$. Other definitions and notations used in the following are summarized in [10]; they are closely connected with the terminology in [19] (see also [15]) and will not be repeated here.

2. Commutative semigroups

Let X be a KB-space with a wou $\mathbf{1}$, and let G be a semigroup (i.e. $T, S \in G$ implies $TS \in G$) of linear bounded positive operators in X (as usual, positive means that $x \geqslant 0$ implies $Tx \geqslant 0$). Further, let G be commutative and bounded with $C := \sup_{T \in G} \|T\| \ (< \infty)$. In this section we give conditions for the existence of a fixed point $x_0 \in X$ of G (i.e. $Tx_0 = x_0$ for all $T \in G$) which is a wou of X and, moreover, we treat fixed point properties of G.

We define

$$K := \Big\{M = \frac{1}{n}\sum_{k=1}^{n} T_k\colon\ T_k \in G;\ k = 1, 2, \ldots, n;\ n \in N\Big\},$$

where N is the set of positive integers, and in K a relation $\leqslant$ by

$$M_1 \leqslant M_2 \ :\Leftrightarrow\ M_2K \subset M_1K \quad (M_1, M_2 \in K).$$

Evidently, K is equipped with the relation $\leqslant$ a directed set. In the following it will be used as index set for some nets.

From a mean ergodic theorem and a criterion of $\sigma(X, X^*)$-compactness in X we get

THEOREM 2.1. *If $u \in X^{**}$ is an element with the properties $T^{**}u \leqslant u$ for all $T \in G$ and $\langle u, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$, then there exists a fixed point x_0 of G, where the net $(M\mathbf{1})_{M \in K}$ converges (in norm) to x_0.*

COROLLARY. *If the net $(M\mathbf{1})_{M\in K}$ has a $\sigma(X^{**}, X^*)$-cluster point $u \in X^{**}$ with $\langle u, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$, then*

(i) *u is a* wou *in X and a fixed point of G,*

(ii) *the net $(M\mathbf{1})_{M\in K}$ converges (in norm) to u.*

These results imply

THEOREM 2.2. *The following statements are equivalent:*

(i) *G has a fixed point x_0 which is a* wou *in X,*

(ii) *the net $(M\mathbf{1})_{M\in K}$ converges (in norm) to a* wou *$z_0 \in X$,*

(iii) *there exists a $\sigma(X^{**}, X^*)$-cluster point $u \in X^{**}$ of the net $(M\mathbf{1})_{M\in K}$ with the property $\langle u, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(iv) *$r_1(f) := \inf_{T\in G}\langle T\mathbf{1}, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(v) *$r_2(f) := \varliminf_{T\in G}\langle T\mathbf{1}, f\rangle = \sup_{S\in G}\ \inf_{T \geqslant S, T\in G}\ \langle T\mathbf{1}, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(vi) *$r_3(f) := \varliminf_{M\in K}\langle M\mathbf{1}, f\rangle = \sup_{L\in K}\ \inf_{M \geqslant L, M\in K}\ \langle M\mathbf{1}, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$.*

In the case of a cyclic semigroup G the equivalence (i) $\Leftrightarrow$ (iv) was proved by P. C. Shields [16]. For some special cases with $X = L^1(\Omega, \mathfrak{B}, m)$ this and other equivalences are contained e.g. in [1], [7], [8], [9], [13], [17].

The following theorem contains a uniqueness statement.

THEOREM 2.3. *Suppose there exists a* wou *$\mathbf{1}^* \in X^*$ with $T^*\mathbf{1}^* \leqslant \mathbf{1}^*$ for all $T \in G$. If G has a fixed point x_0 which is a wou in X, then there exists one and only one fixed point z_0 with the properties*

(i) *$\langle z_0, f\rangle = \langle \mathbf{1}, f\rangle$ for all $f \in X_+^*$ with $T^*f = f$ $(T \in G)$,*

(ii) *$\langle z_0, f\rangle \geqslant r_1(f) = \inf_{T\in G}\langle T\mathbf{1}, f\rangle$ for all $f \in X_+^*$.*

If G is induced by a semigroup Γ of non-singular measurable transformations in a measure space $[\Omega, \mathfrak{B}, m]$, statement (i) of the theorem was proved by D. L. Hanson and T. T. Wright [8]. In this case statement (ii) contains a positive answer to a question formulated in [8].

In order to describe the general fixed point situation of G we denote by X_f the smallest band (or component) in X containing all positive fixed points of G and define

$$Y_{\inf} := \{f \in X^*\colon \inf_{T\in G}\langle T\mathbf{1}, |f|\rangle = 0\},$$

$$X_{\inf} := \{x \in X\colon \langle |x|, f\rangle = 0 \text{ for all } f \in Y_{\inf}\}.$$

As an example for a characterization of X_f we get

THEOREM 2.4. *The* KB-*space X_f is identical with $X_{\inf}$. The semigroup G has a fixed point $x_f \in X_f$ which is a* wou *in the* KB-*space X_f* (1) *with the property*

$$x_f = \lim_{M\in K} M\mathbf{1}_{\inf},$$

where $\mathbf{1}_{\inf} := \sup\{x \in X_{\inf}\colon x \leqslant \mathbf{1}\}$.

If G is a cyclic semigroup on $X = L^1(\Omega, \mathfrak{B}, m)$, the KB-space X_f is isomorphic to the space $L^1(\Omega_f, \mathfrak{B}, m)$; there $\Omega_f \subset \Omega$ is the corresponding strongly conservative set as defined and studied by J. Neveu [13].

3. Banach means

In the special case of cyclic semigroups induced by measurable transformations P. Calderón [2] used Banach limits in order to prove the equivalence of (i) and (iv) in Theorem 2.2 (see also [13], [17]). An analogous use of Banach means in our case is equivalent to the use of $\sigma(X^{**}, X^*)$-cluster points of the net $(M\mathbf{1})_{M\in K}$, considered in the Corollary of Theorem 2.1. This assertion follows from the next proposition.

Let Σ be an arbitrary commutative semigroup with unit ι. A linear functional l on the Banach space $B := B(\Sigma)$ of all bounded real functions on Σ (with the norm $\|f\| := \sup_{\tau\in\Sigma}|f(\tau)|$) is called a *Banach mean* if (i) $\inf_{\tau\in\Sigma} f(\tau) \leqslant l(f) \leqslant \sup_{\tau\in\Sigma} f(\tau)$ $(f \in B)$ and (ii) $l(T_\tau f) = l(f)$ $(f \in B, \tau \in \Sigma)$; here the linear contractions T_τ $(\tau \in \Sigma)$ in B are defined by $(T_\tau f)(\sigma) = f(\tau\sigma)$ $(\sigma \in \Sigma)$. In the dual B^* we define the element l_ι by $\langle f, l_\iota\rangle := f(\iota)$ $(f \in B)$; further we use the set

$$\mathscr{K} := \Big\{M = \frac{1}{n}\sum_{k=1}^{n} T_{\tau_k}^*\colon\ \tau_k \in \Sigma;\ k = 1, 2, \ldots, n;\ n \in N\Big\}$$

(1) In this formulation we include the case $X_f = \{0\}$, $x_f = 0$.

as a directed set with respect to the relation

$$M_1 \leqslant M_2 \;:\Leftrightarrow\; M_2\mathscr{K} \subset M_1\mathscr{K} \qquad (M_1, M_2 \in \mathscr{K}).$$

THEOREM 3.1. *An element $l \in B^*$ is a Banach mean of Σ if and only if l is a $\sigma(B^*, B)$-cluster point of the net $(Ml_\iota)_{M\in\mathscr{K}}$.*

In this way, the existence of a Banach mean is an immediate consequence of the theorem of Alaoglu. For any $f \in B$ the maximal and minimal values $f_{\max}, f_{\min}$ of all Banach means are characterized by the relations

$$f_{\max} = \varlimsup_{M\in\mathscr{K}} \langle Ml_\iota, f\rangle, \quad f_{\min} = \varliminf_{M\in\mathscr{K}} \langle Ml_\iota, f\rangle.$$

If $\Sigma = N^* = \{0, 1, 2, \ldots\}$ we get the well-known results about Banach limits. The values of the Banach limits for a special sequence $f = (x_n) \in B(N^*)$ are the cluster points of the real net $(x_M)_{M\in\mathscr{K}}$ of the means $x_M = \langle Ml_\iota, f\rangle$ $(M \in \mathscr{K})$.

4. Resolvents

Let X be a KB-space with a wou $\mathbf{1}$ and $(R_\lambda)_{\lambda>0}$ a resolvent (see [20]); i.e., $(R_\lambda)_{\lambda>0}$ is a family of linear bounded operators in X satisfying the resolvent equation $R_\lambda - R_\mu = (\mu-\lambda)R_\lambda R_\mu$ $(\lambda, \mu > 0)$. Suppose the operators R_λ are positive and the family $(\lambda R_\lambda)_{\lambda>0}$ is bounded with $C := \sup_{\lambda>0} \|\lambda R_\lambda\|$ $(< \infty)$. We shall investigate fixed point problems for the family $(\lambda R_\lambda)_{\lambda>0}$. For example, such a problem arises in the study of the invariant measure problem for Markov processes: under certain conditions the invariant measures of a Markov process are the fixed points of a corresponding strongly continuous one-parameter semigroup $G = \{T_t\colon t > 0\}$ of positive linear contractions in a KB-space $X = L^1(\Omega, \mathfrak{B}, m)$. It is well known that the fixed points of G are the fixed points of $(\lambda R_\lambda)_{\lambda>0}$, where $(R_\lambda)_{\lambda>0}$ is the resolvent of G defined by $R_\lambda = \int_0^\infty e^{-\lambda t} T_t\,dt$ $(\lambda > 0)$, being evidently positive with $\|\lambda R_\lambda\| \leqslant 1$ $(\lambda > 0)$.

The following result is based on a mean ergodic theorem of Hille's type.

THEOREM 4.1. *Let (λ_n) be a sequence of positive real numbers converging to 0. For every $x \in X$, one of the following alternative assertions is true:*

(i) *$\lambda R_\lambda x$ converges with $\lambda \to +0$ to an element $x_0 \in X$ (then x_0 is a fixed point of $(\lambda R_\lambda)_{\lambda>0}$), or*

(ii) *the sequence $(\lambda_n R_{\lambda_n} x)$ embedded in the bidual X^{**} has infinitely many $\sigma(X^{**}, X^*)$-cluster points and none of these is an order continuous functional on X^*.*

In a similar way as Theorem 2.2 in Section 2 one can prove

THEOREM 4.2. *Let (λ_n) be a sequence of positive real numbers converging to 0. The following assertions are equivalent:*

(i) *there exists a* wou *$x_0 \in X$ which is a fixed point of the family $(\lambda R_\lambda)_{\lambda>0}$,*

(ii) *there exists a* wou *$z_0 \in X$ with the property $\lim_{\lambda\to+0} \lambda R_\lambda \mathbf{1} = z_0$ (then z_0 is a fixed point of $(\lambda R_\lambda)_{\lambda>0}$),*

(iii) *the sequence $(\lambda_n R_{\lambda_n}\mathbf{1})$ embedded in X^{**} has a $\sigma(X^{**}, X^*)$-cluster point $u \in X^{**}$ with the property $\langle u, f\rangle > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(iv) *$\varliminf_{n\to\infty} \langle \lambda_n R_{\lambda_n}\mathbf{1}, f\rangle > 0$ for all $f \in X_+^*\setminus\{0\}$.*

In case $X = L^1(\Omega, \mathfrak{B}, m)$ and $R_\lambda = \int_0^\infty e^{-\lambda t} T_t\,dt$ for a certain strongly continuous one-parameter semigroup $(T_t)_{t>0}$ of positive operators the equivalence (i) $\Leftrightarrow$ (iv) and related results are proved by R. Sato [14].

If we denote by X_f the smallest band in X containing all positive fixed points of the family $(\lambda R_\lambda)_{\lambda>0}$ and define for an arbitrary sequence of positive real numbers $\lambda_n \to 0$ $(n \to \infty)$ the sets $\boldsymbol{Y} := \{f \in X^*\colon \varliminf_{n\to\infty} \langle \lambda_n R_{\lambda_n}\mathbf{1}, |f|\rangle = 0\}$, $\boldsymbol{X} := \{x \in X\colon \langle |x|, f\rangle = 0$ for all $f \in Y\}$, then we can formulate

THEOREM 4.3. *The KB-space X_f is identical with X. The family $(\lambda R_\lambda)_{\lambda>0}$ has a fixed point $x_f \in X_f$ which is a wou in the KB-space X_f (²) with the property*

$$x_f = \lim_{\lambda\to+0} \lambda R_\lambda \mathbf{1},$$

where $\mathbf{1} := \sup\{x \in \boldsymbol{X}\colon x \leqslant \mathbf{1}\}$.

5. Non-commutative semigroups

Let X be a KB-space with a wou $\mathbf{1}$, and let G be a bounded (not necessarily commutative) semigroup of positive linear operators in X. As in Section 3, we denote by $B(G)$ the Banach space of all bounded real functions on G. Similarly to the terminology in [5] for a subset $D \in B(G)$ a linear real functional l on D is called a *D-mean* if $\inf_{T\in G} \varphi(T) = l(\varphi) = \sup_{T\in G} \varphi(T)$ $(\varphi \in D)$. For every $\varphi \in D$ and $S \in G$ we define the functional ${}_S\varphi$ by ${}_S\varphi(T) := \varphi(ST)$ $(T \in G)$. Let D be *left-invariant*, i.e. ${}_S\varphi \in D$ for all $S \in G$, $\varphi \in D$. Then a D-mean l is called *left-invariant*, if $l({}_S\varphi) = l(\varphi)$ for all $S \in G$, $\varphi \in D$. The semigroup G is said to be *D-left-amenable* if D is left-invariant and G has a left-invariant D-mean. Evidently, a commutative G is $B(G)$-left-amenable (e.g. this follows from Theorem 3.1 and the theorem of Alaoglu). We define for any $x \in X$ the set

$$D_x := \{\varphi \in B(G)\colon \text{there exists an element } f \in X^* \text{ with } \varphi(T) = \langle Tx, f\rangle\ (T \in G)\}.$$

Further, we use the set

$$K := \Big\{M = \frac{1}{n}\sum_{k=1}^{n} T_k\colon T_k \in G;\ k = 1, 2, \ldots, n;\ n \in N\Big\}$$

(²) See footnote to Theorem 2.4.

directed by the relation

$$M_1 \leqslant M_2 :\Leftrightarrow KM_2K \subset KM_1K.$$

As a slight modification of a result of J. Dixmier [3] we get

THEOREM 5.1. *Suppose that for some element $x \in X$ the semigroup G is D_x-amenable and the set $Kx = \{Mx\colon M \in K\}$ is relatively $\sigma(X, X^*)$-compact. Then the net $(Mx)_{M\in K}$ has a (norm-) cluster point $x_0 \in X$, which is a fixed point of G. Moreover, for every left-invariant D_x-mean l, there exists such a cluster point x_l with the property $l(\langle Tx, f\rangle) = \langle x_l, f\rangle$ $(f \in X^*)$.*

(This theorem is also true if X is an arbitrary Banach space.)

On the basis of this theorem, one can prove results corresponding to the theorems in Section 2 (see [12]; similar results in the case of $X = L^1(\Omega, \mathfrak{B}, m)$ were obtained by R. Sato, e.g. [14], A. Hajian and Y. Ito [6], E. Granirer [4] and others). For example, we denote by $A_G \subset X$ the set of all wou's $e \in X$ with the property that G is D_e-left-amenable. The semigroup G is said to belong to the class $\mathfrak{A}$ if $A_G \neq \emptyset$. Evidently, under our assumptions every commutative semigroup belongs to $\mathfrak{A}$.

THEOREM 5.2. *The semigroup G has a fixed point $x_0 \in X$ which is a* wou *in X if and only if the following condition holds: G belongs to the class $\mathfrak{A}$ and one of the following equivalent assertions is true:*

(i) *the net $(M\mathbf{1})_{M\in K}$ has a clusterpoint $y_0 \in X$ which is a* wou *in X and a fixed point of G,*

(ii) *there exists a left-invariant D_1-mean l_0 of G with $l_0(\langle T\mathbf{1}, f\rangle) > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(iii) *every left-invariant D_1-mean l of G has the property $l(\langle T\mathbf{1}, f\rangle) > 0$ for all $f \in X_+^* \setminus \{0\}$,*

(iv) $r_1(f) := \inf_{T\in G} \langle T\mathbf{1}, f\rangle > 0$ *for all* $f \in X_+^* \setminus \{0\}$,

(v) $r_2(f) := \underline{\lim}_{T\in G} \langle T\mathbf{1}, f\rangle = \sup_{S\in G} \inf_{T\geqslant S, T\in G} \langle T\mathbf{1}, f\rangle > 0$ *for all* $f \in X_+^* \setminus \{0\}$,

(vi) $r_3(f) := \underline{\lim}_{M\in K} \langle M\mathbf{1}, f\rangle = \sup_{L\in K} \inf_{M\geqslant L, M\in K} \langle M\mathbf{1}, f\rangle > 0$ *for all* $f \in X_+^* \setminus \{0\}$.

References

[1] J. R. Blum and N. Friedman, *On invariant measures for classes of transformations*, Z. Wahrsch. verw. Geb. 8 (1967), 301–305.

[2] A. Calderón, *Sur les mesures invariantes*, C. R. Acad. Sci. Paris 240 (1955), 1960–1962.

[3] J. Dixmier, *Les moyennes invariantes dans les semigroupes et leurs applications*, Acta Sc. Math., Szeged, 12 (1950), 213–227.

[4] E. Granirer, *On finite equivalent measures for semigroups of transformations*, Duke Math. J. 38 (2) (1971), 395–408.

[5] F. P. Greenleaf, *Invariant means on topological groups and their applications*, New York 1969.

[6] A. Hajian and Y. Ito, *Weakly wandering sets and invariant measures for a group of transformations*, J. Math. Mech. 18 (12) (1969), 1203–1216.

[7] A. Hajian and S. Kakutani, *Weakly wandering sets and invariant measures*, Trans. Amer. Math. Soc. 110 (1964), 136–151.

[8] D. L. Hanson and T. T. Wright, *On the existence of equivalent finite measures*, Z. Wahrsch. verw. Geb. 14 (1970), 200–202.

[9] Y. Ito, *Invariant measures for Markov processes*, Trans. Amer. Math. Soc. 110 (1964), 152–184.

[10] R. Kühne, *Fixpunkte positiver Kontraktionen in KB-Räumen*, Math. Nachr. 65 (1975), 259–280.

[11] —, *Invariante Maße als Fixpunkte von Halbgruppen positiver Operatoren in KB-Räumen*, Wiss. Sitzungen zur Stochastik, Berlin, April 1977.

[12] —, *Fixpunkteigenschaften einer Klasse positiver Operatoren in KB-Räumen*, Math. Nachr. 90 (1979), 149–157.

[13] J. Neveu, *Existence of bounded invariant measures in ergodic theory*, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Berkeley and Los Angeles, Vol. II, Part II, (1967), 461–472.

[14] R. Sato, *Invariant measures for ergodic semigroups of operators*, Pacific J. Math. 71 (1) (1977), 173–192.

[15] H. H. Schaefer, *Banach lattices and positive operators*, Berlin, Heidelberg, New York 1974.

[16] P. C. Shields, *Invariant elements for positive contractions on a Banach lattice*, Math. Z. 96 (1967), 189–195.

[17] L. Sucheston, *On existence of finite invariant measures*, ibid. 86 (1964), 327–336.

[18] W. Takahashi, *Invariant functions for amenable semigroups of positive contractions on L^1*, Kōdai Math. Sem. Rep. 23 (1971), 131–143.

[19] Б. З. Вулих, *Введение в теорию полуупорядоченных пространств*, Москва 1961.

[20] K. Yosida, *Functional analysis*, Berlin, Göttingen, Heidelberg 1965.

Presented to the semester
Spectral Theory
September 23–December 16, 1977