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SOME USES OF SUBHARMONICITY IN FUNCTIONAL ANALYSIS

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The aim of this paper is to present a summary of the eight lectures I gave at the Banach Center in Warsaw on this subject. I shall speak only about applications in the theory of Banach algebras and in the theory of polynomial approximation in several complex variables. All the details and other results will be found in my book *Propriétés Spectrales des algèbres de Banach.*

For the definition and the main properties of subharmonic functions see [5], [10], [11], [18], [22].

1. Banach algebras theory

In the following pages $\mathrm{Sp}\,x$ denotes the *spectrum of* x, $\sigma(x)$ denotes the *full spectrum*, i.e. the union of $\mathrm{Sp}\,x$ with its holes, $\varrho(x)$, $\delta(x)$, $c(x)$ denote respectively the *radius*, the *diameter* and the *capacity* of the spectrum of x. $\mathrm{Rad}\,A$ is the Jacobson radical of the algebra A.

The fundamental starting point is

THEOREM 1 (Vesentini). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\mathbf{C}$ into a complex Banach algebra A, then $\lambda \to \varrho(f(\lambda))$ and $\lambda \to \log\varrho(f(\lambda))$ are subharmonic.*

For the proof, see [19], [20]. A more elementary proof not using Radó's theorem is given in [5]. With that result the well-known theorem of Kleinecke and Shirokov and related results are coming more naturally.

COROLLARY 1 (Kleinecke–Shirokov). *Let A be a Banach algebra and a, b elements of A verifying $a(ab-ba) = (ab-ba)a$, then $ab-ba$ is quasi-nilpotent.*

COROLLARY 2. *Let a, b be elements of A verifying $a(ab-ba) = 0$ or $(ab-ba)a = 0$ and suppose that 0 is on the exterior boundary of the spectrum of a (i.e. the boundary of the full spectrum), then $ab-ba$ is quasi-nilpotent.*

COROLLARY 3. (*Principle of maximum for full spectrum*) (Vesentini). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\mathbf{C}$ into a complex Banach algebra*

A and suppose there exists α in D such that $\sigma(f(\lambda))$ is included in $\sigma(f(\alpha))$, for every λ in D, then we have $\sigma(f(\lambda))$ equal to $\sigma(f(\alpha))$ for every λ in D.

If N denotes the set of quasi-nilpotent elements in a Banach algebra, we know that N includes the Jacobson radical of A. For commutative algebras we have equality but the converse is false. R. A. Hirschfeld and S. Rolewicz, H. Behncke, A. S. Nemirovskiĭ, J. Duncan and A. W. Tullo, have built non-commutative algebras where N is reduced to zero. In 1968, R. A. Hirschfeld and W. Żelazko conjectured that this last condition, with continuity of the spectrum, implies commutativity. But in [3] we showed it is false even for rather regular algebras.

Using Vesentini's theorem and Jacobson's density theorem J. Zemánek [25] has given a very interesting spectral characterization of the radical.

COROLLARY 4 (Zemánek). *If A is a complex Banach algebra, the following properties are equivalent:*

(i) *a is in the Jacobson radical of A.*

(ii) *$a+x$ is quasi-nilpotent for every x quasi-nilpotent in A.*

(iii) $\mathrm{Sp}(a+x) = \mathrm{Sp}\,x$, *for every x in A.*

(iv) $(1+x)a$ *is quasi-nilpotent for every x quasi-nilpotent in A.*

It is possible to give a proof not using subharmonicity with the help of ideas developed in [7] or in [14], this proof works even in the real case. As an immediate corrolary we get:

COROLLARY 5 (Słodkowski, Wojtyński, Zemánek [16]). *If A is a complex Banach algebra, the following properties are equivalent:*

(i) *N is equal to the Jacobson radical of A.*

(ii) *N is stable by addition.*

(iii) *N is stable by multiplication.*

If I is a closed two-sided ideal in a Banach algebra we denote by $kh(I)$ the intersection of all primitive ideals containing I. In fact it is the set of x for which the class $\bar{x}$ defined by I is in the Jacobson radical of A/I. Using a rather complicated proof, J. Zemánek [26] obtained:

COROLLARY 6. *If I is a closed two-sided ideal of a Banach algebra A, then the following properties are equivalent:*

(i) *a is in $kh(I)$.*

(ii) $\mathrm{Sp}\,\bar{x}$ *is included in* $\mathrm{Sp}(a+x)$, *for every x in A.*

(iii) $\varrho(\bar{x}) \leqslant \varrho(a+x)$, *for every x in A.*

With the help of Corollary 4, in [5], we gave a very simple proof of this fact.

Corollary 4 characterizing the Jacobson radical has an interesting application which allows us to generalize Gleason–Kahane–Żelazko theorem.

THEOREM 2. *Let A be a complex Banach algebra and B a complex semi-simple Banach algebra with continuous spectral radius. If T is a linear mapping from A into B such that $\varrho(Tx) \leqslant \varrho(x)$, for every x in A and such that the image of T is dense in the set of quasi-nilpotent element of B, then T is continuous.*

Rather intricate calculus on several complex variables gives:

THEOREM 3. *If A is a complex Banach algebra with identity and T a linear mapping from A onto $M_n(C)$ such that $T1 = 1$ and x invertible implies Tx invertible, then T is a morphism or an antimorphism of algebras.*

For numerous applications, see [4].

Several persons (C. Le Page, R. A. Hirschfeld and W. Żelazko, G. Mocanu) showed that complex algebras satisfying $\varrho(x) \geqslant k\|x\|$, for every x in A and a convenient $k \leqslant 1$, are commutative. But these condition characterizes only function algebras. For a general spectral characterization of commutative algebras we got in [1] the following:

THEOREM 4. *If A is a complex Banach algebra then the following properties are equivalent:*

(i) *$A/\mathrm{Rad}\,A$ is commutative.*

(ii) *The spectrum is uniformly continuous on A.*

(iii) *The spectral radius is uniformly continuous on A.*

(iv) *There exists $\alpha > 0$ such that $\varrho(x+y) \leqslant \alpha(\varrho(x)+\varrho(y))$, for every x and y in A.*

(v) *There exists $\alpha > 0$ such that $\varrho(xy) \leqslant \alpha\varrho(x)\varrho(y)$, for every x and y on A.*

(vi) *$\varrho(x+y) \leqslant \varrho(x)+\varrho(y)$ in a neighborhood of identity if A has an identity.*

(vii) *$\varrho(xy) \leqslant \varrho(x)\varrho(y)$ in a neighborhood of identity if A has an identity.*

The proof uses Theorem 1 and Liouville's theorem for subharmonic functions, nevertheless it is possible to prove the equivalence of the first five conditions by using purely algebraic methods and no subharmonicity, as made by V. Pták and J. Zemánek [15]. But their method is useless for the last local conditions. Recently J. Zemánek and myself discovered an even more simple proof [7].

For real Banach algebras the previous theorem is not true (even in the case of quaternions K) but by a very sinuous way it is possible to prove:

THEOREM 5. *If A is a real Banach algebra then the following properties are equivalent:*

(i) *For every irreducible representation π of A, $\pi(A)$ is isomorphic to R, C or K.*

(ii) *Each one of the conditions* (ii) *to* (vii) *in the previous theorem.*

For the proof it is necessary not to use analytic theory directly. It uses two rather deep theorems: Corollary 7, to be seen later, about spectral characterization of finite-dimensional real Banach algebras and Kaplansky's theorem about locally algebraic rings (see [7]).

Now by the use of classical potential theory, we shall give deeper results which are contained mainly in [2], [5].

THEOREM 6 (Spectral pseudo-continuity theorem). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in C, containing α, into a complex Banach algebra A and let E be a subset of D which is non-thin at α, then there exists a sequence (λ_n)*

of elements of E, converging to α, with $\lambda_n \neq \alpha$, such that the full spectrum of $f(\lambda_n)$ converges to the full spectrum of $f(\alpha)$ for the Hausdorff metric.

The analog result for the spectrum is not true. We shall not give the definition of *non-thin at a point* (see [10]) but we keep in mind that a Jordan arc is not thin at every of its points (Oka–Rothstein theorem) and that a domain is not thin at every of its boundary points. Theorem 5 permits to extend some spectral properties even when the spectrum is not continuous, for example it is very useful in the proof of Theorem 9. To understand that this result us not evident let us mention the example given by V. Müller [13] of a Banach algebra A, of two elements a, b in A and of a sequence of real numbers μ_n converging to zero such that $a+\mu_n b$ is quasinilpotent and a is not.

Vesentini's theorem can be extented by:

THEOREM 7. *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\boldsymbol{C}$ into a complex Banach algebra A, then $\lambda \to \delta(f(\lambda))$ and $\lambda \to \log \delta(f(\lambda))$ are subharmonic.*

Using the following lemma, which is an extension of a result of Hartogs, and previous theorem we get Theorem 8.

LEMMA (Aupetit–Wermer). *Let $\lambda \to h(\lambda)$ be a bounded function from a domain D in $\boldsymbol{C}$ into $\boldsymbol{C}$ such that $\lambda \to \log|h(\lambda)-\alpha|$ is subharmonic for every α in $\boldsymbol{C}$, then h is holomorphic or antiholomorphic.*

THEOREM 8. *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\boldsymbol{C}$ into a complex Banach algebra A and let α_0 be an isolated point of the spectrum of $f(\lambda_0)$ for some λ_0 in D. There exists a neighborhood V of λ_0 such that the set of λ in V for which the spectrum of $f(\lambda)$ has an isolated point in the neighborhood of α_0 is of outer capacity zero and otherwise, for every λ in V, the spectrum of $f(\lambda)$ has an isolated point $h(\lambda)$ in the neighborhood of α_0 and h is holomorphic on V.*

If A is a Riesz algebra (see [17]) or equivalently a modular annihilator algebra (see [8]), which implies that the spectrum of every element has at most 0 as limit point, we conclude in this case that every isolated point of the spectrum of $f(\lambda)$ varies holomorphically outside of a countable set of singularities.

The following scarcity result has been invented to extend theorems of Kaplansky, Hirschfeld and Johnson concerning spectral characterization of complex finite-dimensional Banach algebras to the real and involutary cases. The proof would be more easy if we could prove subharmonicity of the functions $\lambda \to$ $\to \log \delta_n(f(\lambda))$, where δ_n denotes the *n-th diameter* and even less, the subharmonicity of $\lambda \to \log c(f(\lambda))$ (this last result would have a lot of interesting applications in Banach algebras theory).

THEOREM 9 (on scarcity of operators with finite spectrum). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\boldsymbol{C}$ into a complex Banach algebra A, then:*

(i) *either the set of λ in D for which* $\operatorname{Sp} f(\lambda)$ *is finite is of outer capacity zero*

(ii) *or there exists an integer $n \geqslant 1$ such that* $\operatorname{Sp} f(\lambda)$ *has exactly n elements for every λ in D except on a closed discrete subset of D where* $\operatorname{Sp} h(\lambda)$ *has less than n elements. In this case, for every λ outside of this subset,* $\operatorname{Sp} f(\lambda)$ *varies holomorphically.*

It would be very interesting to know if there exists an analog to this theorem in the countable case.

COROLLARY 7. *Let A be a real Banach algebra, then $A/\mathrm{Rad}\,A$ is finite-dimensional if and only if A contains an absorbing set on which the spectrum of each element is finite.*

COROLLARY 8. *Let A be a complex Banach algebra with involution, then $A/\mathrm{Rad}\,A$ is finite-dimensional if and only if the set of hermitian elements contains an absorbing set on which the spectrum of each element is finite.*

COROLLARY 9. *Let A be a real Banach algebra containing a non-void open set of invertible elements for which $\|x\|\cdot\|x^{-1}\| = 1$, then A is equal to $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{K}$.*

This last result is a nice local generalization of Gelfand–Mazur theorem.

Using Corollaries 7 and 8 it is possible to improve greatly, in the case of real and involutary Banach algebras, the results of B. A. Barnes [8] about the existence of minimal ideals and the characterization of modular annihilator algebras.

A. Pełczyński has raised the following conjecture for B^*-algebras: if the spectrum of every hermitian element is countable then the spectrum of every element of the algebra is countable.

This problem, obvious in the commutative case, is still unsolved.

In relation with Corollary 8 it would be convenient to make the more general conjecture:

Let A be a complex Banach algebra with involution such that the spectrum of every hermitian element is countable, is it true that the spectrum of every element of A is countable?

Using the method of subharmonicity with Corollaries 7 and 8 we have been able to give a partial answer to this question. It is true if the spectrum of every hermitian element has a finite number of limit points and in this case $A/kh(\mathrm{soc}(A))$ is finite-dimensional, where $\mathrm{soc}(A)$ denotes the socle of A.

For a lot of other applications, see [5].

2. Function algebras theory

By G. Stolzenberg's example we know that in general the set of characters of a function algebra has no analytic structure, but for several examples we know more.

Let K be a compact set, A be a function algebra on K and M be the set of characters of A. For f in A and λ complex we denote by $\overset{-1}{f}(\lambda)$ the set of characters χ satisfying $\chi(f) = \lambda$.

E. Bishop has proved the following result which is fundamental in all the proofs of polynomial approximation on arcs (see [24]).

THEOREM 1. *Let K, A, M, f be defined as previously and let W be a component of $C \setminus f(K)$. Suppose there exists a subset G of W such that:*

(i) *G is of positive planar measure,*

(ii) *$\overset{-1}{f}(\lambda)$ is finite on G.*

Then there exists an integer $n \geqslant 1$ such that $\overset{-1}{f}(\lambda)$ has at most n elements for every λ in W. Consequently $\overset{-1}{f}(W)$ has the structure of a complex analytic variety of dimension one on which each element of A is analytic.

A partial generalization of this theorem has been obtained by R. Basener [9].

THEOREM 2. *Let K, A, M, f, W be defined as previously. Suppose there exists a subset G of W such that:*

1° *G is of positive planar measure,*

2° *$\overset{-1}{f}(\lambda)$ is countable on G.*

Then there exists a non-void open subset of $\overset{-1}{f}(W)$ having the structure of a complex analytic variety of dimension one on which each element of A is analytic.

For g in A if we introduce the functions:

$$\varrho_g(\lambda) = \max\left|g\left(\overset{-1}{f}(\lambda)\right)\right|, \qquad \delta_g(\lambda) = \mathrm{diameter}\left(g\left(\overset{-1}{f}(\lambda)\right)\right)$$

as in Vesentini's theorem and Theorem 7 of Section 1 we can prove that $\lambda \to \log \varrho_g(\lambda)$ and $\lambda \to \log \delta_g(\lambda)$ are subharmonic (the first case has been proved in [23]). In [6], using the techniques developed in [2], we have been able to improve the results of Bishop and Basener by a completely new method, replacing the condition *G of positive planar measure* by the weakest condition *G of positive outer capacity*. The theorem obtained is strangely similar to the scarcity theorem of Section 1. It is also the best result which can be obtained, because, if E is a compact of capacity zero in the unit disk Δ, there exists a function algebra A on a compact K and f in A such that $f(K) =$ boundary of Δ, $\overset{-1}{f}(\lambda)$ has one element on E and $\overset{-1}{f}(\lambda)$ is uncountable on $\Delta \setminus E$. The construction of such an example is very technical. The theorem of R. Basener about analytic structure for several variables can also be extended. Recently D. Kumagai [12] has also proved subharmonicity of more general functions $\lambda \to \log \varphi_g(\lambda)$, with some hypothesis on the algebra A.

All of this can be used to simplify the proof of the Alexander–Björk theorem which says that every continuous function on a rectifiable Jordan arc in C^n is uniformly approximated by polynomials in n variables.

Subharmonicity has also proved to be interesting for problems of holomorphic automorphisms of the unit ball in Banach spaces (see [21]).

Remarks added in December 1981. All the unsolved problems mentioned in this paper are now proved and also many results have been improved. See the subsequent publications of B. Aupetit and Z. Słodkowski.

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