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NOTES ON THE TAYLOR JOINT SPECTRUM OF COMMUTING OPERATORS

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In this paper we shall present some considerations concerning the remarkable definition of a joint spectrum for a finite number of commuting linear operators introduced by J. L. Taylor in [3] and [4]. In the first part we shall show that the Taylor construction of functional calculus can easily be described in terms of sheaf theory. This permits us to give a short proof of the main Taylor theorems. Further we give a characterization of the Taylor spectrum and of the corresponding functional calculus as the best in some sense, satisfying some natural conditions. Finally, we announce a definition of the essential Taylor spectrum and some of its properties, and in some cases the existence of a corresponding functional calculus in Calkin algebra.

A part of these results was obtained in [2].

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The standard construction for functional calculus in a commutative Banach algebra (see, for example, [1]), can be described as follows. For the finite set $a_1, \ldots, a_n$ of elements of the commutative Banach algebra A one can choose n smooth functions $u_1(z), \ldots, u_n(z)$, with values in A, defined for $z \in C^n \setminus K$, where K is the joint spectrum of $a_1, \ldots, a_n$ in A, such that

$$\sum_{i=1}^{n} (a_i - z_i e) u_i(z) = e, \quad z = (z_1, \ldots, z_n).$$

The A-valued differential form of type $(0, n-1)$, defined by the expression

$$R = n!\, u_1 \wedge \bar{\partial} u_2 \wedge \ldots \wedge \bar{\partial} u_n, \tag{1}$$

is $\bar{\partial}$-closed on $C^n \setminus K$. To each function $f(z)$ which is analytic in some neighbourhood of K corresponds an element $\hat{f} \in A$, given by the formula

$$\hat{f} = (2\pi i)^{-n} \int_{bV} f \cdot R \wedge dz, \quad dz = dz_1 \ldots dz_n, \tag{2}$$

where V is some neighbourhood of K with a smooth boundary bV. Since the choice of R is important only up to the $\bar{\partial}$-exact form, R can be considered as an element of the $\bar{\partial}$-cohomological group $H^{n-1}(\tilde{A}, C^n \setminus K)$ with coefficients in the sheaf $\tilde{A}$ of germs of A-valued holomorphic functions.

In the case where A is subalgebra of the algebra $L(X)$ of all bounded linear operators acting in some Banach space X the operator-valued form R determines for each element x of X a closed X-valued form $R(x)$. Therefore R can be considered as a correspondence between the space X and the cohomology group $H^{n-1}(\tilde{X}, C^n \setminus K)$ with coefficients in the sheaf $\tilde{X}$ of germs of holomorphic X-valued functions. More formally, R can be considered as an element of the group $\mathrm{Ext}^{n-1}(\tilde{X}, \tilde{X}, C^n \setminus K)$. The functional calculus is given by the formula

$$\hat{f}(x) = (2\pi i)^{-n} \int_{bV} f(z) \cdot R(x) \wedge dz \tag{3}$$

where $\hat{f}$ is the linear operator in X corresponding to the holomorphic function f.

Now, we shall construct a similar correspondence $x \to R(x)$ for the Taylor spectrum of commuting operators. Taylor's definition of joint spectrum does not depend of the choice of the enveloping Banach algebra of operators. We shall give this definition. Further in the paper we will denote by $T_1, \ldots, T_n$ a finite set of commuting linear bounded operators in the Banach space X. Let Λ_n denote the free anticommutative algebra with n generators $s_1, \ldots, s_n$, and let Λ_n^p be a linear space of homogeneous elements of order p. Put $X_p = \Lambda_n^p \otimes X$ for $0 \leqslant p \leqslant n$. X_p is isomorphic to the direct sum of finitely many copies of the space X. Any element x of X_p has the form

$$x = \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant n} x_{i_1, \ldots, i_p} s_{i_1} \wedge \ldots \wedge s_{i_p}$$

where $x_{i_1, \ldots, i_p}$ are elements of X.

Let $z = (z_1, \ldots, z_n)$ be a point in C^n. Consider the mapping $\alpha_p(z) \colon X_{p-1} \mapsto X_p$ defined by the formula

$$\alpha_p(z)\,[x_{i_1, \ldots, i_{p-1}} s_{i_1} \wedge \ldots \wedge s_{i_{p-1}}] = \sum_{i=1}^{n} (T_i - z_i)\,[x_{i_1, \ldots, i_{p-1}}] s_{i_1} \wedge \ldots \wedge s_{i_{p-1}} \wedge s_i.$$

It is easy to prove, by using the commutativity of operators T_i, that $\alpha_p(z) \cdot \alpha_{p-1}(z) = 0$. So, the spaces X_p and the mappings $\alpha_p(z)$ form a complex

$$E(z) \colon 0 \mapsto X_0 \overset{\alpha_1(z)}{\mapsto} X_1 \overset{\alpha_2(z)}{\mapsto} \ldots \overset{\alpha_n(z)}{\mapsto} X_n \to 0. \tag{4}$$

Note that $X_0 \cong X_n \cong X$, $X_{n-1} \cong X_1 \cong \bigoplus_n X$, α_1 and α_n are given by the formulas

$$\alpha_1(z)[x] = ((T_1 - z_1)x, \ldots, (T_n - z_n)x),$$

$$\alpha_n(z)\,[x_1, \ldots, x_n] = \sum_{i=1}^{n} (T_i - z_i) x_i.$$

Definition (Taylor [3]). The *joint spectrum of the operators* $T_1, \ldots, T_n$ consists of all points z in C^n such that the complex $E(z)$ is not exact.

Theorem 1.1 (Taylor [4]). *There exists a functional calculus in X for the algebra of functions holomorphic in some neighbourhood of the Taylor spectrum.*

The proof of this theorem is the main result in Taylor's paper [4]. We shall give a short proof of the theorem, based on cohomology sheaf theory.

Denote by $\tilde{X}_p$ the sheaf of X_p-valued holomorphic functions on C^n and by $\mathscr{E}(X_p)$ the sheaf of X_p-valued smooth functions. We have complexes

$$\tilde{E} \colon 0 \to \tilde{X}_0 \overset{\alpha_1(z)}{\mapsto} \ldots \overset{\alpha_n(z)}{\mapsto} \tilde{X}_n \to 0,$$

$$\mathscr{E}(E) \colon 0 \to \mathscr{E}(X_0) \overset{\alpha_1(z)}{\mapsto} \ldots \overset{\alpha_n(z)}{\mapsto} \mathscr{E}(X_N) \to 0.$$

We shall use a preliminary Taylor result asserting that the complexes $\tilde{E}$ and $\mathscr{E}(E)$ are exact at all points z for which the complex $E(z)$ is exact.

In sheaf theory, the correspondence between the exact complexes of sheaves and suitable groups Ext is well known. Let K be the Taylor spectrum of $T_1, \ldots, T_n$ and let V be some neighbourhood of it with a smooth boundary bV. Then the functional calculus can be defined by formula (3), where R is an element of the group $\mathrm{Ext}^n(\tilde{X}, \tilde{X}, C^n \setminus K)$ corresponding to the complex $\tilde{E}$, which is exact on $C^n \setminus K$. More precisely, the mapping $x \to R(x)$ can be obtained by the following standard procedure. Let $\mathscr{W}^q(X_p)$ denote the sheaf of germs of differential forms of type $(0, q)$ on C^n with values in the Banach space X_p and let $\Omega^q(X_p, C^n \setminus K)$ denote the linear space of sections of $\mathscr{W}^q(X_p)$ on $C^n \setminus K$. (We consider in this paper only forms with smooth coefficients.) In other words, $\Omega^q(X_p, C^n \setminus K)$ is the space of all X_p-valued differential forms on $C^n \setminus K$ of type $(0, q)$. We can construct the complex of sheaves

$$\mathscr{W}^p(E) \colon 0 \to \mathscr{W}^p(X_0) \overset{\alpha_1}{\to} \ldots \overset{\alpha_n}{\to} \mathscr{W}^p(X_n) \to 0$$

and the complex of linear spaces

$$\Omega^q(E, C^n \setminus K) \colon 0 \to \Omega^q(X_0, C^n \setminus K) \overset{\alpha_1}{\to} \ldots \overset{\alpha_n}{\to} \Omega^q(X_n, C^n \setminus K) \to 0.$$

Since the sheaves $\mathscr{W}^p$ are thin and the complex $\mathscr{W}^p(E)$ is exact on $C^n \setminus K$, the complex $\Omega^p(E, C^n \setminus K)$ is also exact (this is not true for the complex of sections of $\tilde{E}$).

Now let x be an element of X. It follows from the exactness of $\Omega^0(E, C^n \setminus K)$ that there exists a smooth function $\mathscr{W}_0 = \mathscr{W}_0(z, x)$ on $C^n \setminus K$ with values in X_{n-1} such that

$$\alpha_n(z)[\mathscr{W}_0(z, x)] = x.$$

Applying the operator $\bar{\partial}$ to this equality, we have $\alpha_n[\bar{\partial}\mathscr{W}_0] = 0$. (Since the operators $\alpha_i(z)$ depend analytically on z, they commute with the operator $\bar{\partial}$.) Using the exactness of $\Omega^1(E, C^n \setminus K)$, we obtain the form $\mathscr{W}_1$ belonging to $\Omega^1(X_{n-2}, C^n \setminus K)$ such that $\alpha_{n-1}[\mathscr{W}_1] = \bar{\partial}\mathscr{W}_0$. In this way one can construct the forms $\mathscr{W}_i = \mathscr{W}_i(x)$, $i = 0, 1, \ldots, n-1$, $\mathscr{W}_i$ belonging to $\Omega^i(X_{n-i-1}, C^n \setminus K)$ such that

$$\alpha_{n-i}[\mathscr{W}_i] = \bar{\partial}\mathscr{W}_{i-1} \quad \text{for} \quad i = 1, \ldots, n, \quad \alpha_n[\mathscr{W}_0] = x, \tag{5}$$
$$\bar{\partial}\mathscr{W}_{n-1} = 0.$$

We define $R(x)$ as a class of $\bar\partial$-cohomology of $\mathscr{W}_{n-1}(x)$. This definition does not depend of the choice of $\mathscr{W}_0, \ldots, \mathscr{W}_{n-1}$. In fact, if $\mathscr{W}'_0, \ldots, \mathscr{W}'_{n-1}$ are differential forms satisfying the same equalities (5), we can prove, using induction by i, the existence of forms $\tau_i \in \Omega^i(X_{n-i-2}, C^n \setminus K)$ such that

$$\bar\partial\mathscr{W}_i - \bar\partial\mathscr{W}'_i = \alpha_{n-i-1}[\bar\partial\tau_i].$$

Indeed, if this is true for given i, it follows that

$$\alpha_{n-i-1}[\mathscr{W}_{i+1} - \mathscr{W}'_{i+1} - \bar\partial\tau_i] = 0;$$

hence

$$\mathscr{W}_{i+1} - \mathscr{W}'_{i+1} - \bar\partial\tau_i = \alpha_{n-i-2}[\tau_{i+1}],$$
$$\bar\partial\mathscr{W}_{i+1} - \bar\partial\mathscr{W}'_{i+1} = \alpha_{n-i-2}[\bar\partial\tau_{i+1}].$$

For $i = n-1$ we have

$$\alpha_1[\mathscr{W}_{n-1} - \mathscr{W}'_{n-1} - \bar\partial\tau_{n-2}] = 0$$

and

$$\mathscr{W}_{n-1} - \mathscr{W}'_{n-1} = \bar\partial\tau_{n-2}.$$

This means that $\mathscr{W}_{n-1}$ and $\mathscr{W}'_{n-1}$ are in the same class of $\bar\partial$-cohomology. From this immediately follows the linearity of the mapping $x \to R(x)$.

In some cases it is more convenient to use another variant of this construction. Let V and U be neighbourhoods of K such that the closure $\bar U$ of U is contained in V. Denote by $\tilde{\mathscr{W}}_i$ a differential form, defined on C^n, such that its restriction on $C^n \setminus U$ coincides with $\mathscr{W}_i$. Put

$$\bar{\mathscr{W}}_i = \bar\partial\tilde{\mathscr{W}}_{i-1} - \alpha_{n-i}[\tilde{\mathscr{W}}_i], \quad \bar{\mathscr{W}}_0 = x - \alpha_n[\tilde{\mathscr{W}}_0], \quad \bar{\mathscr{W}}_n = \bar\partial\tilde{\mathscr{W}}_{n-1}.$$

$\bar{\mathscr{W}}_i$ is an element of $\Omega^i(X_{n-i}, V)$. It is easy to check that

$$\bar\partial\bar{\mathscr{W}}_i = \alpha_{n-i-1}[\bar{\mathscr{W}}_{i+1}], \quad \bar\partial\bar{\mathscr{W}}_n = 0, \quad x - \bar{\mathscr{W}}_0 \in \operatorname{Im}\alpha_n. \tag{6}$$

The whole of $\bar{\mathscr{W}}_i$ has a compact support contained in V. We denote by $\bar R(x)$ the class of $\bar{\mathscr{W}}_n$ in the group $H^n_K(\tilde X, V)$ of cohomologies with a compact support. For every function f holomorphic in V we have by the Stokes formula

$$\hat f(x) = \int_{bV} fR(x) \wedge dz = \int_V f\bar R(x) \wedge dz. \tag{3'}$$

If $\bar{\mathscr{W}}'_0, \ldots, \bar{\mathscr{W}}'_n$ are differential forms with a compact support satisfying equalities (6), then $\bar{\mathscr{W}}_n$ and $\bar{\mathscr{W}}'_n$ are in the same class of cohomologies in the group $H^n_K(X, V)$. We omit the proof, because, it is similar to the proof in the preceding case.

Let $T_1, \ldots, T_n, S_1, \ldots, S_k$ be commuting operators in the Banach space X. Let $\bar{\mathscr{W}}'_0(x), \ldots, \bar{\mathscr{W}}'_n(x)$ be the differential forms, constructed above, for the operators $T_1, \ldots, T_n$, and let $\bar{\mathscr{W}}''_0(x), \ldots, \bar{\mathscr{W}}''_k(x)$ be corresponding forms for $S_1, \ldots, S_k$. Denote by $z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_k$ the coordinate functions in C^{n+k}. The forms $\bar{\mathscr{W}}'_i$ can be considered as differential forms in C^{n+k}, depending only on $z_1, \ldots, z_n$. For fixed $z = (z_1, \ldots, z_n)$ the coefficients of the components of these forms are elements of X and we can give to it the form $\bar{\mathscr{W}}''_j$ with coefficients depending on $\lambda = (\lambda_1, \ldots, \lambda_k)$. So, we obtain the superposition of the mappings $x \to \bar{\mathscr{W}}'_i(x)$ and $x \to \bar{\mathscr{W}}''_j(x)$, denoted by

$$\bar{\mathscr{W}}'_i \circ \bar{\mathscr{W}}''_j(x) = \bar{\mathscr{W}}'_i(\bar{\mathscr{W}}''_j(x))$$

as a differential form on the space C^{n+k}. $\bar{\mathscr{W}}'_i \circ \bar{\mathscr{W}}''_j$ is a differential form of type $(0, i+j)$ with values in the space $\Lambda^n_{n-i} \otimes \Lambda^k_{k-j} \otimes X$.

Now, we can give a new proof of the "projection property" for the Taylor spectrum and for corresponding functional calculus.

LEMMA 1.2. *Let K be the Taylor spectrum of the operators $T_1, \ldots, T_n, S_1, \ldots, S_k$ and let $\bar R$ be the mapping $X \to H^n_K(X, V)$ constructed above. Let $K', \bar R'$ and $K'', \bar R''$ be the corresponding objects for $T_1, \ldots, T_n$ and for $S_1, \ldots, S_k$. Then we have:*

(a) *$p(K) = K'$, where p is the projection of C^{n+k} on C^n, $p(z, \lambda) = z$;*

(b) *Let V' and V'' be neighbourhoods of K' and K''. Then in the group $H^{n+k}_K(\tilde X, V' \times V'')$ the equality*

$$R' \circ R''(x) = R(x)$$

is satisfied.

Note that in (b) the supperposition $R' \circ R''$ is in fact the Ioneda product of corresponding elements in the groups Ext.

Proof. Denote by $X_p, \alpha_p(z, \lambda)$ the elements of complex (4), constructed for the operators $T_1, \ldots, T_n, S_1, \ldots, S_k$. Let $X'_q, \alpha'_q(z)$ and $X''_r, \alpha''_r(\lambda)$ be the elements of corresponding complexes, constructed for $T_1, \ldots, T_n$ and for $S_1, \ldots, S_k$. We have an isomorphism $\Lambda^{n+k} \cong \Lambda^n \otimes \Lambda^k$. Hence

$$\Lambda^{n+k}_p = \bigoplus_{q+r=p} \Lambda^n_q \otimes \Lambda^k_r.$$

Write

$$X_{q,r} = \Lambda^n_q \otimes \Lambda^k_r \otimes X.$$

Then

$$X_p = \bigoplus_{q=0}^{p} X_{q,p-q}.$$

We have a morphisms $\alpha'_q(z)\colon X_{q-1,r} \to X_{q,r}$ and $\alpha''_r(\lambda)\colon X_{q,r-1} \to X_{q,r}$. If $x \in X_{q,r} \subset X_{q-1,r}$, then

$$\alpha_{q+r+1}(z, \lambda)[x] = (-1)^r \alpha'_{q+1}(z)[x] + \alpha''_{r+1}(\lambda)[x].$$

In order to prove (b), we shall construct differential forms $\bar{\mathscr{W}}_0, \ldots, \bar{\mathscr{W}}_{n+k}$, satisfying (6). Put

$$\bar{\mathscr{W}}_p = \bigoplus_{q+r=p} \bar{\mathscr{W}}'_q \circ \bar{\mathscr{W}}''_r$$

where $\bar{\mathscr{W}}'_0, \ldots, \bar{\mathscr{W}}'_n$ and $\bar{\mathscr{W}}''_0, \ldots, \bar{\mathscr{W}}''_k$ are differential forms satisfying (6), constructed for the operators $T_1, \ldots, T_n$ and for $S_1, \ldots, S_k$. $\bar{\mathscr{W}}_p$ is a form with a compact support with values in X_p. It is easy to check that they satisfy equalities (6). In fact,

$$x - \bar{\mathscr{W}}_0(x) = x - \bar{\mathscr{W}}'_0 \circ \bar{\mathscr{W}}''_0(x) = x - \bar{\mathscr{W}}''_0(x) + \bar{\mathscr{W}}''_0(x),$$
$$\bar{\mathscr{W}}'_0 \bar{\mathscr{W}}''_0(x) \in \operatorname{Im}\alpha'_n + \operatorname{Im}\alpha''_k = \operatorname{Im}\alpha_{n+k},$$

and

$$\bar{\partial}\mathscr{W}_p = \bigoplus_{q+r=p} [(-1)^r(\bar{\partial}_z\mathscr{W}'_q)\circ\mathscr{W}''_r + \mathscr{W}'_q\circ\bar{\partial}_\lambda\mathscr{W}'']$$
$$= \bigoplus_{q+r=p} [(-1)^r\alpha'_{n-q}[\mathscr{W}'_{q+1}]\circ\mathscr{W}''_r + \mathscr{W}'_q\circ\alpha''_{k-r}[\mathscr{W}''_{r+1}]]$$
$$= \alpha_{n+k-p}[\mathscr{W}_{p+1}].$$

Hence

$$R'R''(x) = \mathscr{W}'_n\circ\mathscr{W}''_k(x) = \mathscr{W}_{n+k}(x) = R(x)$$

and (b) is proved.

In order to prove (a), it is sufficient to consider the case $k = 1$. Then we have

$$X_p \cong X'_p \oplus X'_{p-1} \text{ for } 1 \leqslant p \leqslant n, \quad X_0 \cong X_{n+1} \cong X.$$

At the point $(z, \lambda)(z_1, \ldots, z_n, \lambda) \in C^{n+1}$ the mapping α_i has the form

$$\alpha_i(z, \lambda) = \begin{bmatrix} \alpha'_i(z) & S-\lambda \\ 0 & -\alpha'_{i-1}(z) \end{bmatrix}.$$

The complex α_i, X_i is exact for the point (z, λ) in X_i if and only if for every pair $x_i \in X_i$, $x_{i-1} \in X_{i-1}$ such that

$$\alpha'_{i+1}(z)[x_i] = 0, \quad (S-\lambda)x_i = \alpha'_i(z)[x_{i-1}],$$

there exists and element $y \in X_{i-1}$ such that

$$\alpha'_i(z)[y] = x_i, \quad (S-\lambda)y - x_{i-1} \in \operatorname{Im}\alpha_{i-1}(z).$$

Evidently this is satisfied if $z \in K'$. On the contrary, suppose that for fixed $z^0 \in C^n$ the complex α_i, X_i is exact at the point (z^0, λ) for every $\lambda \in C$. We shall prove that z^0 does not belong to K. Fix an integer i, $0 \leqslant i \leqslant n$. Let x be an element of $\operatorname{Ker}\alpha'_i(z^0)$. Then the element $(0, x)$ of the space $X_i = X'_i \oplus X'_{i-1}$ belongs to $\operatorname{Ker}\alpha_{i+1}(z^0, \lambda)$ for every $\lambda \in C$. It follows that there are elements $y(\lambda) \in X'_{i-1}$ and $u(\lambda) \in X'_{i-2}$ such that

(7) $$(S-\lambda)[y(\lambda)] + \alpha'_{i-1}(z^0)[u(\lambda)] = x, \quad \alpha'_i(z^0)[y(\lambda)] = 0.$$

We can choose $y(\lambda)$ and $u(\lambda)$ to be smooth functions of λ such that for $|\lambda| > \|S\|$ $y(\lambda) = (S-\lambda)^{-1}x$, $u(\lambda) = 0$. From (7) we obtain by derivation

$$(S-\lambda)[\bar{\partial}y] + \alpha'_{i-1}[\bar{\partial}u] = 0, \quad \alpha'_i[\bar{\partial}y] = 0.$$

So, the pair of functions $y(\lambda)$, $u(\lambda)$ belong to $\operatorname{Ker}\alpha_i$ and there are smooth functions $v(\lambda) \in X'_{i-2}$, $w(\lambda) \in X'_{i-3}$ such that

$$\alpha'_{i-1}(z^0)[v(\lambda)] = y(\lambda), \quad (S-\lambda)[v(\lambda)] + \alpha'_{i-2}(z^0)[w(\lambda)] = u(\lambda).$$

For $\varrho > \|S\|$ we have

$$x = \int_{|\lambda|=\varrho} (S-\lambda)^{-1}x\,d\lambda = \int_{|\lambda|=\varrho} y(\lambda)\,d\lambda = \int_{|\lambda|\leqslant\varrho} \bar{\partial}y(\lambda)\,d\lambda\,d\bar{\lambda}$$
$$= \int_{|\lambda|\leqslant\varrho} \alpha'_{i-1}(z^0)[v(\lambda)]\,d\lambda\,d\bar{\lambda} = \alpha'_{i-1}(z^0)\Big[\int_{|\lambda|\leqslant\varrho} v(\lambda)\,d\lambda\,d\bar{\lambda}\Big].$$

Hence, x belongs to $\operatorname{Im}\alpha'_{i-1}$. It follows that for every i $\operatorname{Ker}\alpha'_i(z^0) = \operatorname{Im}\alpha'_{i-1}(z^0)$ and z^0 does not belong to the Taylor spectrum K. Assertion (a) is proved.

COROLLARY 1.2. *If the function f is holomorphic in V', we have*

$$\int_{V'} f(z)\bar{R}'\wedge dz = \int_{V'\times V''} f(z)\bar{R}\wedge dz\wedge d\lambda.$$

In order to prove that the correspondence $f \to \hat{f}$ given by formula (3) or (3′) possesses the properties of functional calculus, we shall prove two lemmas.

LEMMA 1.3. *Let B be a compact subset of C^n such that for every point $z \in C^n \setminus B$ there exist operators $U_1(z), \ldots, U_n(z)$ commuting with $T_1, \ldots, T_n$ such that*

$$\sum_{i=1}^n (T_i - z_i)U_i(z) = I_X.$$

Then B contains the Taylor spectrum of $T_1, \ldots, T_n$ and the restriction of R on $C^n \setminus B$ coincides with the class of cohomologies of the form (1).

Proof. Put

$$\mathscr{W}_p = \sum_{0<i_1,\ldots,i_{p+1}} U_{i_1}\bar{\partial}U_{i_2}\wedge \ldots \wedge \bar{\partial}U_{i_p}s_{j_1}\ldots s_{j_{n-p-1}},$$

where $\{i_1, \ldots, i_{p+1}\} \cup \{j_1, \ldots, j_{n-p-1}\} = \{1, 2, \ldots, n\}$, and s_i are generators of Λ^n. $\mathscr{W}_p$ is a form of type $(0, p)$ with values in the space $X_{n-p-1} = \Lambda^n_{n-p-1}\otimes X$. It is easy to check that the forms satisfy equalities (5). Hence

$$R = \mathscr{W}_{n-1} = (p-1)!\sum_{i=1}^n U_i\bar{\partial}U_1\wedge \ldots \wedge \widehat{\bar{\partial}U_i}\wedge \ldots \wedge\bar{\partial}U_n(-1)^i,$$

which is equivalent to (5).

LEMMA 1.4. *For the mapping $R\colon X \to H^{n-1}(\check{X}, C^n \setminus K)$ the following equalities are satisfied*:

(i) $\int_{bV} R(x)\wedge dz = x$,

(ii) $(T_i - z_i)R(x) = 0$,

(iii) $R(Ax) = AR(x)$ *for any operator A in X, commuting with $T_1, \ldots, T_n$.*

Proof. (i) Let B denote a ball in C^n such that $B \supset K$. We have

$$\int_{bV} R(x)\wedge dz = \int_{bB} R(x)\wedge dz.$$

The requirement of Lemma 1.2 is satisfied on $C^n \setminus B$ and here R can be defined by formula (1). Assertion (i) for form (1) is given in [4].

(ii) In the construction above we have $\alpha_1[\mathscr{W}_{n-1}] = \bar{\partial}\mathscr{W}_{n-2}$. Since $\alpha_1\mathscr{W}_{n-1} = ((T_1 - z_1)\mathscr{W}_{n-1}, \ldots, (T_n - z_n)\mathscr{W}_{n-1})$, all $(T_i - z_i)\mathscr{W}_{n-1}$ are $\bar{\partial}$-exact forms.

(iii) If we choose $\mathscr{W}_0, \ldots, \mathscr{W}_{n-1}$ as those in the construction of $R(x)$, then $R(Ax)$ can be constructed by using $A\mathscr{W}_0, \ldots, A\mathscr{W}_{n-1}$, which proves the assertion.

Proof of Theorem 1.1. Let $O(K)$ be the algebra of functions, holomorphic in some neighbourhood of K, and let $L(X)$ be the algebra of bounded linear operators in X. Consider the mapping $f\to\hat{f}$ of $O(K)$ in $L(X)$ given by formula (3) or (3′). Obviously it is a linear mapping with the following properties:

(i) $\hat{1} = I_X$,

(ii) $\hat{z}_i = T_i$, $i = 1, 2, \ldots, n$,

(iii) For any f the operator $\hat{f}$ belongs to the bicommutant of $T_1, \ldots, T_n$.

This follow immediately from assertions (i), (ii), (iii). Now we have only to prove the multiplicativity of the mapping $f\to\hat{f}$, i.e. to prove that

$$\hat{f}\circ\hat{g} = \widehat{f\cdot g}, \quad f, g\in O(K), \tag{8}$$

which is equivalent to

$$\int_V\int_V f(z)g(\lambda)\bar{R}\circ\bar{R}\,dz\,d\lambda = \int_V f(z)g(z)\bar{R}\,dz, \tag{8′}$$

where V is a neighbourhood of the Taylor spectrum K of $T_1, \ldots, T_n$ and $\bar{R}$ is the form with support in V constructed above. Denote by $\bar{R}^2$ the analogous differential form for $2n$ operators $T_1, \ldots, T_n, T_1, \ldots, T_n$. Using Lemma 1.2 (b), we infer that (8) is equivalent to

$$\int_{V\times V}\big(f(z)-f(\lambda)\big)g(z)\bar{R}^2(z)\,dz\,d\lambda = 0.$$

It is easy to see that the Taylor spectrum of $T_1, \ldots, T_n, T_1, \ldots, T_n$ in C^{2n} is contained in the diagonal

$$\Delta = \{(z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_n)\colon z_i = \lambda_i,\ i = 1, 2, \ldots, n\}.$$

Indeed, for the point $(z, \lambda)\in C^{2n}$, $z_i\neq\lambda_i$, we have

$$I_X = (T_i-z_i)(\lambda_i-z_i)^{-1}I_X-(T_i-\lambda_i)(\lambda_i-z_i)^{-1}I_X$$

and by Lemma 1.3 (z, λ) does not belong to the joint spectrum.

For fixed $f\in O(K)$ we can choose $\varepsilon > 0$ in such a way that at all points $(z, \lambda)\in C^{2n}$ such that $|z-\lambda| < \varepsilon$ we have

$$f(z)-f(\lambda) = \sum_{i=1}^{n}(z_i-\lambda_i)h_i(z, \lambda),$$

where $h_i(z, \lambda)$ are holomorphic functions defined in some neighbourhood V' of $\Delta(K) = \{(z, \lambda), z = \lambda, z\in K\}$. By (ii) we obtain

$$(z_i-\lambda_i)\bar{R}^2 = (T_i-\lambda_i)\bar{R}^2-(T_i-z_i)\bar{R}^2 = 0$$

and

$$\int_{V\times V}\big(f(z)-f(\lambda)\big)g(z)\bar{R}^2\,dz\,d\lambda = \sum_{i=1}^{n}\int_{V'}h_i g(z)(z_i-\lambda_i)\bar{R}^2\,dz\,d\lambda = 0.$$

The proof is finished.

2

In this section we give some properties of maximality for the Taylor spectrum.

Let K be a compact subset of C^n. By $\hat{K}$ we denote the hull of holomorphy of K, i.e. the compact of maximal ideals of $O(K)$. The mapping

$$p\colon m\to\big(m(z_1), \ldots, m(z_n)\big)$$

defines projection of $\hat{K}$ on a subset of C^n.

Let $T_1, \ldots, T_n$ be commuting linear operators, let $K = \sigma_T(T_1, \ldots, T_n)$ be their Taylor spectrum, and let $f\to\hat{f}$ be the Taylor functional calculus. Suppose that K' is another compact subset of C^n such that there exists a functional calculus, i.e. a homomorphism $O(K)\to L(X)\colon f\to\check{f}$ with properties (i), (ii), (iii).

THEOREM 2.1. *Under the assumptions and notations of this section we have*

(a) *$p(\hat{K}')$ contains K.*

(b) *For any $f\in O(K')\cap O(K)$ we have $\hat{f} = \check{f}$.*

Proof. Suppose that $O(K')$ has a finite number of generators $f_1(z), \ldots, f_k(z)$ (in the general case we can consider K' as a limit of a decreasing system of compact sets with this property). Then $\hat{K}'$ can be considered as a subspace of C^{n+k} and the mapping

$$i\colon (z_1, \ldots, z_n)\to\big(z_1, \ldots, z_n, f_1(z), \ldots, f_k(z)\big)$$

is an injection of K' into $\hat{K}'$. Denote by T_{n+i} the operator $\check{f}_i$, $i = 1, 2, \ldots, k$. For any point $z_0 = (z_1^0, \ldots, z_{n+k}^0)\notin\hat{K}'$ there exist holomorphic functions $u_1, \ldots, u_{n+k}$, defined in some neighbourhood of $\hat{K}'$ in C^{n+k}, and such that

$$\sum_{i=1}^{n+k}(z_i-z_i^0)u_i(z) = 1.$$

This means that

$$\sum_{i=1}^{n+k}(T_i-z_i^0)\circ\check{u}_i = I_X,$$

and therefore $\sigma_T(T_1, \ldots, T_{n+k})$ is a subset of $\hat{K}'$. Lemma 1.2 (a) implies that $\sigma_T(T_1, \ldots, T_n)\subset p(K)$. Since the set of all polynomials of $z_1, \ldots, z_n, f_1, \ldots, f_k$ is dense in $O(\hat{K}')$, the functional calculus $f\to\check{f}$ coincides with the Taylor functional calculus for $T_1, \ldots, T_{n+k}$. Using Corollary 1.2, we obtain assertion (b).

EXAMPLE. Let a Banach space X be represented as $X_1\oplus X_2$. Let T_1, S_1 be a pair of commuting linear operators in X_1 such that $\sigma_T(T_1, S_1) = K = \{(z_1, z_2)\in C^2\colon |z_1|\leqslant 1, |z_2|\leqslant 1\}$ and let T_2, S_2 be commuting operators in X_2 such that $\sigma_T(T_2, S_2) = \{(0, 0)\}$. Let $T = T_1\oplus T_2$, $S = S_1\oplus S_2$ be operators in X. Then

$$\sigma_T(T, S) = \{(z_1, z_2), |z_1|\leqslant 1, |z_2|\leqslant 1\}.$$

Put $K' = \{z_1, z_2)\colon |z_1| = 1, |z_2|\leqslant 1\}\cup\{(z_1, z_2)\colon |z_1|\leqslant 1, |z_2| = 1\}\cup\{(0, 0)\}$. Any function $f\in O(K')$ can be represented as a pair (f_1, f_2), where $f_1\in O(K)$, f_2 is holomorphic in a neighbourhood of $(0, 0)$. We can define $\check{f} = \hat{f}_1(T_1, S_1)\oplus\hat{f}_2(T_2, S_2)$.

This is a functional calculus with properties (i), (ii), (iii) for the operators T, S which is larger than the Taylor functional calculus.

Now we give an axiomatic characterization of the Taylor spectrum. Let $\mathfrak{X}$ be a category of Banach spaces with a fixed action of n commuting operators $T_1, \ldots, T_n$. Let all the morphisms in $\mathfrak{X}$ be bounded linear mappings, commuting with $T_1, \ldots, T_n$. Denote by $\sigma_\pi(T, X) = \sigma_\pi(T_1, \ldots, T_n, X)$ the complement in C^n of the set of all points $z = (z_1, \ldots, z_n)$ such that the linear span of $\operatorname{Im}(T_i - z_i)$, $1, 2, \ldots, n$, contains all X.

THEOREM 2.2. *Suppose that for any $X \in \mathfrak{X}$ we have a compact subset $\sigma(T, X) \subset C^n$ with the following properties:*

(a) *If X' is a direct sum of finitely many copies of X, then $\sigma(T, X') = \sigma(T, X)$.*

(b) *For every X we have $\sigma(T, X) \supset \sigma_\pi(T, X)$.*

(c) *If Y is an invariant Banach subspace of the Banach space X, then*

$$\sigma(T, Y) \subset \sigma(T, X) \cup \sigma(T, X/Y).$$

Then, for every $X \in \mathfrak{X}$, $\sigma(T, X)$ contains the Taylor spectrum of $T_1, \ldots, T_n$ in X.

Remark. Obviously the Taylor spectrum has properties (a) and (b); property (c) was proved by Taylor in [3].

Proof. Let X be an object of $\mathfrak{X}$, and let z be a point in the Taylor spectrum of $T_1, \ldots, T_n$ in X. Then the complex $E(z)$, defined by (4), is not exact. Let i be an integer such that

$$H_i(E(z)) \neq 0, \quad H_{i+1}(E(z)) = \ldots = H_n(E(z)) = 0$$

(here $H_i(E(z)) = \operatorname{Ker}\alpha_{i+1}(z)/\operatorname{Im}\alpha_i(z)$). Put $Y_i = \operatorname{Ker}\alpha_{i+1}(z)$ and consider the corresponding isomorphisms

(9) $\quad X_k/Y_k \cong \operatorname{Im}\alpha_{k+1}(z) = Y_{k+1}, \; k = i, i+1, \ldots, n-2, \quad X_{n-1}/Y_{n-1} = X.$

If $y \in Y_i$, we have $(T_l - z_l)y \in \operatorname{Im}\alpha_{l-1}(z)$, $l = 1, \ldots, n$. In fact, let

$$y = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_l \leqslant n} y_{i_1 \ldots i_l} s_{i_1} \ldots s_{i_l} = y' + y'',$$

where

$$y' = \sum_{l \in \{i_1, \ldots, i_l\}} \ldots, \quad y'' = \sum_{l \notin \{i_1, \ldots, i_l\}} \ldots$$

Then $\alpha_{i+1}(z)y' = (T_l - z_l)y''$. Let $u \in X_{i-1}$ be such that $u \wedge s_l = y'$. Then $\alpha_i(z)u = y$.

Since $\operatorname{Im}\alpha_i \neq \operatorname{Ker}\alpha_{i+1}$, we have

$$z \in \sigma_\pi(T, \operatorname{Ker}\alpha_{i+1}(z)) \subset \sigma(T, \operatorname{Ker}\alpha_{i+1}(z)).$$

Suppose that $z \notin \sigma(T, X)$. Since all X_k are direct sums of finitely many copies of X, we have $z \notin \sigma(T, X_k)$. Using the isomorphisms (9), we infer by (c) that $z \notin \sigma(T, Y_{n-1}), \ldots, z \notin \sigma(T, Y_i)$ — a contradiction. The theorem is proved.

3

DEFINITION. We shall say that the point z *belongs to the Fredholm spectrum* $\sigma_F(T_1, \ldots, T_n)$ of the operators $T_1, \ldots, T_n$ if z belongs to $\sigma_T(T_1, \ldots, T_n)$ and all homology groups $H_i(E(z))$ are finite-dimensional. By $\sigma_e(T_1, \ldots, T_n)$ (essential Taylor spectrum) we shall denote the complement of $\sigma_F(T_1, \ldots, T_n)$ in $\sigma_T(T_1, \ldots, T_n)$.

THEOREM 3.1. *The essential Taylor spectrum possesses the "projection property".*

THEOREM 3.2. *$\sigma_F(T_1, \ldots, T_n)$ is an analytic subset of C^n.*

It is easy to prove that to every function holomorphic in some neighbourhood of σ_e corresponds an element of the Calkin algebra $L(X)/K(X)$ (quotient algebra of the algebra of all bounded operators over the ideal of all compact operators). This result is interesting only in the one-dimensional case. In fact, if $\dim\sigma_F \geqslant 2$, then σ_e contains the boundary of σ_F, and any function holomorphic in some neighbourhood of σ_e can be extended holomorphically to σ_T. Hence, the holomorphic functional calculus in $L(X)/K(X)$ is trivial, i.e. it is induced by some functional calculus in $L(X)$. In order to obtain a nontrivial functional calculus, it is necessary to consider a larger algebra of functions, and to assume the corresponding requirements for the "growth of the resolvent". Another obstacle to the existence of nontrivial functional calculus can be the geometry of the spectrum. There exists a Banach space and operators T_1, T_2 such that their joint spectrum coincides with the unit polydisc, the essential spectrum is the boundary of the polydisc, and there exists no nontrivial functional calculus. Now we have the following

THEOREM 3.3. *Let $T_1, \ldots, T_n$ be commuting bounded linear operators in X such that*

(a) *$\sigma_T(T_1, \ldots, T_n) = \bar{D}$, $\sigma_e(T_1, \ldots, T_n) = \mathrm{b}D$, where D is a strongly pseudoconvex domain in C^n.*

(b) *For every K and sufficiently small open set U we can find finite dimentional subspaces $H_1(z) \subset X_{k-1}$, $H_2(z) \subset X_k$, such that for every holomorphic vector-function $x(z) \in \operatorname{Ker}\alpha_k(z)$ on U there exist a sequence $y^n(z) \in X_{k-1}$ of holomorphic vector-functions such that:*

$$\lim_n \alpha_k(z)[y^n(z)] = x(z) \bmod H_2(z) \quad \text{on} \quad U$$

and

$$\|y^n(z) \bmod H_1(z)\| \leqslant M(\operatorname{dist}(z, \mathrm{b}D)) \quad \text{on} \quad U \setminus \mathrm{b}D$$

where $M(t)$ is a positive function satisfying the estimate

$$\int_0^\delta \ln\ln M(t)\,dt < \infty.$$

Then there exists a nontrivial functional calculus $B \to L(X)/K(X)$ for some regular Banach algebra B of functions, defined on $\sigma_e(T_1, \ldots, T_n)$.

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THE STRUCTURE OF A CLASS OF BANACH ALGEBRAS GENERATED BY A BANACH SPACE

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1

Let E be a complex Banach space. We shall investigate the structure of a commutative Banach algebra with identity A which has the following three properties:

(i) A contains (an isometric copy of) E.

(ii) Every linear functional (l.f.) on E of norm at most one can be extended to a multiplicative linear functional (m.l.f.) on A.

(iii) The algebra generated by E is dense in A.

Property (ii) will be called the *multiplicative extension property* (m.e.p.).

2

It was historically first proved that some subspaces of certain given Banach algebras had the multiplicative extension property. The problem of giving a general characterization of subspaces with the m.e.p. was then raised. There are examples of algebras which have no subspaces with the m.e.p.: for instance, every finite-dimensional algebra and the algebra of continuous functions on a compact scattered space.

On the other hand, the following are two positive examples:

(a) Let X be a compact convex and balanced subset of a Hausdorff locally convex topological vector space. Then the subspace of continuous linear functionals on X has the m.e.p. in $C(X)$.

(b) If A is any function algebra on an uncountable compact metrizable space there exists an isometry $T\colon A \to A$ such that $T(A)$ has the m.e.p. in A.

We now give some necessary and sufficient conditions for a subspace E of a Banach algebra A to have the m.e.p.

(1) Let $\{x_i\colon i \in I\}$ be a set of linearly independent elements of E whose span is dense in E. Then E has the m.e.p. if and only if the joint spectrum of the x_i's is balanced and convex.

(2) E has the m.e.p. if and only if every finite-dimensional subspace of E has the m.e.p.