При $k=1$ из этой теоремы мы получаем теорему М. Г. Гасымова [3] и А. Г. Костюченко, которая обобщает результат М. Г. Крейна и Г. К. Лангера [5] при самосопряженных операторных пучков второго порядка.

Отметим, что при выполнении условия теоремы 4 корневые векторы $K(\Pi_+)$ пучка $P(\lambda)$ отвечающих собственным значениям из правой полуплоскости также образует k-кратно полную систему в H.

Теорема 5. *Пусть выполняются условия теоремы* 2 (*теоремы* 3) *и* $A^{-1} \in C_p$ $(0<p<\infty)$, $A_j A^{-j} \in C_\infty$ $(j=1, \ldots, n)$. *Тогда система* $K(\Pi_-)$ *пучка* (1) $(2k+1)$-*кратно* ($(2k-1)$-*кратно*) *полна в* H.

Отметим, что если в теоремах 4 и 5 оператор $A^{-1} \in C_1$, то для любого набора m векторов $(m=k, 2k+1, (2k-1))$, $f_j \in \mathscr{D}(A^{n-j-1/2})$ разложение по системе $K(\Pi_-)$ m-кратно суммируема методом Абеля.

Литература

[1] М. В. Келдыш, *О собственных значениях и собственных функциях некоторых классов несамосопряженных уравнений*, ДАН СССР, 77 (1) (1951), 11–14.

[2] Дж. Э. Аллахвердиев, *О многократно полных системах и несамосопряженных операторах зависящих от параметра*, ДАН СССР 166 (1) (1966), 11–14.

[3] М. Г. Гасымов, *О кратной полноте части собственных и присоединенных векторов полиномиальных операторных пучков*, Изв. АН Арм. ССР, математика VI, 2–3 (1971), 131–147.

[4] Ю. А. Палант, *Об одном признаке полноты системы собственных и присоединенных векторов полиномиального пучка операторов*, ДАН СССР 141 (3) (1961), 558–560.

[5] М. Г. Крейн, Г. К. Лангер, *О некоторых математических принципах линейной теории демпфированных колебаний континумов*, Труды Международного симпозиума по применению теории функций комплексного переменного в механике сплошной среды, Москва 1965, 283–322.

[6] М. Г. Гасымов, *К теории полиномиальных операторных пучков*, ДАН СССР 199 (4) (1971), 747–750.

[7] A. Friedman, M. Shinbrot, *Nonlinear eigenvalue problems*, Acta. Math. 121 (1–2) (1968), 77–125.

[8] R. E. L. Turner, *A class of nonlinear eigenvalue problems*, J. Funct. Anal. 2 (1968), 297–322.

[9] С. С. Мирзоев, *Двукратная полнота части собственных и присоединенных векторов полиномиальных пучков четвертого порядка*, Изв. АН Азерб. ССР, сер. физ. мат. наук 6 (1974).

[10] Г. Е. Радзиевский, *Об одном методе доказательства полноты векторов оператор функций*, ДАН СССР 214 (2) (1974), 291–294.

Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

A NOTE ON GENERAL DILATION THEOREMS

W. MLAK

Institute of Mathematics of the Polish Academy of Sciences,
Kraków, Poland

Summary. Positive definite scalar functions on homogeneous spaces are one of the main objects of representation theory of groups. Functions of this type are on the other hand closely related to induced representations of groups. Several generalizations of results concerning such functions to the case of operator functions have been extensively studied by Kunze in [1], whose paper inspired the investigations discussed below. Inspiration goes also back to recent trends of general dilation theory as presented in [2], [3], [6].

We introduce the notation: Let E be a complex Banach space and E^* its topological dual. By $\overline{L}(E)$ we denote the space of all antilinear bounded operators from E into E^*. Given two Banach spaces M, N, we denote by $L(M, N)$ the space of all linear bounded operators from M into N. We write $L(M) \overset{\mathrm{df}}{=} L(M, M)$.

Definition (see [2], [6]). Let Z be a set and $B(\cdot, \cdot)\colon Z\times Z \to \overline{L}(E)$ an operator valued function. We say that $B(\cdot, \cdot)$ is *positive definite* (abbreviated: p.d.) and write $B \gg 0$ if for every n, every $f_1, \ldots, f_n \in E$ and every $z_1, \ldots, z_n$ the inequality

$$\sum_{i,k=1}^{n} \big(B(z_i, z_k) f_i\big)(f_k) \geqslant 0$$

holds true.

The factorization property of positive definite functions presented below plays a basic role. For p.d. scalar functions it is atributed to Aroszajn and Kolmogorov, and its abstract operator version appears in papers [1] and [7] for Hilbert space valued operator functions. In both these papers the factorization is used essentially in connection with certain dilation problems. The extension of factorization theorem to Banach space valued operator functions and an explicit use of it to dilation problems appears in [2] and [6] and reads as follows:

(A–K) Let $B(\cdot, \cdot)\colon Z\times Z \to \overline{L}(E)$ be a positive definite operator valued function. Then there is a Hilbert space K and an operator valued function $X(\cdot)\colon Z \to L(K)$ such that $B(u, v) = X(v)^* X(u)$ for all $u, v \in Z$. If K is minimal, i.e. $K = \bigvee_{u\in Z} X(u)E$, then $X(\cdot)$ and K are unique up to unitary equivalence. This means that if $X_i(\cdot)$: $Z \to L(K_i)$, $K_i = \bigvee_{u\in Z} X_i(z)E$ and $X_i(v)^*X_i(u) = B(u, v)$ for $i = 1, 2$ and $u, v \in Z$, then there is a unitary map U: $K_1 \to K_2$ such that $UX_1(u) = X_2(u)$ for all $u \in Z$.

If K in (A–K) is minimal, then the expression of B in the form $B(u, v) = X(v)^*X(u)$ is called canonical. When identifying canonical expressions, which are unitarily equivalent, we can say that B has a unique canonical expression.

Let S be a certain semi-group of mappings of the set Z into itself, and $B(\cdot,\cdot)$: $Z\times Z \to \overline{L}(E)$. The main idea involved in [1], [2], [6] concerning the construction of dilation π is to define it directly by formula $\pi(u)X(t)f = X\big(u(t)\big)f$ for $u \in S$, $t \in Z$ and $f \in E$, $B(t, s) = X(s)^*X(t)$ being the canonical expression of B. The linear operator $\pi(u)$ will be well defined if the following Getoor type condition holds true:

(G) For every n, $z_1, \ldots, z_n \in Z$, $f_1, \ldots, f_n \in E$, the equality

$$\sum_{i,k=1}^{n} \big(B(z_i, z_k)f_i\big)(f_k) = 0$$

implies the equality

$$\sum_{i,k=1}^{n} \Big(B\big(u(z_i), u(z_k)\big)f_i\Big)(f_k) = 0.$$

To be more precise, (G) implies that $\pi(u)$ given by

$$\pi(u) \sum_{j=1}^{n} X(t_j)f_j = \sum_{j=1}^{n} X\big(u(t_j)\big)f_j$$

is well defined and maps the linear manifold spanned by $X(t)f$ ($t \in Z, f \in E$) into itself and, moreover, $\pi(u)\circ\pi(v) = \pi(uv)$ for $u, v \in S$. If we require $\pi(u)$ to be bounded, we have to assume that the following boundedness condition (of type introduced by Sz.-Nagy in his pioneer paper [5]) is fulfilled:

(B) For every $n, z_1, \ldots, z_n \in Z$, $f_1, \ldots, f_n \in E$

$$\sum_{i,k=1}^{n} \Big(B\big(u(z_i), u(z_k)\big)f_i\Big)(f_k) \leqslant \varrho(u) \sum_{i,k=1}^{n} \big(B(z_i, z_k)f_i\big)(f_k)$$

with some finite $\varrho(u)$ independent of n, z_i and f_i.

Indeed, if (B) holds true, then (G) holds true and $\pi(u)$ is well defined and, again by (B), $\|\pi(u)\|^2 \leqslant \varrho(u)$. We are then able to extend $\pi(u)$ onto the whole space $K = \bigvee_{z\in Z} X(z)E$ and then get a representation $\pi(\cdot)$: $S \to L(K)$ (provided (B) holds for all $u \in S$), which satisfies the equality

$$B\big(u(t), v(s)\big) = X(s)^*\pi(v)^*\pi(u)X(t). \tag{1}$$

Formula (1) in this general setting appears implicitly in [2] for semi-groups and in [6] for $S = Z =$ a group ($u(t) = ut$). This formula is basic for dilation theory.

To give an illustration, assume that Z is a multiplicative semi-group and define $S = \{u(\cdot)\colon u(t) = ut,\ u \in S\}$. Suppose e is the unit of Z. Then taking $s = t = e$, we derive from formula (1)

$$B(u, v) = R^*\pi(v)^*\pi(u)R$$

where $R = X(e)$, and we call $\pi(\cdot)$ an R-dilation for $B(\cdot,\cdot)$.

Following the ideas of the theory of group representations in homogeneous spaces and the related so-called multipliers used by Kunze in [1], we assume in the sequel that we are given the following objects:

(a) A positive definite function $B(\cdot,\cdot)$: $Z\times Z \to \overline{L}(E)$.

(b) A semi-group S of mappings of Z into itself.

(c) An operator valued function $V(\cdot,\cdot)$: $S\times Z \to L(E)$ which satisfies the equa ion

$$V(uv, t) = V\big(u, v(t)\big)V(v, t) \quad \text{for} \quad u, v \in S \text{ and } t \in Z.$$

The function $V(\cdot,\cdot)$ satisfying (c) will be called a *multiplier*.

Let $B(t, s) = X(s)^*X(t)$ ($X(t) \in L(E, K)$) be the canonical form of B. Then the formula

$$\pi(u) \sum_{i=1}^{n} X(t_i)f_i = \sum_{i=1}^{n} X\big(u(t_i)\big)V(u, t_i)f_i$$

will define an operator $\pi(u)$ on the manifold M spanned by $X(t)f$ ($t \in Z, f \in E$) if the following analogue of Getoor condition will be satisfied:

(G_1) For every n, $t_1, \ldots, t_n \in Z$, $f_1, \ldots, f_n \in E$ the equality

$$\sum_{i,k=1}^{n} \big(B(t_i, t_k)f_i\big)(f_k) = 0$$

implies the equality

$$\sum_{i,k=1}^{n} \Big(V(u, t_k)^*B\big(u(t_i), u(t_k)\big)V(u, t_i)f_i\Big)(f_k) = 0.$$

If (G_1) holds true for every $u \in S$ then $\pi(u)\circ\pi(v) = \pi(uv)$, where $\pi(u)$, $\pi(v)$ are linear operators in M. Indeed, since

$$\Delta = \pi(u)\Big(\pi(v)\sum_j X(t_j)f_j\Big) = \pi(u)\sum_j X\big(v(t_j)\big)V(v, t_j)f_j$$
$$= \sum_j X\big((uv)(t_j)\big)V\big(u, v(t_j)\big)V(v, t_j)f_j$$

and since, by (c),

$$V\big(u, v(t_j)\big)V(v, t_j) = V(uv, t_j),$$

we get by the definition of $\pi(uv)$

$$\Delta = \sum_j X\big((uv)(t_j)\big)V(uv, t_j)f_j = \pi(uv)\sum_j X(t_j)f_j,$$

which proves the claim. The natural analogue of the boundedness condition (B) now reads as follows:

(B_1) There is a finite positive function $\varrho\colon S \to R^+$ such that for every $u \in S$, every n and every $t_1, \ldots, t_n \in Z$, $f_1, \ldots, f_n \in E$ the inequality

$$\sum_{i,k=1}^{n} \big(V(u,t_i)^* B\big(u(t_i), u(t_k) V(u,t_k) f_i\big)(f_k)\big) \leqslant \varrho(u) \sum_{i,k=1}^{n} \big(B(t_i,t_k)f_i\big)(f_k)$$

holds true.

It is plain that (B_1) implies (G_1), and so $\pi(u)$ makes sense. Then, since the left-hand side of the inequality in (B_1) equals to $\|\pi(u)\sum_j X(t_j)f_j\|_K^2$ and the right-hand side to $\varrho(u)\|\sum_j X(t_j)f_j\|_K^2$, $\pi(u)$ extends uniquely to a bounded operator in $L(K)$, which we denote also by $\pi(u)$. Certainly $\pi(\cdot)$ is a representation of S, i.e. $\pi(uv) = \pi(u)\pi(v)$ for $u, v \in S$. Notice, by the way, that if $V(u,s) \equiv I_E$ (= the identity operator in E), then the above construction of $\pi(\cdot)$ is just the same as that performed under condition (B).

We are now able to formulate our dilation theorem.

THEOREM 1. *Suppose that Z, $B(\cdot,\cdot)$, S and $V(\cdot,\cdot)$ satisfy* (a), (b) *and* (c). *If the boundedness condition* (B_1) *is fulfilled then there is a representation $\pi(\cdot)\colon S \to L(K)$ $(K = \bigvee_{z\in Z} X(z)E)$ such that*

$$V(v,s)^* B\big(u(t), v(s)\big) V(u,t) = X(s)^*\pi(v)^*\pi(u)X(t) \tag{2}$$

for all $t, s \in Z$; $u, v \in S$, where $B(s,t) = X(s)^*X(t)$ is the canonical representation of $B(\cdot,\cdot)$.

Remark 1. We can always extend S to a unital semi-group $\tilde{S}$ by adjoining the identity map $e\colon Z \to Z$ $(e(t) = t$ for $t \in Z)$ and to extend $V(\cdot,\cdot)$ by writing $V(e,t) \equiv I_E$. It is then plain that (c) is satisfied for $V(\cdot,\cdot)$ extended as above to $\tilde{S}$ and the boundedness conditions for $u = e$ remains valid, for it reduces simply to equality with $\varrho(e) = 1$. The representation $\pi(\cdot)$ of S in Theorem 1 then extends to the unital representation $\tilde{\pi}(\cdot)\colon \tilde{S} \to L$ and (2) remains true with π being replaced by $\tilde{\pi}$. Notice that if Z is itself a semi-group without unit and S is the semi-group of actions on Z by formula $u(s) = us$, $\tilde{S}$ is usually different (up to semi-group isomorphism) from the formal algebraic unitization of Z.

The above theorem includes the general theorem of [2] (simply by taking $V(u,t) \equiv I_E$), and consequently also the theorem of [5], and is a slight extension to semi-groups of Theorem 2 of [1], which we will briefly discuss below.

To begin with, suppose that assumptions of Theorem 1 are satisfied and Z is a semi-group and define S to be the semi-group of actions by elements of Z, namely $S = \{u\colon Z \to Z \mid u(s) = us \text{ for } s \in Z\}$. Assume additionally that Z is unital, i.e. S has a unit $e(s) = es = s$ $(s \in Z)$, where e is the unit of Z, and

$$V(e,s) = I_E \quad \text{for} \quad s \in Z. \tag{d}$$

Then $\pi(e)X(t)f = X(et)V(e,t)f = X(t)f$, which proves that $\pi(\cdot)$ is unital, i.e. $\pi(e) = I_K$ in this case. Moreover, (c) now implies also

$$V(u,e) = V(u,e)^2.$$

If $V(u,e)$ = the constant operator, then by (d) we conclude that $V(u,e) = I_E$. n this case the relation (2) can be written in an orthodox dilation form as

$$B(u,v) = R^*\pi(v)^*\pi(u)R \tag{e}$$

where $R = X(e)$.

Let us now go back to general assumptions of Theorem 1. It is obvious that if $B(\cdot,\cdot)$ is projectivity invariant, i.e. if (a), (b) hold true and

$$V(u,s)^* B\big(u(t), u(s)\big) V(u,t) = B(t,s) \tag{3}$$

for $u, v \in S$; $t, s \in Z$, then (B_1) holds true, and $\pi(\cdot)$ are isometries. We hence derive the following theorem:

THEOREM 2. *Let $B(\cdot,\cdot)\colon Z\times Z \to \overline{L}(E)$ be a positive definite operator function which satisfies* (3), *where $V(\cdot,\cdot)$ satisfies* (a), (b), (c). *Then* (2) *holds true and $\pi(\cdot)\colon S \to L(K)$ is an isometric representation of S. If, additionally, S is unital with unit e $(e(t) \equiv t$ for $t \in Z)$ and $V(e,s) = I_E$ for $s \in Z$, then $\pi(\cdot)$ is a unital representation. Also* (e) *holds true if Z is a unital semi-group with S = semi-group of actions.*

Proof. Only the last statement requires a proof. By definition,

$$\pi(e)X(t)f = X\big(e(t)\big)V(e,t)f = X(t)f. \qquad \blacksquare$$

Assume now that S is a group of transformations of Z onto Z $\big(e(t) = t\big)$ and let all assumptions of Theorem 2 be satisfied. Then:

$$\pi(\cdot)\colon S \to L(K) \text{ is a unitary representation.} \tag{4}$$

The above statement is just the algebraic part of Theorem 2 of [1], slightly extended, namely to Banach space valued operator functions. To prove (4), we first notice that $\pi(\cdot)$ is by Theorem 2 an isometric unital representation. It is then sufficient to prove that the range of every $\pi(u)$ is all of $K = \bigvee_{t\in Z} X(t)E$. We have by definition

$$\pi(u^{-1})X\big(u(t)\big)f = X(t)V(u^{-1},t)f.$$

By (c) and (d) we get

$$V(v,t)V\big(v^{-1},v(t)\big) = V\Big(v, v^{-1}\big(v(t)\big)\Big)V\big(v^{-1},v(t)\big) = V\big(e,v(t)\big) = I_E$$

and

$$V\big(v^{-1},v(t)\big)V(v,t) = V(v^{-1}v,t) = I_E.$$

It follows that $V(v,t)$ has a 2-sided inverse in $L(E)$. Consequently $V(u^{-1},t)E = E$, which by the previous equality involving $\pi(u^{-1})$ shows that the range of $\pi(u^{-1})$ is dense in K, as was to be proved.

Remark 2. If $Z(=S) =$ a group, and $V(u, t) \equiv I_E$, then the above statement is just the assertion of Naĭmark dilation theorem — see [4], [5], [6].

Remark 3. Using standard arguments of dilation theory — see [3], [5], one can prove suitable theorems, which explain how some continuity properties of B and $V(\cdot, \cdot)$ imply such properties of $\pi(\cdot)$.

References

[1] R. A. Kunze, *Positive definite operator valued kernels and unitary representation*, Functional Analysis, Proc. Conf. Univ. California, Irvine; Washington 1967, 235–247.
[2] P. Masani, *An explicit treatment of dilation theory* (preprint, 1975, Autumn).
[3] W. Mlak, *Dilations of Hilbert space operators* (to appear in Dissert. Math. Ser.).
[4] M. A. Naĭmark, *Positive definite operator functions on a commutative group* (in Russian with English summary), Bull. Izv. Ac. Sc. USSR ser. mat. (1943), 237–244.
[5] B. Sz.-Nagy, *Prolongements des transformations de l'espacee de Hilbert qui sortent de ces espace.* Appendix au livre *Leçons d'analyse fonctionelle* par F. Riesz et B. Sz.-Nagy, Akadémiai Kiado, Budapest 1955.
[6] B. Sz.-Nagy and A. Koranyi, *Operator theoretische Behandlung...* Acta Math. 100 (1958), 171–202.
[7] A. Weron, *Prediction theory in Banach spaces* (Karpacz Winter School on Probability), Lect. Notes in Math., Springer Verlag, 1975, 213–234.

Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

ON JORDAN MODELS OF C_0-CONTRACTIONS

VLADIMIR MÜLLER

Czechoslovak Academy of Sciences, Institute of Mathematics, Praha, Czechoslovakia

The lecture was devoted to the results of paper [4].

In [1] the following theorem was proved:

THEOREM. *Let T be a contraction of the class C_0 on a separable Hilbert space Then there exists a unique (up to constant factors of modulus* 1) *sequence* $\{m_i\}_{i=1}^{\infty}$ *of inner scalar functions such that*:

(1) m_{i+1} *divides* m_i *for each* i,

(2) T *is quasisimilar to* $S(m_1) \oplus S(m_2) \oplus \ldots$ *(the "Jordan model" of* T).

In [3] and [5] it was proved that if T has finite defect indices $\delta_T = \delta_{T*} = n$ then, for $i = 1, 2, \ldots, n$, m_i is equal to the $(n-i+1)$-th invariant factor of the characteristic function of T.

In [7] the problem was raised what is the relation of the functions m_i to the characteristic function of T in the general case. An answer to this question was given independently in [4] and [2]. The main result of [4] is the following theorem:

THEOREM. *Let T be an operator of class C_0 acting on a separable Hilbert space, Θ its characteristic function and let Ω be a contractive analytic function such that $\Theta\Omega = \Omega\Theta = \psi \cdot I_n$ where $\psi \in H^\infty$ is inner and n is the defect index of T (such an Ω exists by* [6], VI.3.1). *Let $S(m_1) \oplus S(m_2) \oplus \ldots$ be the Jordan model of T. Then $m_r = \psi/\varepsilon_r(\Omega)$ for every natural number $r \leqslant n$, where $\varepsilon_r(\Omega)$ denotes the r-th invariant factor of Ω (if n is finite then in this notation $m_i = 1$ for $i > n$).*

References

[1] H. Bercovici, C. Foiaş and B. Sz.-Nagy, *Compléments à l'étude des opérateurs de classe C_0*, III, Acta Sci. Math. 37 (1975), 313–322.
[2] H. Bercovici and D. Voiculescu, *Tensor operations on characteristic functions of C_0 contractions*, Acta Sci. Math. 39 (1977), 205–231.
[3] B. Moore III and E. A. Nordgren, *On quasi-equivalence and quasi-similarity*, Acta Sci. Math. 34 (1973), 311–316.
[4] V. Müller, *On Jordan models of C_0-contractions*, Acta Sci. Math. 40 (1978), 309–313.