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# THE SZEGÖ AND BEURLING THEOREMS AND OPERATOR FACTORIZATIONS

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In 1949 Beurling gave the now famous factorization of analytic functions on the unit disk as the product of an inner function and an outer function. Years earlier Szegö had solved the somewhat similar problem of describing the positive functions on the unit circle that can be written as the modulus of an analytic function. (The best known contemporary reference for these theorems is Hoffman's book [3]). Within the past few decades many theorems in operator theory have exhibited some resemblance to the above results. Not always has the relationship between the operator theory and the classical function theory been clear. The purpose here is to describe one direction in which such research has moved, to tie together some theorems that have appeared in the past few years, and to answer a few previously unanswered questions concerning these theorems. Most of the ideas contained herein arose in conversations with Ralph Gellar. In particular, the multiplicity theorem (Theorem 4) was his conjecture.

First we briefly review the necessary language of Hardy spaces. For each of the $L^p$ spaces of functions on the unit circle, $H^p$ is the subspace consisting of those functions whose Fourier coefficients of negative index are all zero. In $H^2$, an *outer function* is a function $f$ having the property that $fH^\infty$ is dense in $H^2$, and an *inner function* is a function that is of modulus one almost everywhere on the unit circle. The *bilateral shift* is the operator $U\colon L^2 \to L^2$ defined by $U\colon f(z) \to zf(z)$. The *unilateral shift* $S$ on $H^2$ is the operator $S = U|H^2$.

Recall also that there is an easy description of the operators that commute with $U$ or $S$. (We shall assume throughout that "operator" means "bounded operator." Also, for simplicity in the multiplicity discussion surrounding Theorem 4 we shall assume that the underlying Hilbert space is separable.) The commutant of $U$ is $L^\infty$, i.e. each operator $B$ that commutes with $U$ is of the form $A = A_\varphi$: $f \to \varphi f$ for all $f \in L^2$, where $\varphi \in L^\infty$. In the same sense, $H^\infty$ is the commutant of $S$.

The original factorization theorems of Beurling and of Szegö are as follows:

THEOREM 1. *If $f \in H^2$, then $f = gh$ where $g$ is inner and $h$ is outer.*

THEOREM 2. *If $f \in L^1$ and $f \geqslant 0$, then $f = |h|^2$ where $h \in H^2$ if and only if $\int \log f(e^{it})\,dt > -\infty$.*

Theorem 1 is easily stated in the language of operator theory. There is a slight loss of generality since we discuss only bounded operators. In effect this means that in Theorems 1 and 2 we shall be factoring only bounded functions. Then, in the language of operator theory, Theorem 1 becomes the following:

THEOREM 1′. *If $A$ is an operator that commutes with the unilateral shift $S$ on $H^2$, then $A = WB$ where*

(i) *$W$ and $B$ commute with $S$,*
(ii) *$W$ is an isometry, and*
(iii) $(BH^2)^- = H^2$.

It is not so clear just what one can do with Theorem 2 in the language of operator theory because of the log integrability condition. Suppose $\varphi \in L^\infty$. Let $A_\varphi$ be the operator on $L^2$ that is "multiplication by $\varphi$." If $\int \log|\varphi(e^{it})|\,dt > -\infty$, from the Beurling and Szegö theorems one gets that $|\varphi|$ is the modulus of an outer function $h \in H^\infty$, or $\varphi = wh$ where $w$ is of constant modulus one (*not* analytic in general). If $D$ then is the operator "multiplication by $1/w$" on $L^2$, it is easy to check that $D$ has the property that $DA_\varphi H^2 \subseteq H^2$. The preceeding discussion is reversible so as to permit the following simple "Szegö-type" theorem.

THEOREM 2′. *A non-zero operator $A$ on $L^2$ that commutes with the bilateral shift $U$ can be factored into a product $A = WB$ where*

(i) *$W$ and $B$ commute with $U$,*
(ii) *$W$ is unitary, and*
(iii) $(BH^2)^- = H^2$,

*if and only if there exists an invertible operator $D$ that commutes with $U$ and maps $AH^2$ into $H^2$.*

It is now known that the Beurling factorization theorem for operators that commute with the unilateral shift can be extended so as to provide a factorization for operators commuting with any isometry. The following extension may be found in either [4] or [2].

THEOREM 1″. *If $S$ is any isometry in Hilbert space and $AS = SA$, then $A$ can be factored $A = WB$ where*

(i) *$W$ and $B$ commute with $S$,*
(ii) *$W$ is a partial isometry with initial space equal to $R(B)^-$, and*
(iii) *$R(B)^-$ reduces $S$.*

Theorem 1″ bears an interesting relationship to the polar factorization. Let $\mathscr{A}$ be the algebra of operators that commute with a given isometry. The theorem asserts that if $A \in \mathscr{A}$, then $A = WB$ where (i) $W, B \in \mathscr{A}$, (ii) $W$ is a partial isometry with initial space equal to $R(B)^-$, and (iii) $R(B)^-$ corresponds to an orthogonal projection in $\mathscr{A}$. By way of comparison, if $\mathscr{A}$ is a $w^*$-algebra, then the standard polar factorization of an operator $A \in \mathscr{A}$ satisfies these three properties. It would be of interest to know what other kinds of unstarred algebras admit such a factorization.



Two recent and related generalizations of the Szegö factorization theorem have appeared. Theorem 2″ below appeared in [2].

THEOREM 2″. *Let $U$ be unitary on $K$ and $H$ an invariant subspace for $U$. If $AU = UA$, then $A$ can be factored $A = WB$ where*

(i) *$W$ and $B$ commute with $U$,*
(ii) *$W$ is a partial isometry with initial space $(BK)^-$,*
(iii) *$W|H$ is a partially isometric mapping of $H$ into $K$ with initial space $(BH)^-$, and*
(iv) *$BH \subseteq H$ and $(BH)^-$ is a reducing subspace for the isometry $U|H$ on $H$,*

*if there exists an invertible operator $D$ commuting with $U$ and mapping $AH$ into $H$.*

Moore, Rosenblum, and Rovnyak [4] consider the Szegö problem from a different viewpoint. Rather than factor operators that commute with some given unitary operator, they factor operators which are "Toeplitz" with respect to a given isometry; if $V$ is an isometry, $T$ is "$V$-Toeplitz" if $V^*TV = T$. Of course, it is elementary to "lift" this equation to the space of the minimal unitary extension of $V$ in which case the "lifted" version of $T$ commutes with the unitary extension of $V$. Establishing the relation between the factorization studied in Theorem 2″ and the Moore, Rosenblum, Rovnyak factorization of [4], however, is a non-trivial matter. The next theorem does this.

THEOREM 3. *Let $U$ be a unitary operator on $K$ and $H$ an invariant subspace for $U$. Suppose $AU = UA$. Let $P_H$ be the orthogonal projection of $K$ onto $H$, and $T = P_H A^* A|H$. Finally, let $V = U|H$.*

*Then $A = WB$ as in Theorem 2″ if and only if $T = C^*C$ where $CV = VC$ and $(CH)^-$ reduces $V$.* (This factorization of $T$ is the factorization found in [4].)

*Proof.* One implication is trivial. If $A = WB$ as in Theorem 2″, then clearly $A^*A = B^*B$. Simply let $C = B|H$ and it follows that $T = C^*C$ and $C$ has the desired properties.

The proof of the converse implication is similar to the proof of Theorem 2″ given in [2]. For that reason some of the details are omitted.

Assume $T = C^*C$ where $CV = VC$ and $(CH)^-$ reduces $V$. Let $H_0$ be the smallest reducing subspace for $U$ containing $H$. Then $C$ may be extended to an operator $B$ on $H_0$ which has the properties that $BH \subseteq H$ and $(BH)^-$ reduces $V$, and $(BH_0)^-$ reduces $U$. ($B$ is determined by the fact that $BU^{-n}f = U^{-n}Cf$ when $f \in H$ and $n \geqslant 0$.)

It is easy now to define $W$ on $BH$. Since $B|H = C$, $W$ can be defined by $WBf = Af$ on $BH$, where $f \in H$. Then $W$ is well-defined and norm-preserving on $BH$. So $W$ extends uniquely by continuity to an isometric linear map on $(BH)^-$ into $K$. Furthermore, $W$ "lifts" to $(BH_0)^-$ in a natural way, i.e. there is a unique way to extend $W$ to an operator on $(BH_0)^-$ into $K$ that is isometric, that commutes with $U$ on $(BH_0)^-$, and satisfies now $Af = WBf$ for all $f \in H_0$. (Keep in mind that $(BH_0)^-$ reduces $U$.)

The remainder of the proof uses a technique found in [2]. Since $H_0$ reduces $U$ and $AU = UA$, $(AH_0)^-$ reduces $U$. Let $P$ be the orthogonal projection onto $(AH_0)^-$ and let $Q = I-P$. Then $QA$ commutes with $U$ and $QAH_0 = \{0\}$. Consider now the polar factorization of $QA$; $QA = JR$ where $R \geqslant 0$ and $J$ is a partial isometry with initial space $(RK)^-$. Extend $B$, which is presently defined on $H_0$, to all of $K$ by defining $Bf$ to be $Rf$ for $f \in H_0^\perp$. Since $QAH_0 = \{0\}$, $RK \perp H_0$. Thus the closure of the range of $B$ splits into an orthogonal sum

$$(BK)^- = (BH_0)^- \oplus (RK)^-.$$

Finally, to extend $W$ to the entire space $K$, recalling that $W$ is already defined on $(BH_0)^-$, define $W$ to be equal to $J$ on $(RK)^-$ and zero on $(BK)^\perp$. This makes $W$ a partial isometry with initial space $(BK)^-$. It can be checked now that $W$ and $B$ satisfy the conditions of Theorem 2″ and that $Af = WBf$ for all $f \in K$.

The existence of an invertible operator $D$ as in Theorem 2″ is a condition that merits some examination. In the original setting where $U$ is the simple bilateral shift (Theorem 2′), the existence of $D$ is necessary and sufficient for the Szegö factorization. In fact it is shown in [2] that if $U$ is any unitary of finite spectral multiplicity, then the existence of $D$ in Theorem 2″ is necessary for the factorization. Also in [2] is the conjecture that the multiplicity of $U$ *completely determines* whether the existence of $D$ is both necessary and sufficient for the factorization to exist for all commuting operators. Below we shall show that this is true, given the proper interpretation of multiplicity.

In this discussion, a *measure* will always mean a positive finite Borel measure with compact support in the complex plane. Also, a "part" of an operator is simply the restriction of the operator to some reducing subspace.

Suppose now that $\mu$ is a measure; let $M_\mu$ denote the operator on $L^2(\mu)$ which is multiplication by the identity function. The *$\mu$-multiplicity* of $U$ is the maximum number $K$ such that an orthogonal sum of $K$ copies of $M_\mu$ is isomorphic to a part of $U$. To say that $U$ is of *weakly finite multiplicity* means that the $\mu$-multiplicity of $U$ is finite for every measure $\mu$. (This criteria can also be described via the multiplicity function of $U$. Such a viewpoint, however, does not seem quite so well suited for the construction needed in Theorem 4 below as does the above.)

THEOREM 4. *The sufficient condition in Theorem 2″ that $D$ exist is also a necessary condition for the factorization if and only if $U$ is of weakly finite multiplicity.*

In [2] it is proved that if $U$ is of weakly finite multiplicity, then the existence of $D$ in Theorem 2″ is both necessary and sufficient for the factorization. What remains then is to show that if $U$ does *not* have weakly finite multiplicity, then there is a commuting operator $A$ that factors $A = WB$ as in Theorem 2″ but for which *no such $D$* exists. The construction is not difficult, but rests on the following propositions. We shall use "$\sim$" to denote (i) unitary equivalence between operators, (ii) the correspondence between an operator and a spectral representation for the operator, or (iii) equivalence of two spectral representations, in the sense that they represent the same operator.



PROPOSITION 1. *If $\mu \ll \nu$ and $\nu_a$ is the absolutely continuous part of $\nu$ with respect to $\mu$, then $M_\mu \sim M_{\nu_a}$.*

*Proof.* It is easily checked that the hypothesis implies that $\mu$ and $\nu_a$ are equivalent measures (each absolutely continuous with respect to the other). This implies that $M_\mu \sim M_{\nu_a}$. (This is the cornerstone of the uniqueness part of the spectral representation theorem.)

PROPOSITION 2. *If $\mu$ and $\nu$ are orthogonal measures, then there are no non-zero operators $A$ that intertwine $M_\mu$ and $M_\nu$, i.e. that satisfy $AM_\mu = M_\nu A$.*

This result is known. A simple proof may be given involving only elementary topological considerations. Basically, the idea is that two orthogonal measures "almost" live on disjoint compact sets.

PROPOSITION 3. *Suppose $\{\mu_i\}$ and $\{\nu_j\}$ are sequences of measures such that $\mu_i \perp \nu_j$ for all $i, j$, and suppose that $U$ is unitary operator which has a spectral representation $\sum L^2(\mu_i) \oplus \sum L^2(\nu_j)$. (Here of course the symbol "$\sum$" denotes orthogonal direct sum, and $M_{\mu_i}$ is the operator acting on $L^2(\mu_i)$, $M_{\nu_j}$ the operator acting on $L^2(\nu_j)$.)*

*Then if $AU = UA$, the subspaces $M = \sum L^2(\mu_i)$ and $N = \sum L^2(\nu_j)$ must be reducing subspaces for $A$. Consequently, for $A$ to be invertible it is necessary and sufficient that $A|M$ be invertible on $M$ and $A|N$ be invertible on $N$.*

*Proof.* Let $i$ and $j$ be fixed, let $P$ denote the orthogonal projection onto $L^2(\nu_j)$, and let $A_{ij} = PA|L^2(\mu_i)$. Then for any $f \in L^2(\mu_i)$,

$$M_{\nu_j} A_{ij} f = UA_{ij} f = UPAf = PUAf = PAUf = PAM_{\mu_i} f = A_{ij} M_{\mu_i} f.$$

It follows then from Proposition 2 that $A_{ij} = 0$. This proves $AM \subseteq M$. Similarly $AN \subseteq N$.

*Proof of Theorem* 4. Now it is possible to perform the indicated construction, i.e. if $U$ is not of weakly finite multiplicity to construct an operator $A$ which factors $A = WB$ as in Theorem 2″ but for which no operator $D$ exists as in Theorem 2″.

If $U$ is not of weakly finite multiplicity, there is a measure $\mu$ such that the orthogonal direct sum of infinitely many copies of $M_\mu$ forms a part of $U$. A Zorn's Lemma argument easily shows that $U = U_1 \oplus U_2$ where $U_1$ is an orthogonal direct sum of countably infinitely many copies of $M_\mu$ and where $M_\mu$ does not form a part of $U_2$. Let $\sum L^2(B_n, \nu)$ be an ordered spectral representation of $U_2$ ([1], p. 916), i.e. $\{B_n\}$ is a decreasing sequence of Borel sets in the plane. Finally, let $\mu = \mu_a + \mu_s$ be the decomposition of $\mu$ relative to $\nu$ so that $\mu_a \ll \nu$ and $\nu \perp \mu_s$. Observe now that $\mu_s$ cannot be zero. The reason is that if $\mu_s$ were zero, then we would have $\mu \ll \nu$ and Proposition 1 would then say that $M_\mu \sim M_{\nu_a}$. But since $L^2(\nu) \sim L^2(\nu_a) \oplus \oplus L^2(\nu_s)$ it would then follow that $M_\mu$ is a part of $U_2$.

Now $U$ has a spectral representation $U \sim \sum L^2(\mu) \oplus \sum L^2(B_n, \nu)$. But since $L^2(\mu) \sim L^2(\mu_a) \oplus L^2(\mu_s)$ we can just as well write for a spectral representation of $U$, $U \sim \sum L^2(\mu_s) \oplus \sum L^2(\mu_a) \oplus L^2(B_n, \nu)$. Letting $M$ be the subspace $M = \sum L^2(\mu_s)$ and $N$ the subspace $N = \sum L^2(\mu_a) \oplus \sum L^2(B_n, \nu)$, we are in a position where Proposition 3 is applicable; every operator that commutes with $U$ has $M$ and $N$ as reducing subspaces.

Let $A$, $B$, and $W$ be the following operators: $A$ is the backward shift on $M = \sum L^2(\mu_s)$, i.e. $A$: $(f_1, f_2, \ldots) \to (f_2, f_3, \ldots)$. (The representation of elements of $\sum L^2(\mu_s)$ as sequences should be self-explanatory.) On $N$, define $A$ to be zero. On $M$ let $B$ be the operator $B$: $(f_1, f_2, f_3, \ldots) \to (0, f_2, f_3, \ldots)$, and on $N$ let $B$ equal zero. ($B$ is an orthogonal projection.) And let $W = A$. Finally, let the role of $H$ in Theorem 2″ be played here by the subspace which is the range of $B$.

It is straightforward to check that $A = WB$ and that all conditions of the factorization of Theorem 2″ are met. But can there be an invertible operator $D$ that commutes with $U$ and maps $AH$ into $H$? From Proposition 3 we have seen that such an operator $D$ would have to map $M$ one-to-one onto $M$. But $AH = M$ whereas $H$ is a proper subspace of $M$. Thus $D$ could not map $AH$ into $H$.

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# DISTRIBUTION OF EIGENVALUES AND NUCLEARITY

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In this paper we shall use the terminology introduced in [6]. In particular, $\mathfrak{L}(E, F)$ denotes the set of all (bounded linear) operators from the Banach space $E$ into the Banach space $F$. Since we are concerned with spectral properties of operators, all Banach spaces under consideration are supposed to be complex.

## 1. $\mathfrak{S}_p^{\mathrm{eig}}$-operators

Let $S \in \mathfrak{L}(E, E)$ and put

$$N(\lambda, S) := \bigcup_{k=1}^{\infty} \{x \in E\colon (\lambda I_E - S)^k x = 0\}.$$

Here $I_E$ denotes the identity map of $E$. If $N(\lambda, S) \neq \{0\}$, then $\lambda \in C$ (complex field) is called an *eigenvalue* of $S$ and

$$\alpha(\lambda, S) := \dim N(\lambda, S)$$

is said to be its *algebraic multiplicity*.

Let $0 < p < \infty$. An operator $S \in \mathfrak{L}(E, F)$ is of *Riesz type* $l_p$ if

$$\sum_{\lambda \in C} \alpha(\lambda, LS)|\lambda|^p < \infty \quad \text{for all} \quad L \in \mathfrak{L}(F, E).$$

The class of these operators will be denoted by $\mathfrak{S}_p^{\mathrm{eig}}$.

*Remark*. If $S \in \mathfrak{S}_p^{\mathrm{eig}}(E, E)$, then we have

$$\sum_{\lambda \in C} \alpha(\lambda, S)|\lambda|^p = \sum_I |\lambda_i(S)|^p,$$

where $(\lambda_i(S)\colon i \in I)$ is the (countable!) family of all eigenvalues $\lambda \neq 0$ repeated according to their (finite!) algebraic multiplicities.

In order to check the following result we need an elementary consequence of the spectral mapping theorem; [1], VII.3.19.

LEMMA. *Let* $0 < p < \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\sum_{\mu \in C} \alpha(\mu, S^n)|\mu|^{p/n} = \sum_{\lambda \in C} \alpha(\lambda, S)|\lambda|^p \quad \textit{for all } S \in \mathfrak{L}(E, E).$$