Now U has a spectral representation $U \sim \sum L^2(\mu) \oplus \sum L^2(B_n, \nu)$. But since $L^2(\mu) \sim L^2(\mu_a) \oplus L^2(\mu_s)$ we can just as well write for a spectral representation of U, $U \sim \sum L^2(\mu_s) \oplus \sum L^2(\mu_a) \oplus L^2(B_n, \nu)$. Letting M be the subspace $M = \sum L^2(\mu_s)$ and N the subspace $N = \sum L^2(\mu_a) \oplus \sum L^2(B_n, \nu)$, we are in a position where Proposition 3 is applicable; every operator that commutes with U has M and N as reducing subspaces.

Let A, B, and W be the following operators: A is the backward shift on $M = \sum L^2(\mu_s)$, i.e. A: $(f_1, f_2, \ldots) \to (f_2, f_3, \ldots)$. (The representation of elements of $\sum L^2(\mu_s)$ as sequences should be self-explanatory.) On N, define A to be zero. On M let B be the operator B: $(f_1, f_2, f_3, \ldots) \to (0, f_2, f_3, \ldots)$, and on N let B equal zero. (B is an orthogonal projection.) And let $W = A$. Finally, let the role of H in Theorem 2″ be played here by the subspace which is the range of B.

It is straightforward to check that $A = WB$ and that all conditions of the factorization of Theorem 2″ are met. But can there be an invertible operator D that commutes with U and maps AH into H? From Proposition 3 we have seen that such an operator D would have to map M one-to-one onto M. But $AH = M$ whereas H is a proper subspace of M. Thus D could not map AH into H.

References

[1] N. Dunford and J. Schwartz, *Linear Operators*, part II, John Wiley, New York 1964.

[2] R. Gellar and L. Page, *Inner-outer factorizations of operators*, J. Math. Anal. and Appl. 61 (1977), 151–158.

[3] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

[4] B. Moore, M. Rosenblum, J. Rovnyak, *Toeplitz operators associated with isometries*, Proc. Amer. Math. Soc. 49 (1975), 189–194.

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DISTRIBUTION OF EIGENVALUES AND NUCLEARITY

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In this paper we shall use the terminology introduced in [6]. In particular, $\mathfrak{L}(E, F)$ denotes the set of all (bounded linear) operators from the Banach space E into the Banach space F. Since we are concerned with spectral properties of operators, all Banach spaces under consideration are supposed to be complex.

1. $\mathfrak{S}_p^{\mathrm{eig}}$-operators

Let $S \in \mathfrak{L}(E, E)$ and put

$$N(\lambda, S) := \bigcup_{k=1}^{\infty} \{x \in E\colon (\lambda I_E - S)^k x = 0\}.$$

Here I_E denotes the identity map of E. If $N(\lambda, S) \neq \{0\}$, then $\lambda \in C$ (complex field) is called an *eigenvalue* of S and

$$\alpha(\lambda, S) := \dim N(\lambda, S)$$

is said to be its *algebraic multiplicity*.

Let $0 < p < \infty$. An operator $S \in \mathfrak{L}(E, F)$ is of *Riesz type* l_p if

$$\sum_{\lambda \in C} \alpha(\lambda, LS) |\lambda|^p < \infty \quad \text{for all} \quad L \in \mathfrak{L}(F, E).$$

The class of these operators will be denoted by $\mathfrak{S}_p^{\mathrm{eig}}$.

Remark. If $S \in \mathfrak{S}_p^{\mathrm{eig}}(E, E)$, then we have

$$\sum_{\lambda \in C} \alpha(\lambda, S) |\lambda|^p = \sum_I |\lambda_i(S)|^p,$$

where $(\lambda_i(S)\colon i \in I)$ is the (countable!) family of all eigenvalues $\lambda \neq 0$ repeated according to their (finite!) algebraic multiplicities.

In order to check the following result we need an elementary consequence of the spectral mapping theorem; [1], VII.3.19.

LEMMA. *Let* $0 < p < \infty$ *and* $n = 1, 2, \ldots$ *Then*

$$\sum_{\mu \in C} \alpha(\mu, S^n) |\mu|^{p/n} = \sum_{\lambda \in C} \alpha(\lambda, S) |\lambda|^p \quad \textit{for all } S \in \mathfrak{L}(E, E).$$

We are now prepared to prove

PROPOSITION 1. *Let $0 < p < \infty$ and $n = 1, 2, \dots$ Then for every operator ideal $\mathfrak{A}$ the inclusions $\mathfrak{A} \subseteq \mathfrak{S}_p^{eig}$ and $\mathfrak{A}^n \subseteq \mathfrak{S}_{p/n}^{eig}$ are equivalent.*

Proof. Suppose that $\mathfrak{A} \subseteq \mathfrak{S}_p^{eig}$. If $S \in \mathfrak{A}^n(E, F)$ and $L \in \mathfrak{L}(F, E)$, then there exists a factorization

$$LS\colon E = M_0 \xrightarrow{T_1} M_1 \xrightarrow{T_2} \dots \xrightarrow{T_n} M_n = E$$

such that $T_k \in \mathfrak{A}(M_{k-1}, M_k)$ for $k = 1, \dots, n$. Form the Cartesian product $M := M_1 \times \dots \times M_n$ equipped with any suitable norm. Then by

$$T\colon (x_1, \dots, x_{n-1}, x_n) \to (T_1 x_n, T_2 x_1, \dots, T_n x_{n-1})$$

we define an operator $T \in \mathfrak{A}(M, M)$. Observe that E can be identified with the subspace $\{0\} \times \dots \times \{0\} \times M_n$ of M which is invariant under T^n. Moreover, the restriction of T^n to E coincides with $LS = T_n \dots T_1$. So, by the preceding lemma, we have

$$\sum_{\mu \in C} \alpha(\mu, LS)|\mu|^{p/n} \leqslant \sum_{\mu \in C} \alpha(\mu, T^n)|\mu|^{p/n} = \sum_{\lambda \in C} \alpha(\lambda, T)|\lambda|^p < \infty.$$

Therefore $S \in \mathfrak{S}_{p/n}^{eig}(E, F)$. This proves that $\mathfrak{A}^n \subseteq \mathfrak{S}_{p/n}^{eig}$. In order to check the converse implication we suppose that $\mathfrak{A}^n \subseteq \mathfrak{S}_{p/n}^{eig}$. If $S \in \mathfrak{A}(E, F)$ and $L \in \mathfrak{L}(F, E)$, then $(LS)^n \in \mathfrak{A}^n(E, E)$. Hence

$$\sum_{\lambda \in C} \alpha(\lambda, LS)|\lambda|^p = \sum_{\mu \in C} \alpha\big(\mu, (LS)^n\big)|\mu|^{p/n} < \infty.$$

Therefore $S \in \mathfrak{S}_p^{eig}(E, F)$. This proves that $\mathfrak{A} \subseteq \mathfrak{S}_p^{eig}$.

PROPOSITION 2. *If $X \in \mathfrak{L}(E_0, E)$, $S \in \mathfrak{S}_p^{eig}(E, F)$, and $B \in \mathfrak{L}(F, F_0)$ then $BSX \in \mathfrak{S}_p^{eig}(E_0, F_0)$.*

Proof. Let $L_0 \in \mathfrak{L}(F_0, E_0)$. Then the operators $L_0 BSX$ and $XL_0 BS$ are related; cf. [6], 27.3.1. Therefore we have

$$\sum_{\lambda \in C} \alpha(\lambda, L_0 BSX)|\lambda|^p = \sum_{\lambda \in C} \alpha(\lambda, XL_0 BS)|\lambda|^p < \infty.$$

This proves the assertion.

Next we show that $\mathfrak{S}_p^{eig}$ is not an operator ideal. This yields a negative answer to a problem which has been posed in 1969; cf. [5].

PROPOSITION 3. *Let $0 < p < \infty$. Then there are a Banach space E as well as operators $S_1, S_2 \in \mathfrak{S}_p^{eig}(E, E)$ such that $S_1 + S_2 \notin \mathfrak{S}_p^{eig}(E, E)$.*

Proof. Choose a natural number n and a real number q such that $2np > 2q > (2n-1)p \geqslant 4$. Take any sequence $(\sigma_i) \in l_{2q}$ not belonging to $l_{(2n-1)p}$ and define the diagonal operator $S \in \mathfrak{L}(l_\infty, l_q)$ by $S(\xi_i) := (\sigma_i^2 \xi_i)$. Furthermore, let $J \in \mathfrak{L}(l_q, l_\infty)$ be the canonical embedding.

In the following $\mathfrak{P}_r$ stands for the *ideal of absolutely r-summing* operators. Obviously we have $S \in \mathfrak{P}_{np}(l_\infty, l_q)$. It has been proved in [2] that $\mathfrak{P}_{np} \subseteq \mathfrak{S}_{np}^{eig}$. So from Proposition 1 we get $\mathfrak{P}_{np}^n \subseteq \mathfrak{S}_p^{eig}$. Therefore $S(JS)^{n-1} \in \mathfrak{S}_p^{eig}$.

On the other hand, by [6], 22.4.2, we have $\mathfrak{L}(l_\infty, l_q) = \mathfrak{P}_{np}(l_\infty, l_q)$. This implies that $L(JS)^{n-1}J \in \mathfrak{S}_p^{eig}$ for all $L \in \mathfrak{L}(l_\infty, l_q)$. Therefore $(JS)^{n-1}J \in \mathfrak{S}_p^{eig}$.

Form the Cartesian product $E := l_q \times l_\infty$ equipped with any suitable norm. Then the operators $S_1, S_2 \in \mathfrak{L}(E, E)$ defined by

$$S_1\colon (x, y) \to \big(S(JS)^{n-1}y, 0\big)$$

and

$$S_2\colon (x, y) \to \big(0, (SJ)^{n-1}Jx\big)$$

are of Riesz type l_p. It follows from

$$S_1 + S_2\colon (\sigma_i e_i, e_i) \to \sigma_i^{2n-1}(\sigma_i e_i, e_i)$$

that $\lambda_i(S_1 + S_2) := \sigma_i^{2n-1}$ is an eigenvalue of $S_1 + S_2$. Now

$$\sum_{i=1}^{\infty} |\lambda_i(S_1 + S_2)|^p = \sum_{i=1}^{\infty} |\sigma_i|^{(2n-1)p} = \infty$$

implies that $S_1 + S_2 \notin \mathfrak{S}_p^{eig}(E, E)$.

Remark. If $p = 2$ or $p = 1$, then the above proof can be essentially simplified.

In contrast to the preceding result it turns out that $\mathfrak{S}_p^{eig}(H, H)$ is an ideal in the operator algebra $\mathfrak{L}(H, H)$ of the separable infinite-dimensional Hilbert space H. More precisely, if $\mathfrak{S}_p(H, H)$ denotes the Schatten ideal of type l_p, then we have

PROPOSITION 4. *Let $0 < p < \infty$. Then $\mathfrak{S}_p^{eig}(H, H) = \mathfrak{S}_p(H, H)$.*

Proof. Obviously, $\mathfrak{S}_p(H, H) \subseteq \mathfrak{S}_p^{eig}(H, H)$ is an immediate consequence of Weyl's Theorem; cf. [6], 27.4.3.

The converse inclusion can be checked in two steps. First we observe that every operator $S \in \mathfrak{S}_p^{eig}(H, H)$ is approximable. Otherwise, by [6], 5.1.1 (Lemma 3), there would exist operators $B, X \in \mathfrak{L}(H, H)$ such that $BSX = I_H$. This is a contradiction by Proposition 2. Now it is clear that every operator $S \in \mathfrak{S}_p^{eig}(H, H)$ admits a Schmidt factorization; cf. [6], D.3.3. In other terms, there are operators $U \in \mathfrak{L}(l_2, H)$ and $V \in \mathfrak{L}(l_2, H)$ as well as a diagonal operator $S_0 \in \mathfrak{L}(l_2, l_2)$ generated by a sequence $(\sigma_i) \in c_0$ such that $S = VS_0 U^*$ and $S_0 = V^*SU$. Therefore $S_0 \in \mathfrak{S}_p^{eig}(l_2, l_2)$, and it follows from

$$\sum_{i=1}^{\infty} |\sigma_i|^p = \sum_{\lambda \in C} \alpha(\lambda, S_0)|\lambda|^p < \infty$$

that $S_0 \in \mathfrak{S}_p(l_2, l_2)$. So we also have $S \in \mathfrak{S}_p(H, H)$. This completes the proof.

PROPOSITION 5. *If $\mathfrak{A}$ is an operator ideal such that $\mathfrak{A} \subseteq \mathfrak{S}_1^{eig}$, then $\mathfrak{A} \subseteq \mathfrak{P}_2$.*

Proof. Suppose that $S \in \mathfrak{A}(E, F)$. Let (x_i) be any weakly 2-summable sequence in E. Choose functionals $b_i \in F'$ such that $\langle Sx_i, b_i\rangle = \|Sx_i\|$ and $\|b_i\| = 1$. Take $(\beta_i) \in l_2$. Then by

$$X\colon (\xi_i) \to \sum_{i=1}^{\infty} \xi_i x_i$$

and

$$B\colon y \to (\beta_i \langle y, b_i\rangle)$$

we define operators $X \in \mathfrak{L}(l_2, E)$ and $B \in \mathfrak{L}(F, l_2)$. By Proposition 4 it follows that $BSX \in \mathfrak{A}(l_2, l_2) \subseteq \mathfrak{S}_1^{\mathrm{ei}}\ l_2, l_2) = \mathfrak{S}_1(l_2, l_2)$. Using [6], 15.4.3 we see that

$$\sum_{i=1}^{\infty} |\beta_i|\,\|Sx_i\| = \sum_{i=1}^{\infty} |\beta_i \langle Sx_i, b_i\rangle| = \sum_{i=1}^{\infty} |\langle BSXe_i, e_i\rangle| < \infty$$

for all $(\beta_i) \in l_2$. Hence the sequence (Sx_i) is absolutely 2-summable. This proves that $S \in \mathfrak{P}_2(E, F)$.

In the following $\mathfrak{N}$ denotes the *ideal of nuclear operators.*

THEOREM. *Let* $0 < p < \infty$. *If* $\mathfrak{A}$ *is an operator ideal such that* $\mathfrak{A} \subseteq \mathfrak{S}_p^{\mathrm{eig}}$, *then* $\mathfrak{A}^{2n} \subseteq \mathfrak{N}$ *whenever* $n \geqslant p$.

Proof. By Proposition 1, we have $\mathfrak{A}^n \subseteq \mathfrak{S}_1^{\mathrm{eig}}$. Now Proposition 5 implies that $\mathfrak{A}^n \subseteq \mathfrak{P}_2$. Therefore $\mathfrak{A}^{2n} \subseteq \mathfrak{P}_2^2 \subseteq \mathfrak{N}$; cf. [6], 24.6.5.

Let us recall that $\mathfrak{R}$, the *ideal of Gohberg operators,* is the largest operator ideal possessing the property that every $S \in \mathfrak{R}(E, E)$ is a Riesz operator; cf. [6], 26.7.2. It is well known that $\mathfrak{R}$ contains all operators $S \in \mathfrak{L}(E, E)$ which have some compact power S^n.

COROLLARY. *Let* $0 < p < \infty$. *If* $\mathfrak{A}$ *is an operator ideal such that* $\mathfrak{A} \subseteq \mathfrak{S}_p^{\mathrm{eig}}$ *then* $\mathfrak{A} \subseteq \mathfrak{R}$.

2. Examples

Let $\mathfrak{P}_{(r,2,2)}$ with $1 \leqslant r < \infty$ denote the *ideal of absolutely* $(r, 2, 2)$-*summing operators*; cf. [6], 17.1.1.

Clearly $\mathfrak{P}_{(1,2,2)} \subseteq \mathfrak{S}_2^{\mathrm{eig}}$. On the other hand, since $\mathfrak{P}_{(2,2,2)}$ contains the identity map of l_1, we have $\mathfrak{P}_{(2,2,2)}$ non $\subseteq \mathfrak{S}_p^{\mathrm{eig}}$ for $0 < p < \infty$. These borderline cases support König's

CONJECTURE 1. *If* $1 < r < 2$ *and* $1/p = 1/r - 1/2$, *then* $\mathfrak{P}_{(r,2,2)} \subseteq \mathfrak{S}_p^{\mathrm{eig}}$.

As shown in [3] we have a somewhat weaker inclusion, namely $\mathfrak{P}_{(r,2,2)} \subseteq \mathfrak{S}_{p+\varepsilon}^{\mathrm{eig}}$ for all $\varepsilon > 0$. This, however, is enough to establish

PROPOSITION 6. *If* $1 < r < 2$ *and* $n > 2r/(2-r)$, *then* $\mathfrak{P}_{(r,2,2)}^{2n} \subseteq \mathfrak{N}$.

Let $\mathfrak{P}_{(p,2)}$ with $2 \leqslant p < \infty$ denote the *ideal of absolutely* $(p, 2)$-*summing operators*; cf. [6], 17.2.1.

For these operator ideals we now formulate König's

CONJECTURE 2. *If* $2 < p < \infty$, *then* $\mathfrak{P}_{(p,2)} \subseteq \mathfrak{S}_p^{\mathrm{eig}}$.

Remark. At present it seems to be unknown whether every $\mathfrak{P}_{(p,2)}$ is contained in some $\mathfrak{S}_q^{\mathrm{eig}}$. The only result along this line is the inclusion $\mathfrak{P}_{(p,2)} \subseteq \mathfrak{S}_q^{\mathrm{eig}}$ for $q > 2p/(4-p)$ and $2 < p < 4$ which has been recently checked by König. Moreover, we have $\mathfrak{P}_{(p,2)}^n \subseteq \mathfrak{K}$ for $n > p/2$, where $\mathfrak{K}$ denotes the *ideal of compact operators*; cf. [4].

Remark (added in proof). During the printing of this paper several related results have been obtained, cf. [7] and [8]. In particular, it is now proved that $\mathfrak{P}_{(p,2)}$ non $\subseteq \mathfrak{S}_p^{\mathrm{eig}}$ but $\mathfrak{P}_{(p,2)} \subseteq \mathfrak{S}_{p+\varepsilon}^{\mathrm{eig}}$ for $2 < p < \infty$ and $\varepsilon > 0$. Moreover, we have $\mathfrak{P}_{(p,2)}^{2n} \subseteq \mathfrak{N}$ for $n > p/2$.

References

[1] N. Dunford and J. T. Schwartz, *Linear operators I,* New York 1958.

[2] W. B. Johnson, H. König, B. Maurey and J. R. Retherford, *Eigenvalues of p-summing and l_p-type operators in Banach spaces,* J. Funct. Anal. 32 (1979), 353–380.

[3] H. König, *On the spectrum of products of operator ideals,* Math. Nachr. 93 (1979), 223–232.

[4] B. Maurey and A. Pełczyński, *A criterion for composition of (p, q)-absolutely summing operators,* Studia Math. 54 (1976), 291–300.

[5] A. Pietsch, *Ideale von S_p-Operatoren in Banachräumen,* ibid. 38 (1970), 59–69.

[6] —, *Operator ideals,* Berlin 1978.

[7] H. König, J. R. Retherford, and N. Tomczak-Jägermann, *On the eigenvalues of $(p, 2)$-summing operators and constants associated with normed spaces,* J. Funct. Anal. 37 (1980), 88–126.

[8] A. Pietsch, *Weyl numbers and eigenvalues of operators in Banach spaces,* Math. Ann. 247 (1980), 149–168.

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