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SURVEY OF SOME RESULTS IN SPECTRAL THEORY OBTAINED IN PRAGUE

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The paper is devoted to a series of investigations centered about the notion of the spectral radius of an operator on a Banach space and its relation to the norm of the underlying space. The formula

$$|T|_\sigma = \lim |T^r|^{1/r}$$

suggests the following question: can anything be said about the rate of convergence of this sequence? One of the main results is the existence, for some finite-dimensional spaces, of the critical exponent. This is a number q, depending on the space but independent of the operator such that the qth term of the sequence $|T^q|^{1/q}$ contains significant information about $|T|_\sigma$. In particular, estimates of the form

$$|T|\, g\left(\frac{|T^q|^{1/q}}{|T|}\right) \leqslant |T|_\sigma$$

may be obtained valid for all operators on the given space.

We refer the reader to a separate paper in this volume which is devoted to this question and is entitled *Universal estimates of the spectral radius.* In particular, the critical exponent of an n-dimensional Hilbert space equals n.

The case of a Hilbert space receives, deservedly, more attention. The relation between the norm and the spectral radius is a very close one there: for every operator T, we have the equality

$$|T^*T| = |T^*T|_\sigma.$$

The involution and the norm in C^* algebras are connected by the equation

$$|T|^2 = |T^*T|.$$

Let us observe that this equation is equivalent to the inequality

$$|T| \leqslant |T^*T|^{1/2}$$

since the inequality $|T^*T| \leqslant |T^*||T| = |T|^2$ is immediate. The classical result of Gelfand and Naĭmark may thus be stated as follows. A Banach $*$-algebra which satisfies $|x| \leqslant |x^*x|^{1/2}$ for every x is isometrically *isomorphic to a C^* algebra. An algebraic analogue of this inequality is the inequality

$$|x|_\sigma \leqslant |x^*x|_\sigma^{1/2}.$$

The present author has shown [16] that the following conditions are equivalent for any Banach algebra A with involution (no continuity conditions on the involution are imposed):

(i) the inequality $|x|_\sigma \leqslant |x^*x|_\sigma^{1/2}$ holds for every $x \in A$;

(ii) the algebra is hermitian, i.e. self-adjoint elements have real spectra.

The analogy between the two inequalities is much more than a formal similarity. By using the fundamental inequality $|x|_\sigma \leqslant |x^*x|_\sigma^{1/2}$, it is possible to develop a theory of hermitian algebras which is parallel to that of C^* algebras. Of course, since no metric conditions are present, essentially different methods of proof have to be found. See [16] and [17].

Returning to operators on a Hilbert space, the idea suggests itself of examining a little more deeply the relation of the spectral radius of an operator T and that of T^*T. We consider the modulus of T, the positive square root of T^*T. There is another possible definition, $(TT^*)^{1/2}$, but both have the same spectral radius. Indeed, we have

$$|T| = |T^*T|^{1/2} = |T^*T|_\sigma^{1/2} = |(T^*T)^{1/2}|_\sigma.$$

If M is any positive operator on a Hilbert space and if U is any unitary operator, then

$$|UM|_\sigma \leqslant |UM| = |M| = |M|_\sigma.$$

Now if H is n-dimensional, apart from the obvious inequality $|UM|_\sigma \leqslant |M|_\sigma$, there is also an estimate from below

$$|M^{-1}|_\sigma^{-(n-1)/n}|M|_\sigma^{1/n} \leqslant |UM|_\sigma \leqslant |M|_\sigma;$$

see the author's remark [25]. An immediate corollary is the inequality

$$|A|^{1/n}(|A^{-1}|^{-1})^{(n-1)/n} \leqslant |A|_\sigma \leqslant |A|,$$

valid for any invertible operator A on an n-dimensional Hilbert space [25].

Thus far we have studied one operator with the aim of obtaining bounds for its spectrum in terms of the norm of its iterates. A related problem is to examine the spectrum as a function of the operator, in particular to examine the change of the spectrum where an operator is replaced by another one, which is in some sense close to it.

If P is a projection, we might try to relate the spectrum of the "submatrix" PAP (which might be easier to compute) to the spectrum of the operator A. In the general case, let us mention an early paper of M. Fiedler and the present author [11]. The main result of that paper is based on an iterative construction. In a recent paper [20] the rate of convergence of this process was found. The details can be found in [20]. The general form of results of this type is as follows. We are given

a projection P and consider the smaller operator PAP restricted to the range of P. Its spectrum is denoted by σ_P. Furthermore, we assume that the complementary operator $(I-P)A(I-P)$ restricted to the range of $I-P$ is sufficiently well invertible. Let us denote it by B and let $\gamma = \{\inf|By|;\ |y| \geqslant 1\} > 0$. Set $\beta = |PA(I-P)| \times |(I-P)AP|$. If β is sufficiently small compared with γ, then there exists a function r depending on β and γ and such that each disc with centre λ_P in σ_P and radius $r(\beta, \gamma)$ intersects the spectrum of A.

In the case of a Hilbert space and of a normal operator A it turns out [22] that the radius of the inclusion disc is just $|PA(I-P)|$ and that no conditions on B need be imposed. This is all the more surprising since B need not be normal even if A is.

For normal operators on a Hilbert space it is not difficult to see that the Hausdorff distance of the spectra is majorized by the norm of the difference of the operators,

$$d\big(\sigma(A_1),\ \sigma(A_2)\big) \leqslant |A_1 - A_2|.$$

For a proof and some applications, see [23].

It is interesting to examine the converse problem. Suppose A is a Banach algebra in which the inequality

$$d\big(\sigma(a_1), \sigma(a_2)\big) \leqslant \alpha|a_1 - a_2|$$

is satisfied for every pair of elements. It has been shown by B. Aupetit and independently by J. Zemánek and the present author [24] that this condition implies that A is commutative (of course, modulo the radical). The method of proof adopted in [24] is a combination of an algebraic identity and of the classical use of the exponential function. The main result shows that the only algebras in which the spectrum (or the spectral radius) behaves nicely are the commutative ones.

We only state here some of the interesting conditions which turn out to be equivalent. Details are to be found in [24] or in the work of B. Aupetit. A survey of the work of B. Aupetit together with a complete bibliography of his work appear in this volume, pp. 31–37.

Let A be a Banach algebra. Then the following conditions are equivalent (the inequalities are to be satisfied for all x, y in A with suitable constants):

(i) the spectral radius is subadditive

$$|x+y|_\sigma \leqslant \alpha(|x|_\sigma + |y|_\sigma);$$

(ii) the spectral radius is submultiplicative

$$|xy|_\sigma \leqslant \beta|x|_\sigma|y|_\sigma;$$

(iii) $\sigma(x+y) \subset \sigma(x)+\sigma(y)$;

(iv) $\sigma(xy) \subset \sigma(x)\sigma(y)$;

(v) the spectrum is uniformly continuous

$$d\big(\sigma(x), \sigma(y)\big) \leqslant |x-y|;$$

(vi) $d(\sigma(x), \sigma(y)) \leqslant \gamma|x-y|_\sigma$;

(vii) the spectral radius is uniformly continuous

$$||x|_\sigma - |y|_\sigma| \leqslant \delta|x-y|;$$

(viii) $||x|_\sigma - |y|_\sigma| \leqslant \delta'|x-y|_\sigma$;

(ix) the algebra $A/\mathrm{rad}\,A$ is commutative.

If any of the conditions is satisfied, then they are also satisfied with involved constants equal to one.

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