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UNIVERSAL ESTIMATES OF THE SPECTRAL RADIUS

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Let A be a linear operator defined on a finite-dimensional linear space E. Its spectral radius will be denoted by $|A|_\sigma$. If the space E is equipped with a norm $x \to |x|$ and if we denote by $|T|$ the corresponding operator norm of an operator T, we have the well-known formula

$$|A|_\sigma = \lim |A^q|^{1/q} = \inf |A^q|^{1/q}.$$

This equality links two quantities which have — at first sight — a substantially different character. The left-hand side in defined purely algebraically as the maximum of the moduli of the proper values of a certain matrix whereas the right-hand side is defined in terms of an infinite process. (The equality becomes a little less surprising if we realize the relation of the left-hand side to the fundamental theorem of algebra, and, in particular, its proof based on complex function theory and radii of convergence of series; nevertheless one fact is still worthnoticing: the left-hand side is independent of the choice of the norm on E and the notion itself does not require E to be normed while the process on the right-hand side (though not its limit) depends in a significant way on the choice of the norm on E.)

In view of the finite-dimensionality of E and the finite character of the eigenvalue problem it is natural to ask whether a finite section of the sequence

$$|A|,\ |A^2|^{1/2},\ |A^3|^{1/3}, \ldots$$

would not be sufficient to obtain significant information about the spectral radius. This has to be explained more carefully. Of course, for a particular A, the spectral radius may be arbitrarily well approximated by $|A^r|^{1/r}$ if r is sufficiently large so that we do indeed use only a finite section of the sequence; our task, however, is a more delicate one: we ask whether there exists an r independent of A such that $|A^r|^{1/r}$ contains significant information about the spectral radius.

We intend to show in this paper that an adequate description of this problem is contained in the following definition:

DEFINITION. Let E be a Banach space. We shall say that q is the *critical exponent of the space E* if q is the smallest natural number which has the following property: if A is a linear operator on E for which $1 = |A| = |A^q|$ then $|A|_\sigma = 1$.

The definition of the critical exponent in its full generality appears first in the present author's 1960 paper [11]. The first critical exponent to be computed (although not described as such) was that of the n-dimensional l_∞ space; the result, n^2-n+1,was obtained in 1957 by J. Mařík and the present author [7]. The result of [11] says that an n-dimensional Hilbert space has the critical exponent n.

To explain the meaning of the definition, let us recall the following theorem:

THEOREM. *Let E be a finite-dimensional linear space and A a linear operator on E. Then the following conditions are equivalent:*

(i) *for each $y \in E$ and each $x_0 \in E$ the sequence $x_{r+1} = Ax_r+y$ is convergent (to a solution of $x = Ax+y$);*

(ii) *the series $I+A+A^2+ \ldots$ is convergent;*

(iii) $A^r \to 0$,

(iv) $|A|_\sigma < 1$,

(v) *there exists a norm p_A on E such that $p_A(A) < 1$ (for the corresponding operator norm);*

(vi) *if p is any norm on E then there exists an m such that $p(A^m) < 1$.*

Most implications here are almost immediate. The crucial implication is based on the Jordan canonical form. The theorem is a deep one; a careful analysis of its proof shows that it also proves the spectral radius formula.

To return to the definition of the critical exponent, we shall restate it in another form, from which it will become evident that the proof of the existence of the critical exponent requires fairly delicate geometric considerations; in particular, the geometric properties of the unit ball of $B(E)$, the algebra of all operators on E, play a decisive role. Now consider a fixed norm on E. The equivalence of (iv) and (vi) shows that for each A with $|A|_\sigma < 1$ there exists an m with $p(A^m) < 1$. Denote by $m(A)$ the smallest exponent with this property. It is not difficult to see that:

The critical exponent of the space E is the maximum of $m(A)$ on the set C of all operators A on E for which $|A| = 1$ and $|A|_\sigma < 1$.

The set C is far from being compact; already this superficial observation makes it clear that the existence of q is a delicate matter. Indeed, an ingenious example of a two-dimensional space due to L. Danzer (unpublished) shows that the function m may be unbounded on C.

The meaning of the critical exponent may be described less precisely but more intuitively as follows. Consider an operator A of norm 1 and construct the sequence $A, A^2, A^3, \ldots$ Clearly

$$|A| \geqslant |A^2| \geqslant |A^3| \geqslant \ldots$$

Roughly speaking, the fact that q is the critical exponent of the space means that the sequence A^r either starts converging to zero before its qth term or does not converge to zero at all (in the second case $|A^r| = 1$ for all r).

At this point it might be useful to sketch briefly some ideas concerning general iterative processes as proposed by the present author at the 1968 Gatlinburg Symposium.

To explain the main problem, it will be convenient to consider the following situation. Let us first explain what we mean by an iterative process. The objects to which we will apply the iterative process will be taken from a given complete metric space A. The expression iterative process will be given a slightly more general meaning than is usual. An iterative process P on A will be a sequence $P = \{P_0, P_1, P_2, \ldots\}$ of mappings of A into itself which yields, for each $a \in A$, a sequence

$$P_0(a),\ P_1(a),\ P_2(a),\ \ldots$$

In applications frequently $P_k = P_0^k$, in which case the process is an iteration in the classical sense. The first question to be considered in this context is to find conditions under which, for a given $a \in A$, the sequence $P_k(a)$ will be convergent. In many cases theoretical considerations enable us to state such conditions, at least sufficient conditions for convergence. The verification of the conditions may turn out, in some cases, to be equally difficult if not more difficult than the solution of the problem for which the iterative process has been designed. Another approach which suggests itself is the following: to decide the question as to whether the process converges for a given $a \in A$ on the basis of the behaviour of a certain number of elements of the sequence $P_0(a), P_1(a), \ldots$, itself. It is usually possible to find a criterion of convergence which may be verified by performing less costly and time-consuming operations on a finite sequence $P_0(a), \ldots, P_q(a)$ than a complicated operation performed on a itself.

Thus, instead of verifying the conditions for convergence on a we simply start computing the sequence of iterations and apply simple criteria on its finite sections. There are, essentially, two cases possible. Either the beginning of the sequence shows marked convergence phenomena, which makes it possible to verify the simpler criteria mentioned above, or the behaviour of the initial steps is unstable, there are violent oscillations and the question of the suitability of the method arises: is it reasonable to continue, in the hope of encountering, later in the process, a considerable improvement of the convergence, or is the method to be abandoned? The main problem of course is the following: how many steps are necessary in order to make this decision with a reasonable degree of reliability?

Clearly two problems may be formulated immediately:

1. *The qualitative problem.* Does there exist a number q and a simple criterion C such that $P_n(a)$ converges if and only if $\{P_0(a), \ldots, P_q(a)\} \in C$?

2. *The quantitative problem.* Does there exist a number q and a simple operation on sequences of type $P_0(a), \ldots, P_q(a)$ which yields significant information about the rate of convergence of $P_n(a)$?

In both problems of course q is to be independent of $a \in A$.

The second question is clearly much more ambitious. It might seem surprising at first sight that — in the case of linear operators considered above — the purely qualitative property of q already implies quantitative consequences. One of the aims of this paper is to explain this in some detail.

The paper is divided into three sections. The first two contain material not published before (although announced already at the 1968 Gatlinburg Symposium). The third section surveys results obtained thus far and concludes with some open questions.

1. Composition of functions

In the following sections we shall need a notion of an inverse function (in the sense of superposition) for non-decreasing functions which might have intervals of constancy. The present section is devoted to the discussion of what seems to be a suitable notion; we collect here the basic properties of it. The proofs are straightforward — the only reason for including this section is the fact that these notions might also be used in other investigations. As far as our applications are concerned, in all concrete situations investigated thus far the functions considered are strictly increasing, so that the new notions coincide with the ordinary inverse functions. It is even conceivable that all the functions occurring in our applications will be strictly monotonous (see Problem 7); nevertheless the proof has not been given, so that these notions have to be used and, at the same time, they seem to have an intrinsic interest.

Let T be a completely regular topological space; let u and v be two continuous real-valued functions on T. We shall suppose that

$$0 \leqslant u(t) \leqslant 1, \quad 0 \leqslant v(t) \leqslant 1$$

for all $t \in T$. We shall assume that both u and v assume at least the two values 0 and 1. Also, it will be convenient to denote by S the segment $\{s;\ 0 \leqslant s \leqslant 1\}$. We define two functions, $\left(\frac{v}{u}\right)^*$ and $\left(\frac{v}{u}\right)_*$ from S into S as follows: For each $p \in S$ we set

$$\left(\frac{v}{u}\right)^*(p) = \sup\{v(t);\ u(t) \leqslant p\},$$

$$\left(\frac{v}{u}\right)_*(p) = \inf\{v(t);\ u(t) \geqslant p\}.$$

PROPOSITION 1.1. *The functions* $\left(\frac{v}{u}\right)^*$ *and* $\left(\frac{v}{u}\right)_*$ *are nondecreasing on* S. *We have*

(i) $\left(\frac{v}{u}\right)_*(p) \leqslant \left(\frac{v}{u}\right)^*(p)$ *for all* p *for which* $u^{-1}(p)$ *is nonvoid;*

(ii) *for each* $t \in T$

$$\left(\frac{v}{u}\right)_*(u(t)) \leqslant v(t) \leqslant \left(\frac{v}{u}\right)^*(u(t));$$

(iii) *if* T *is compact then, for each* $x \in S$,

$$\left(\frac{u}{v}\right)_* \circ \left(\frac{v}{u}\right)^*(x) \leqslant x \leqslant \left(\frac{v}{u}\right)^* \circ \left(\frac{u}{v}\right)_*(x).$$

Proof. For each $p \in S$ denote by p_* the segment $p_* = \{s;\ s \leqslant p\}$ and by p^* the segment $p^* = \{s;\ s \geqslant p\}$. It follows that

$$\left(\frac{v}{u}\right)^*(p) = \sup v\big(u^{-1}(p_*)\big), \quad \left(\frac{v}{u}\right)_*(p) = \inf v\big(u^{-1}(p^*)\big).$$

If $p < q$ we have $p_* \subset q_*$; hence $\sup vu^{-1}p_* \leqslant \sup vu^{-1}q_*$; at the same time $p^* \supset q^*$, so that $\inf vu^{-1}p^* \leqslant \inf vu^{-1}q^*$. This proves that the functions are nondecreasing. If $u^{-1}(p)$ is nonvoid, we have $p \in p^* \cap p_*$ whence

$$\left(\frac{v}{u}\right)_*(p) = \inf vu^{-1}p^* \leqslant \inf vu^{-1}p \leqslant \sup vu^{-1}p \leqslant \sup vu^{-1}p_* = \left(\frac{v}{u}\right)^*(p).$$

Now suppose T to be compact. We have

$$\left(\frac{u}{v}\right)_* \circ \left(\frac{v}{u}\right)^*(x) = \inf\{u(t);\ v(t) \geqslant \sup v\big(u^{-1}(x_*)\big)\}.$$

Since $u^{-1}(x_*)$ is compact, there exists a $t_0 \in u^{-1}(x_*)$ for which $v(t_0) = \sup v\big(u^{-1}(x_*)\big)$. We thus have $u(t_0) \leqslant x$ and $v(t_0) \geqslant \sup v\big(u^{-1}(x_*)\big)$, so that the above infimum is $\leqslant x$. Similarly

$$\left(\frac{v}{u}\right)^* \circ \left(\frac{u}{v}\right)_*(x) = \sup\{v(t);\ u(t) \leqslant \inf u\big(v^{-1}(x^*)\big)\}.$$

Since $v^{-1}(x^*)$ is compact, there exists a $t_0 \in v^{-1}(x^*)$ for which $u(t_0) = \inf u\big(v^{-1}(x^*)\big)$. Thus $v(t_0) \geqslant x$ and $u(t_0) \leqslant \inf u\big(v^{-1}(x^*)\big)$, so that the above supremum is $\geqslant x$.

PROPOSITION 1.2. *Let* T *be compact. Then* $\left(\frac{v}{u}\right)^*$ *is continuous from the right and* $\left(\frac{v}{u}\right)_*$ *is continuous from the left.*

Proof. Suppose that $p < 1$ and that $p_n \in S$, $p_n \downarrow p$. It follows that

$$u^{-1}(p_{n*}) \supset u^{-1}(p_{n+1*}) \quad \text{and} \quad \bigcap u^{-1}(p_{n*}) = u^{-1}(p_*).$$

Since $u^{-1}(p_*) \subset u^{-1}(p_{n*})$ for all n, we have $\left(\frac{v}{u}\right)^*(p) = \sup v\big(u^{-1}(p_*)\big) \leqslant$ $\leqslant \sup v\big(u^{-1}(p_{n*})\big) = \left(\frac{v}{u}\right)^*(p_n)$ whence $\left(\frac{v}{u}\right)^*(p) \leqslant \inf \left(\frac{v}{u}\right)^*(p_n) = \lim \left(\frac{v}{u}\right)^*(p_n)$. Now, T being compact, the sets $u^{-1}(p_{n*})$ are compact, so that there exist $t_n \in u^{-1}(p_{n*})$ for which $\left(\frac{v}{u}\right)^*(p_n) = v(t_n)$. Let t_0 be an accumulation point of the sequence t_n. Since $u(t_n) \leqslant p_n$, we have $u(t_0) \leqslant p$. At the same time $v(t_0)$ is an accumulation point of $v(t_n) = \left(\frac{v}{u}\right)^*(p_n)$, so that $v(t_0) = \lim \left(\frac{v}{u}\right)^*(p_n)$. Hence $\left(\frac{v}{u}\right)^*(p)$ $= \sup v(u^{-1}(p_*)) \geq v(t_0) = \lim \left(\frac{v}{u}\right)^*(p_n)$. The proof of the other statement is analogous.

We have seen that, for compact T, the function $\left(\frac{u}{v}\right)_*$ is a sort of lower inverse for $\left(\frac{v}{u}\right)^*$ in the sense $\left(\frac{u}{v}\right)_* \circ \left(\frac{v}{u}\right)^*(x) \leq x$ for all $x \in S$. We intend to show that $\left(\frac{u}{v}\right)_*$ is the greatest possible of all nondecreasing function φ for which $\varphi\left(\left(\frac{v}{u}\right)^*(x)\right) \leq x$, $x \in S$. We first prove the following simple lemma.

LEMMA 1.3. *Let f be a nondecreasing function mapping S into S. Define g_0: $S \to S$ by the formula*

$$g_0(S) = \inf\{x;\ f(x) \geq s\}.$$

Let g be a nondecreasing function g: $S \to S$. The following conditions are equivalent:

(i) *$p < g(s)$ implies $f(p) < s$,*

(ii) *$g(f(x)) \leq x$ for all $x \in S$,*

(iii) *$g(p) \leq g_0(p)$ for all $p \in S$.*

Proof. Assume that g satisfies (i) and suppose that $g(f(x)) > x$ for some $x \in S$. Setting $p = x$ and $s = f(x)$, we obtain $f(x) = f(p) < s = f(x)$, which is a contradiction. This proves (ii). Now assume (ii). If $f(s) \geq p$, then $s \geq g(f(s)) \geq g(p)$, whence $\inf\{s; f(s) \geq p\} \geq g(p)$. This proves (iii). Now assume (iii) and suppose that $p < g(s)$ and $f(p) \geq s$. It follows that $g_0(s) = \inf\{z;\ f(z) \geq s\} \leq p$, whence $p < g(s) \leq g_0(s) \leq p$, a contradiction. The proof is complete.

The following result together with the preceding lemma shows that, for compact T, the function $\left(\frac{u}{v}\right)_*$ is the largest of the functions g which satisfy $g\left(\left(\frac{v}{u}\right)^*(x)\right)$ $\leq x$.

LEMMA 1.4. *If T is compact, then*

$$\left(\frac{u}{v}\right)_*(p) = \inf\left\{z;\ \left(\frac{v}{u}\right)^*(z) \geq p\right\}$$

for all $p \in S$.

Proof. Let $p \in S$ be given. Consider the following two sets of numbers

$$M_1 = \left\{z;\ \left(\frac{v}{u}\right)^*(z) \geq p\right\}, \qquad M_2 = \{u(t);\ v(t) \geq p\}$$

and let us show that they have the same infimum. First of all we show that M_2 $\subset M_1$. Indeed, consider a $t \in T$ such that $v(t) \geq p$. By Lemma 1.1 we have $\left(\frac{v}{u}\right)^*(u(t)) \geq v(t) \geq p$, so that $u(t) \in M_1$. This proves $M_2 \subset M_1$, whence

$$\inf\left\{z;\ \left(\frac{v}{u}\right)^*(z) \geq p\right\} \leq \inf M_2 = \left(\frac{u}{v}\right)_*(p).$$

To prove the reverse inequality, we intend to show that for each $z \in M_1$ there exists a $z' \in M_2$ for which $z' \leq z$. Fix a $z \in M_1$. Since $\sup\{v(t);\ u(t) \leq z\} \geq p$ and the set $\{t \in T;\ u(t) \leq z\}$ is compact, there exists a $t_0 \in T$ for which $u(t_0) \leq z$ and $v(t_0) \geq p$. Set $z' = u(t_0)$. Then $z' \leq z$ and $z' \in M_2$.

2. Estimates for the spectral radius

Let E be a given finite-dimensional Banach space. We shall denote by A the Banach algebra of all linear operators on E equipped with the operator norm corresponding to E.

NOTATION. For each natural q we set

$$r_q(a) = \frac{|a^q|^{1/q}}{|a|}.$$

Clearly $0 \leq r_q(a) \leq 1$ for all $a \in A$.

It is well known that the limit of the sequence $r_q(a)$ exists and that

$$\lim r_q(a) = \inf r_q(a) = \frac{|a|_\sigma}{|a|}.$$

We shall write

$$r(a) = \frac{|a|_\sigma}{|a|}.$$

It will be useful to remember that

$$r_q(\lambda a) = r_q(a), \qquad r(\lambda a) = r(a)$$

for $\lambda \neq 0$.

We now choose a natural number q and consider the pair of functions

$$f = \left(\frac{r_q}{r}\right)^*, \qquad g = \left(\frac{r}{r_q}\right)_*,$$

so that

$$f(p) = \sup\{r_q(a);\ a \in A,\ r(a) \leq p\},$$
$$g(p) = \inf\{r(a);\ r \in A,\ r_q(a) \geq p\}.$$

The following estimates are immediate from the definition of f and g.

For each $a \in A$

$$|a^q|^{1/q} \leq |a| f\left(\frac{|a|_\sigma}{|a|}\right), \qquad |a|_\sigma \geq |a| g\left(\frac{|a^q|^{1/q}}{|a|}\right).$$

We have to keep in mind that the functions f and g depend on the natural number q. The main result of this section consists in showing that — roughly speaking — these estimates contain nontrivial information if and only if q is greater than or equal to the critical exponent of E.

The results of the preceding section, together with the fact that the unit sphere $\{a \in A;\ |a| = 1\}$ of A is compact (more precisely: the space $A \setminus \{0\}$ divided by the equivalence relation

$$a_1 \sim a_2 \text{ if and only if } a_1 = \lambda a_2 \text{ for some } \lambda$$

is topologically the same as the unit sphere of A), immediately yield the following proposition:

PROPOSITION 2.1. *The functions f and g have the following properties:*
(a) *both f and g are nondecreasing;*
(b) *f is continuous from the right, g is continuous from the left;*
(c) *$g(f(x)) \leqslant x \leqslant f(g(x))$ for all $0 \leqslant x \leqslant 1$;*
(d) *$f \circ g \circ f = f$ and $g \circ f \circ g = g$;*
(e) *g is the largest of all nondecreasing functions φ with the property that $\varphi(f(x)) \leqslant x$ for all $x \in S$.*

THEOREM 2.2. *Let q be a natural number and let*

$$f = \left(\frac{r_q}{r}\right)^*, \quad g\left(\frac{r}{r_q}\right)_*.$$

Then the following conditions are equivalent:
(i) *$|a| = 1$ and $|a^q| = 1$ implies $|a|_\sigma = 1$,*
(ii) *$r_q(a) = 1$ implies $r(a) = 1$,*
(iii) *$f(p) < 1$ for all $p < 1$,*
(iv) *$g(1) = 1$.*

Proof. Assume (i) and consider an $a \in A$, $a \neq 0$ for which $r_q(a) = 1$. Set $b = \dfrac{a}{|a|}$, so that $|b| = 1$ and $r_q(b) = r_q(a) = 1$. It follows that $|b^q|^{1/q} = |b| r_q(b) = 1$ so that, by (i), we have $|b|_\sigma = 1$, whence $r(a) = r(b) = \dfrac{|b|_\sigma}{|b|} = 1$. This proves (ii). Now suppose that (ii) is fulfilled, and suppose that $f(p) = 1$ for some $p < 1$. Consider the set

$$C = \{a \in A;\ |a| = 1,\ |a|_\sigma \leqslant p\}.$$

It is not difficult to see that

$$\sup\{|a^q|^{1/q},\ a \in C\} \geqslant f(p).$$

Indeed, if $b \neq 0$ is such that $r(b) \leqslant p$, then $a = \dfrac{b}{|b|}$ satisfies $|a| = 1$ and $|a|_\sigma = |a| r(a) = |a| r(b) \leqslant p$, so that $a \in C$. At the same time $|a^q|^{1/q} = |a| r_q(a) = r_q(b)$. This establishes the above inequality. The spectral radius being continuous on finite-dimensional algebras, the set C is compact. Let $a_0 \in C$ be the point at which the function $a \to |a^q|^{1/q}$ assumes its maximum on C. We have by the above inequality

$$|a_0^q|^{1/q} \geqslant f(p) = 1,$$

so that $r_q(a_0) = 1$. Since $a_0 \in C$, we have $r(a_0) = \dfrac{|a_0|_\sigma}{|a_0|} \leqslant p$; this contradicts condition (ii), which says that $r(a_0) = 1$. This proves (iii).

To prove the implication (iii) → (iv) let us suppose that $f(p) < 1$ for all $p < 1$ and that $g(1) < 1$. Since $f(g(x)) \geqslant x$ for all $1 \geqslant x \geqslant 0$, we have, by (iii), for $p = g(1)$

$$1 \leqslant f(g(1)) = f(p) < 1,$$

a contradiction.

It remains to prove the implication (iv) → (i). Suppose that $a \in A$ satisfies $|a| = 1$ and $|a^q|^{1/q} = 1$. We thus have $r_q(a) = 1$. Since $g(r_q(x)) \leqslant r(x)$ for all x, we have, in particular,

$$|a|_\sigma = |a| r(a) \geqslant |a| g(r_q(a)) = |a| g(1) = 1.$$

This proves condition (i). The proof is complete.

3. The maximum problem

In this section we summarize a part of the results obtained thus for concerning the quantitative theory of the critical exponent in an n-dimensional Hilbert space. In 1960 the present author established the fact that the critical exponent of a finite-dimensional Hilbert space equals its dimension. The problem of computing the maximum of $|a^n|^{1/n}$ under the conditions $|A| \leqslant 1$ and $|A|_\sigma \leqslant p$ was formulated and solved for the first time by present author in [13], [15]. The author suceeded in describing, for each n and each $0 < p < 1$, a certain operator $S_{n,p}$ on the n-dimensional Hilbert space, such that $S_{n,p}$ realizes the maximum of $|A^n|^{1/n}$ under the conditions $|A| \leqslant 1$, $|A|_\sigma \leqslant p$. Quite recently, by a careful analysis of the proof in [15], Z. Dostál has been able to show that these conditions determine the operator uniquely up to a multiplicative constant of modulus 1 and unitary equivalence. Let us describe briefly the operator $S_{n,p}$ and the method which leads to its construction. The method adopted in [15] consists in dividing the maximum problem into two stages.

The first maximum problem. First, consider a polynomial of degree n whose roots lie in the interior of the unit disc and consider the class $\mathscr{A}$ of all operators T on the n-dimensional Hilbert space with the properties

$$|T| \leqslant 1, \quad \varphi(T) = 0.$$

The spectra of such operators are contained in the spectrum of the polynomial φ. Suppose we have found, for each φ, an operator S_φ which realizes the maximum of $|T^n|$ for $T \in \mathscr{A}_\varphi$. We may then pass to

The second maximum problem. Consider a fixed p, $0 < p < 1$. Denote by $\mathscr{A}(p)$ the class of all operators T with $|T| \leqslant 1$ and $|T|_\sigma \leqslant p$. Consider a $T \in \mathscr{A}(p)$. If φ is the polynomial $\varphi(\lambda) = \det(\lambda - T)$, we have $\varphi(T) = 0$ and the roots of φ lie in the disc

$$D(p) = \{z;\ |z| \leqslant p\}.$$

It follows that

$$\mathscr{A}(p) = \bigcup \mathscr{A}_\varphi,$$

the union being taken over the class $F(p)$ of all polynomials whose roots lie in $D(p)$. It follows that the maximum of $|T^n|$ on $\mathscr{A}(p)$ equals the maximum of the function $\varphi \to |S_\varphi^n|$ on the set $F(p)$.

The solution of the second maximum problem is difficult to prove but easy to state.

The maximum of $|S_\varphi^n|$ on the set $F(p)$ is attained for the polynomial $\varphi(\lambda) = (\lambda - p)^n$.

The first maximum problem has been generalized by B. Sz.-Nagy [8] to completely nonunitary contractions and functions of class H^∞ instead of polynomials.

Let us first state the finite-dimensional case. Let φ be a polynomial of degree $n \geqslant 1$, and denote by $\mathscr{A}_\varphi$ the set of all contractions T on the n-dimensional Hilbert space for which $\varphi(T) = 0$. Let K be the Hilbert space of all sequences $x = (x_0, x_1, \ldots)$ with $|x| = (\sum |x_i|^2)^{1/2}$ and S the backward shift operator on K defined by

$$S(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

Let $K_\varphi = \{x;\ x \in K,\ \varphi(S)x = 0\}$, so that K_φ is an n-dimensional subspace of K invariant with respect to S. Denote by S_φ the restriction of S to K_φ. Clearly $S_\varphi \in \mathscr{A}_\varphi$. We may now state the solution of the first maximum problem as follows.

Let ψ be any polynomial. The maximum of $|\psi(T)|$ for $T \in \mathscr{A}_\varphi$ is attained for $T = S_\varphi$.

To state the result in its full generality, let us recall some notions and facts from the work of B. Sz.-Nagy. A contraction T on a Hilbert space H is called *completely nonunitary* if there exists no nonzero reducing subspace H_0 for T on which T is unitary. For completely nonunitary contractions on H a functional calculus for H^∞ functions may be constructed [8] in the following manner. Given a function $\varphi \in H^\infty$ with $\varphi(z) = \sum a_n z^n$, the operator $\varphi(T)$ is defined as the limit

$$\varphi(T) = \lim_{r \to 1} \sum a_n r^n T^n.$$

This limit exists in the strong topology.

The result of B. Sz.-Nagy may be stated as follows. Let $\varphi \in H^\infty$, $\varphi \neq 0$ and let us denote by $\mathscr{A}_\varphi$ the set of all completely nonunitary contractions T on Hilbert spaces for which $\varphi(T) = 0$. Let K be the Hilbert space of all sequences $x = (x_0, x_1, \ldots)$ with $|x| = (\sum |x_i|^2)^{1/2}$ and S the (backward) shift operator on K defined by $S(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. Let $K_\varphi = \{x;\ x \in K,\ \varphi(S)x = 0\}$, so that K_φ is a subspace of K invariant with respect to S. Let S_φ be the restriction of S to K_φ, so that $S_\varphi \in \mathscr{A}_\varphi$. The solution of the maximum problem is as follows.

Let $\psi \in H^\infty$. The maximum of the norm of $\psi(T)$ for $T \in A_\varphi$ is attained for $T = S_\varphi$.

Once the maximal operator is characterized, it remains to compute the norm of S^n. Since, unfortunately, in a Hilbert space the norm of an operator T equals the square root of the spectral radius of T^*T, the task of finding $|S^n|$ involves finding the maximal eigenvalue of an n-dimensional operator. It might therefore be advisable to look for suitable estimates. Some results of this type have been obtained by N. J. Young [17], [18].

4. The maximal operator

The results of [15] have received only little attention in spite (or possibly because of) their originality. In [15] we have presented the solution of the two maximum problems. The rest, which is sraightforward, more technical and possibly less intriguing from the point of view of functional analysis, was left to the reader. Most readers seem to have missed the possibility of new interesting results. In order to illustrate the further work to be done and one of the possible methods, we intend to present a full description of the maximal operator in two dimensions; we shall use the general method, although for $n = 2$ the result may be obtained much more directly. Even in the case of $n = 2$ an interesting estimate may be obtained.

We begin by writing down the matrix of the maximal operator in an orthonormal basis.

Let φ be the polynomial $\varphi(\lambda) = (\lambda - p)^2$. A natural basis for the space K_φ are the two vectors

$$b_0 = (1, p, p^2, p^3, p^4, \ldots), \quad b_1 = (0, 1, 2p, 3p^2, 4p^3, \ldots),$$

for which $Sb_0 = pb_0$ and $Sb_1 = b_0 + pb_1$. The matrix of S in the basis b_0, b_1 is thus

$$\begin{pmatrix} p & 0 \\ 1 & p \end{pmatrix}.$$

In order to orthogonalize the basis, we shall need the scalar products. It is not difficult to compute

$$(b_0, b_0) = \sum_0 p^{2k} = \frac{1}{1-p^2},$$

$$(b_0, b_1) = \sum_0 np^{2n-1} = \frac{p}{(1-p^2)^2},$$

$$(b_1, b_1) = \sum_0 (np^{n-1})^2 = \frac{p^2+1}{(1-p^2)^3}.$$

If we write, for shortness, w for $(1-p^2)^{1/2}$, it is easy to see that

$$e_0 = wb_0, \quad e_1 = -wpb_0+w^3b_1$$

is an orthonormal system in which the matrix of S assumes the following simple form:

$$\begin{pmatrix} p & 0 \\ (1-p)^2 & p \end{pmatrix}.$$

Its square turns out to be

$$\begin{pmatrix} p^2 & 0 \\ 2p(1-p^2) & p^2 \end{pmatrix} = p^2 \begin{pmatrix} 1 & 0 \\ 2\dfrac{1-p^2}{p} & 1 \end{pmatrix},$$

so that the norm of S^2 will be

$$|S^2| = p^2|M|,$$

where M is the operator given by

$$\begin{pmatrix} 1 & 0 \\ 2b & 1 \end{pmatrix}$$

with $b = (1-p^2)/p$.

Now it is not difficult to see that the norm of the operator given (in an orthonormal basis) by the matrix

$$\begin{pmatrix} 1 & 0 \\ 2b & 1 \end{pmatrix}$$

is $b+(1+b^2)^{1/2}$.

To sum up:

Let H be a two-dimensional Hilbert space. Let p be a given number, $0 < p < 1$. The operator T whose matrix with respect to an orthonormal basis is

$$\begin{pmatrix} p & 0 \\ 1-p^2 & p \end{pmatrix}$$

has the following properties:

(a) T is a contraction, $|T| = 1$,

(b) $|T|_\sigma = p$,

(c) among all the operators on H with properties $|T| \leqslant 1$, $|T|_\sigma \leqslant p$ the operator T has the largest value of $|T^2|$. This maximum equals

$$p\left(1-p^2+\sqrt{1-p^2+p^4}\right).$$

It is now easy to give estimates from below for the spectral radius.

Let us denote by φ the function

$$\varphi(p) = p\left(1-p^2+(1-p^2+p^4)^{1/2}\right).$$

We have shown that, for every operator T,

$$\frac{|T^2|}{|T|^2} \leqslant \varphi\left(\frac{|T|_\sigma}{|T|}\right).$$

Suppose we have a nondecreasing function ψ for which $\psi(\varphi(p)) \leqslant p$ for each p, $0 < p < 1$. It follows that

$$\psi\left(\frac{|T^2|}{|T|^2}\right) \leqslant \frac{|T|_\sigma}{|T|},$$

so that

$$|T|\,\psi\left(\frac{|T^2|}{|T|^2}\right) \leqslant |T|_\sigma \leqslant |T^2|^{1/2}.$$

It is not difficult to verify that $\psi(t) = \frac{1}{2}t$ is such a function. Even this simple choice yields an interesting inequality, valid for all operators on a two-dimensional Hilbert space

$$\frac{1}{2}\frac{|T^2|}{|T|} \leqslant |T|_\sigma \leqslant |T^2|^{1/2}.$$

Let us mention briefly the result in the general case. The maximal operator, taken with respect to an orthonormal basis, assumes the following form

$$\begin{matrix} p & 0 & 0 & 0 & 0 & \dots \\ 1-p^2 & p & 0 & 0 & 0 & \dots \\ -p(1-p^2) & 1-p^2 & p & 0 & 0 & \dots \\ p^2(1-p^2) & -p(1-p^2) & 1-p^2 & p & 0 & \dots \\ -p^3(1-p^2) & p^2(1-p^2) & -p(1-p^2) & 1-p^2 & p & \dots \\ p^4(1-p^2) & -p^3(1-p^2) & \dots & & & \end{matrix}$$

We may speak of *the* maximal operator since Z. Dostál [1] has shown that all solutions of the maximum problem differ only by a multiplicative constant and unitary equivalence. In spite of the undeniable aesthetic qualities of the above matrix, all this seems to indicate that it is more advisable to look for estimates.

5. Some open problems

We intend to conclude this paper by mentioning some of the vast number of questions which should be answered. We begin by restating two of the problems mentioned in the present author's report at the Copenhagen Convexity Symposium in 1965.

A deceptively simple problem which is still unsolved is the following:

PROBLEM 1. Does there exist an infinite-dimensional Banach space with a finite critical exponent?

More important problems arise of course in finite dimensions. We collect first problems for general Banach spaces.

PROBLEM 2. Characterize Banach spaces whose critical exponent is finite.

PROBLEM 3. Determine the asymptotic behaviour of the function $M(n)$ where $M(n)$ is the minimum of the critical exponents of all n-dimensional Banach spaces.

Since for a Hilbert space the critical exponent equals its dimension, we have $M(n) \leqslant n$ for every n. B. Grünbaum and M. Perles have shown that $M(n)$ can be considerably less than n. Hence even the following weaker version of the preceding problem is interesting.

PROBLEM 4. Is $\liminf M(n)$ infinite?

In other words, does there exist a sequence of Banach spaces E_n with $\dim E_n$ tending to infinity and such that the critical exponents of all E_n lie below a certain finite bound?

A deeper investigation of the function f of section two immediately suggests the following problems.

PROBLEM 5. Let E be a Banach space, q a natural number, p a positive number, $p < 1$. Suppose a_0 is the operator which realizes the maximum of $|a^q|$ subject to the constraints $|a| \leqslant 1$ and $|a|_\sigma \leqslant p$. Does it follow that $|a_0| = 1$ and $|a_0|_\sigma = p$? Clearly, at least one of the two equalities must hold.

To clarify the meaning of the constraints $|a| \leqslant 1$, $|a|_\sigma \leqslant p$ it would be useful to solve

PROBLEM 6. Let E be a Banach space. For each $0 < p < 1$ give a description of the set

$$C(p) = \{a \in B(E);\ |a| = 1,\ |a|_\sigma = p\}.$$

Is it always nonvoid?

A closely related problem is

PROBLEM 7. Let E be a Banach space, q a natural number; for each positive $p < 1$ set

$$f(p) = \sup |a^q|,$$

the supremum being taken over all linear operators a on E subject to the constraints $|a| \leqslant 1$, $|a|_\sigma \leqslant p$. Is f a strictly increasing function of p? Is f continuous?

The critical exponent of the n-dimenional l_1 or l_∞ space is n^2-n+1 [7], for l_2 it equals n [11]. It remains to solve

PROBLEM 8. Determine the critical exponent of the n-dimensional l_p space.

Apart from the interesting and ingenious investigations of M. Perles [9], [10], who proved the existence and obtained estimates for some p, little progress was made.

The case of the Hilbert space is both interesting and important. Let us conclude with the following intriguing

PROBLEM 9. Let H be an n-dimensional Hilbert space. Let p be a positive number, $0 < p < 1$, and denote by $D(p)$ the disc $\{z;\ |z| \leqslant p\}$. Let ψ be a given polynomial (or a function holomorphic on a neighbourhood of $D(p)$). Let $F(p)$ be the set of all polynomials whose roots lie in $D(p)$. For each $\varphi \in F(p)$ we know that

$$\max\{|\psi(a)|;\ a \in B(H),\ |a| \leqslant 1,\ \varphi(a) = 0\} = |\psi(S_\varphi)|.$$

Find the polynomial $\varphi \in F(p)$ which realizes the maximum of the function

$$\varphi \to |\psi(S_\varphi)| \text{ on } F(p).$$

The classical result of the present author says that, for $\psi(z) = z^n$ (n being the dimension) this maximum is attained for the polynomial $\varphi(z) = (z-p)^n$.

A d d e d i n p r o o f. A survey of the more recent results in this area is contained in the papers

V. P t á k, *A maximum problem for matrices*, Linear Algebra and Appl. 28 (1979), 193–204.

V. P t á k and N. J. Y o u n g, *Functions of operators and the spectral radius*, ibid. 29 (1980), 357–392.

References

[1] Z. D o s t á l, *Uniqueness of the operator attaining* $C(H_{n,r,n})$, Čas. pěst. mat. 103 (1978), 236–243.

[2] —, *Norms of iterates an the spectral radius of matrices*, ibid. 105 (1980), 256–260.

[3] H. F l a n d e r s, *On the norm and spectral radius*, Lin. and Multilin. Algebra 2 (1974), 239–240.

[4] M. G o l d b e r g and G. Z w a s, *On matrices having equal spectral radius and spectral norm*, Lin. Alg. Appl. 8 (1974), 427–434.

[5] В. К и р з н е р и М. И. Т а б а ч н и к о в, *О критических показателях норм в n-мерном пространстве*, Сиб. Мат. Журнал 12 (1971), 672–675.

[6] Ю. И. Л ю б и ч и М. И. Т а б а ч н и к о в, *Об одной теореме Марика-Птака*, ibid. 10 (1969), 470–473.

[7] J. M a ř í k and V. P t á k, *Norms spectra and combinational properties of matrices*, Czech. Math. J. 85 (1960), 181–196.

[8] B. Sz.-N a g y, *Sur la norme des fonctions de certains operateurs*, Acta Math. Ac. Sci. Hungaricae 20 (1969), 331–334.

[9] M. P e r l e s, *Critical exponents of convex bodies*, The Hebrew University of Jerusalem Ph. D. thesis 1964 (Hebrew with English Summary).

[10] —, *Critical exponents of convex sets*, Proc. Coll. Convexity, Copenhagen 1967, 221–228.

[11] V. P t á k, *Norms and the spectral radius of matrices*, Czech. Math. J. 87 (1962), 553–557.

[12] —, *Critical exponents*, Proc. of the 1965 Copenhagen Coloquium on Convexity, Copenhagen 1967, 244–248.

[13] —, *Rayon spectral, norme des itérés d'un opérateur et exposant critique*, C. R. Acad. Sci. Paris 265 (1967), 257–259.

[14] —, *Isometric parts of operators and the critical exponent*, Časopis pěst. mat. 101 (1967), 383–388.

[15] —, *Spectral radius, norms of iterates and the critical exponent*, Linear Algebra and Appl. 1 (1968), 245–260.

[16] H. W i m m e r, *Spektralradius und Spektralnorm*, Czech. Math. J. 99 (1974), 501–502.

[17] N. J. Y o u n g, *Norms of powers of matrices with constrained spectrum*, Linear Algebra and Appl. 23 (1979), 227–244.

[18] —, *Analytic programmes in matrix algebras*, Proc. London Math. Soc. (3) 36 (1978), 226–242.

[19] —, *Matrices which maximise any analytic function*, Acta Math. Acad. Sci. Hungaricae 34 (1979), 239–243.

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