Borel set Δ onto an interval (c, d) of the real line if $\xi(x) \in (c, d)$ for almost all $x \in \Delta$, and $\xi(x) \notin (c, d)$ for almost all $x \in R \setminus \Delta$. Then

$$(1) \qquad \int_{\Delta} \frac{z-w^*}{(t-z)(t-w^*)} f(\xi(t))dt = \int_c^d \frac{\xi(z)-\xi(w)^*}{(t-\xi(z))(t-\xi(w)^*)} f(t)dt$$

for all nonreal numbers z and w and any function $f(x)$ such that $(1+x^2)^{-1}f(x) \in L^1(c, d)$. If also

$$\lim_{y\to\infty} \xi(iy)/(iy) = \beta > 0,$$

then for any $f(x) \in L_p(c, d)$, $1 \leqslant p < \infty$, and any nonreal number z,

$$(2) \qquad \int_{\Delta} \frac{f(\xi(t))}{t-z} dt = \int_c^d \frac{f(t)}{t-\xi(z)} dt$$

and

$$(3) \qquad \int_{\Delta} |f(\xi(t))|^p dt = \beta^{-1} \int_c^d |f(t)|^p dt;$$

furthermore, in case $p = 1$,

$$(4) \qquad \int_{\Delta} f(\xi(t))dt = \beta^{-1} \int_c^d f(t)dt.$$

Under the stated assumptions, the integrals on the left sides of (1)–(4) converge absolutely. Analogous formulas are proved for singular integrals. Applications are given to approximation theory, distribution function formulas for Hilbert transforms, and orthogonal expansions and isometric operators in L^2 spaces.

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A NOTE ON SEMICHARACTERS

Z. SŁODKOWSKI and W. ŻELAZKO

Institute of Mathematics of the Polish Academy of Sciences, Warszawa, Poland

We introduce here a concept of semicharacter of a complex Banach algebra. We show that the algebra of all $n \times n$ matrices with complex entries possesses a proper semicharacter if and only if $n = 2$. We prove the absence of semicharacters for algebras $B(X)$ of all bounded endomorphisms of some classical Banach spaces. We also investigate relations between semicharacters and minimal subspectra and, finally, we give the definition of a semicharacter of a locally compact group.

1. Semicharacters of Banach algebras

DEFINITION 1.1. Let A be a complex Banach algebra. A *semicharacter on A* is a complex-valued function φ defined on A such that for every commutative subalgebra $\mathscr{A} \subset A$ the restriction of φ to $\mathscr{A}$ is a multiplicative-linear functional (= a character) on $\mathscr{A}$. We do not assume that φ is a continuous function. In case when A possesses the unit element e, we assume also that φ is not identically equal to zero, i.e. $\varphi(e) = 1$.

If A has no unit element and $A_1 = A \oplus \{Ce\}$ is its unital extension, then every semicharacter φ on A extends to a semicharacter φ_1 on A_1 given by $\varphi_1(x+\lambda e) = \varphi(x)+\lambda$, $x \in A$, $\lambda \in C$.

Let us also remark that if a semicharacter is a linear functional on A, then it is multiplicative and linear, i.e. a character on A (cf. [1]). And so, we say that a semicharacter is *proper* if it is not a character, i.e. if it is not a linear functional.

We shall now describe all semicharacters of the algebras of all linear endomorphisms of n-dimensional Euclidean spaces, $n = 2, 3, \ldots$ (for $n = 1$ it is the algebra isomorphic to the field C and it has a character — the identity map onto itself). Since all these algebras are simple, then all possible semicharacters are proper. If X is a Banach space, $B(X)$ will stand for the Banach algebra of all bounded endomorphisms of X. With this notation we have the following

THEOREM 1.2. *Let $A = B(C^n)$, $n = 1, 2, \ldots$ Then A possesses $2^{\mathfrak{c}}$ (proper) semicharacters if $n = 2$ and no semicharacters if $n \geqslant 3$. Here $\mathfrak{c}$ is the cardinality of continuum.*

Proof. We shall identify the elements of $B(C^n)$ with $n\times n$ matrices with complex entries. We shall divide our proof onto three steps.

I. Suppose first that $n=2$. A direct computation gives all commutative maximal subalgebras of $B(C^2)$, in fact all of them are commutants of single matrices. We list out all these algebras in form of parametrized families (here e will stand for the unit matrix $\begin{pmatrix}1&0\\0&1\end{pmatrix}$).

1° Algebras of the form $\mathscr{A}=\mathscr{A}(\alpha,\beta)=\left\{xe+y\begin{pmatrix}0&\alpha\\\beta&1\end{pmatrix}:\ x,y\in C\right\},\quad \alpha,\beta\in C;$ this is a two-parameter family consisting of continuum maximal commutative subalgebras of A.

2° $\mathscr{A}=\mathscr{A}(\gamma,\delta)=\left\{xe+y\begin{pmatrix}0&\gamma\\\delta&0\end{pmatrix}:\ x,y\in C\right\},\ \gamma,\ \delta\in C,\ |\gamma|+|\delta|>0.$ This is another two-parameter family of maximal commutative subalgebras of A.

Every maximal commutative subalgebra of A is either of the form 1° or 2°, different values of parameters give different algebras and all these algebras are "almost disjoint" in the sense that the intersection of any two different algebras consists of scalar multiples of the unit matrix e.

An easy computation shows that every multiplicative-linear functional on the algebra $\mathscr{A}(\alpha,\beta)$ of the form 1° is given by the formula

$$f(xe+yg)=x\pm ys,\quad \text{where}\quad s=(1+4\alpha\beta)^{1/2}\ \text{and}\ g=\begin{pmatrix}0&\alpha\\\beta&1\end{pmatrix}.$$

Similarly, every multiplicative-linear functional f on the algebra $\mathscr{A}(\gamma,\delta)$ of the form 2° is given by

$$f(xe+yh)=x\pm yr,\quad \text{where}\quad r=(\gamma\delta)^{1/2}\ \text{and}\ h=\begin{pmatrix}0&\delta\\\gamma&0\end{pmatrix}.$$

This means that $\mathscr{A}(\alpha,\beta)$ of the form 1°, or $\mathscr{A}(\gamma,\delta)$ of the form 2° has only one character, provided $\alpha\beta=-1/4$ or $\gamma\delta=0$, respectively. Otherwise these algebras have exectly two characters $f_t^{(1)}$, $f_t^{(2)}$, $f_t^{(i)}\in\mathfrak{M}(\mathscr{A}_t)$, $t\in[0,1]$, where by $\mathscr{A}_t$, $t\in[0,1]$, we designate the family of all algebras of the form 1° or 2° possessing exactly two different multiplicative and linear functionals (there are continuum of such algebras). $\mathfrak{M}(\mathscr{A}_t)$ stands for the two-point maximal ideal space of $\mathscr{A}_t$ (we identify maximal ideals with corresponding multiplicative-linear functionals). Designate also by $\mathscr{A}_t$, $t\in[2,3]$, the set of all remaining algebras in the families 1° and 2°, so that for each $t\in[2,3]$ the algebra $\mathscr{A}_t$ possesses exactly one multiplicative and linear functional f_t.

For any subset $S\subset[0,1]$ we put

$$\varphi_S(x)=\begin{cases}f_t^{(1)}(x) & \text{for}\quad x\in\mathscr{A}_t,\ t\in S,\\ f_t^{(2)}(x) & \text{for}\quad x\in\mathscr{A}_t,\ t\in[0,1]\setminus S,\\ f_t(x) & \text{for}\quad x\in\mathscr{A}_t,\ t\in[2,3].\end{cases}\tag{1}$$

Since for any x of the form $x=\lambda e$ it is $\varphi_s(x)=\lambda$ and for x not of this form there is exactly one $t\in[0,1]\cup[2,3]$ with $x\in\mathscr{A}_t$, φ_s is a well defined complex-valued function on A. Moreover, $\varphi_s|\mathscr{A}_t\in\mathfrak{M}(\mathscr{A}_t)$ for all t, so φ_s is a semicharacter on A. On the other hand, every semicharacter on A must be of the form φ_S for some $S\subset[0,1]$, and for $S_1\neq S_2$ it is $\varphi_{S_1}\neq\varphi_{S_2}$. In this way we have exactly $2^{\mathfrak{c}}$ semicharacters on $A=B(C^2)$, where $\mathfrak{c}$ is the cardinality of continuum.

II. Suppose that $n=3$ and put $A=B(C^3)$. We shall show that there are no semicharacters on A. Assume the contrary: there is a semicharacter φ on A, and try to get a contradiction. Consider the set S of all points $s\in C^3$ with $|s|=1$ and with real coordinates. We can identify the set S with the unit sphere in R^3. To each $s\in S$ there corresponds an element $p(s)\in B(C^3)$ which is the orthogonal projection onto the one-dimensional subspace of C^3 spanned by s. Whenever we have three pairwise orthogonal elements $s_1,s_2,s_3\in S$, then $p(s_1)+p(s_2)+p(s_3)=e$ — the unit element of $B(C^3)$, and $p(s_i)p(s_j)=p(s_j)p(s_i)=0$ for $i,j=1,2,3$, $i\neq j$, so the elements $p(s_i)$ commute pairwise. This implies that

$$1=\varphi\big(p(s_1)+p(s_2)+p(s_3)\big)=\varphi\big(p(s_1)\big)+\varphi\big(p(s_2)\big)+\varphi\big(p(s_3)\big).$$

Since $p(s)^2=p(s)$, we have $\varphi\big(p(s)\big)=0$ or 1 for each $s\in S$. Together with the previous formula this means that, given any three pairwise orthogonal points $s_1,s_2,s_3\in S$, one of them, say s_1, satisfies $\varphi\big(p(s_1)\big)=1$, while for the other two we have $\varphi\big(p(s_2)\big)=\varphi\big(p(s_3)\big)=0$.

This also means that if $\varphi\big(p(s)\big)=1$ for some $s\in S$, then $\varphi\big(p(s')\big)=0$ for every $s'\in S$ with $s'\perp s$, i.e. $\varphi\big(p(s')\big)=0$ for all s' in the great circle (equator circle) of $S\subset R^3$, ortogonal to s. We put

$$W=\{s\in S:\ \varphi\big(p(s)\big)=0\};$$

from the above it follows that W contains at least two different circles of the form $S\cap s^{\perp}$, $s\in S$, and together with any two points s_1,s_2, $s_1\perp s_2$, it contains the whole great circle passing through these points. Easy geometric considerations, which we leave to the reader, show that the only set W with these properties is the whole of S. This gives a desired contradiction, since for some $s\in S$ it must be $\varphi\big(p(s)\big)=1$. The contradiction shows that there are no semicharacters on $B(C^3)$.

III. Suppose that $n>3$ and $A=B(C^n)$. We will show that there is no semicharacter on A. Assume, as before, that there is one, denoted by φ, and try to get a contradiction. Consider the commutative subalgebra $\mathscr{A}$ of A consisting of all diagonal matrices of the form

$$x=\begin{pmatrix}\alpha_1(x)&&0\\&\ddots&\\0&&\alpha_n(x)\end{pmatrix}.$$

Such an algebra is isomorphic to the algebra of all complex valued functions defined on a finite set consisting of n points, and so every character on $\mathscr{A}$ must be a point evaluation, i.e. $\varphi(x)=\alpha_i(x)$ for some fixed i. Without loss of generality we may assume that $i = 1$, so that $\varphi(x) = \alpha_1(x)$ for all x in $\mathscr{A}$. Let x_0 be the matrix (a_{ij}), $1 \leqslant i, j \leqslant n$, with $a_{11} = 1$ and $a_{ij} = 0$ for $i \cdot j \neq 1$. Then $x_0 \in \mathscr{A}$ and $\varphi(x_0) = 1$. Let A_0 be the subalgebra of A consisting of all matrices (a_{ij}) with $a_{ij} = 0$ for $\min(i,j) > 3$. The algebra A_0 is isomorphic to $B(\boldsymbol{C}^3)$ and $x_0 \in A_0$. The semi-character φ restricted to A_0 is not identically equal to 0, since $\varphi(x_0) = 1$. This gives a contradiction, since, by II, there are no semicharacters on $B(\boldsymbol{C}^3)$.

Remark 1.3. From the formula (1) it follows that no semicharacter on $B(\boldsymbol{C}^2)$ is continuous. Thus we pose the following

PROBLEM. Suppose that there is a continuous semicharacter φ on a Banach algebra A. Does it follow that φ is a character on A?

We shall now consider the problem of existence of proper semicharacters on the algebras of the form $B(X)$ where X is an infinite-dimensional complex Banach space.

PROPOSITION 1.4. *If a Banach space X can be decomposed into a direct sum*

$$X = X_1 \oplus X_2 \oplus X_3, \tag{2}$$

with X_1, X_2, X_3 isomorphic to each other, then the algebra $B(X)$ has no semicharacter.

Proof. Let p_{ij} be the isomorphism of X_j onto X_i, with p_{ii} being the identity map of X_i, choosen in such a way that

$$p_{ij} p_{jk} = p_{ik}, \quad 1 \leqslant i, j, k \leqslant 3.$$

(If we identify X_i with X_j, we can choose as p_{ij} the identity map.) If $a = (a_{ij})$, $1 \leqslant i, j \leqslant 3$, is a numerical matrix with complex entries, then it defines an operator on X given by

$$T_a(x) = T_a\big((x_1, x_2, x_3)\big) = \Big(\sum_{i=1}^{3} a_{1i} p_{1i} x_i, \sum_{i=1}^{3} a_{2i} p_{2i} x_i, \sum_{i=1}^{3} a_{3i} p_{3i} x_i\Big)$$

where $x = (x_1, x_2, x_3) \in X$ and $x_i \in X_i$.

We have $T_{ab} = T_a T_b$, $T_e = I$ — the identity map of X, and so $a \to T_a$ is a unital isomorphism of $B(\boldsymbol{C}^3)$ into $B(X)$. If Φ is a semicharacter on $B(X)$, then the formula

$$\varphi(a) = \Phi(T_a)$$

would give a semicharacter on $B(\boldsymbol{C}^3)$, which, in view of Theorem 1.2, is impossible. Thus there are no semicharacters on $B(X)$.

There are many Banach spaces admitting decomposition of the form (2) with summands isomorphic to each other. This holds, in particular, for spaces isomorphic to their squares (as e.g. infinite-dimensional Hilbert spaces, or l_p spaces). On the other hand, there are infinite-dimensional Banach spaces X for which the algebra $B(X)$ possesses not only semicharacters but even characters. An example of such a space is the famous James space, which we shall denote by J. Using these spaces we shall obtain infinite-dimensional Banach spaces X for which the algebras $B(X)$ possess proper semicharacters.

PROPOSITION 1.5. *Let Y be a Banach space such that the algebra $B(Y)$ possesses a (non-zero) character Φ. Put $X = Y \oplus Y$. Then $B(X)$ possesses proper semicharacters, and each semicharacter of $B(X)$ is proper.*

Proof. Let $x \in X$, $x = (y_1, y_2)$, $y_i \in Y$, and let $T \in B(X)$. Put $T(y_1, 0) = \big(T_{11}(y_1), T_{21}(y_1)\big)$, $T(0, y_2) = \big(T_{12}(y_2), T_{22}(y_2)\big)$. Here T_{ij} is an element of $B(Y)$. Let Φ be a character on $B(Y)$ and write

$$m(T) = \begin{pmatrix} \Phi(T_{1,1}) & \Phi(T_{1,2}) \\ \Phi(T_{2,1}) & \Phi(T_{2,2}) \end{pmatrix}.$$

It is easy to see that m is a unital homomorphism of $B(X)$ onto $B(\boldsymbol{C}^2)$. Taking any semicharacter φ_s on $B(\boldsymbol{C}^2)$ (see formula (1)), we obtain a semicharacter

$$\Phi_S(T) = \varphi_S\big(m(T)\big)$$

on $B(X)$.

On the other hand, similarly as in Proposition 1.4, we see that there are no characters on $B(X)$, since there exists a unital imbedding of $B(\boldsymbol{C}^2)$ into $B(X)$.

COROLLARY 1.6. *If $X = J \oplus J$, then there are proper semicharacters on $B(X)$.*

2. Semicharacters and minimal subspectra

In this section we explain relations between the concept of semicharacter and previously introduced concept of a subspectrum [2]. Theorem 2.4 obtained here is in fact the source of the concept of a semicharacter.

Let A be a complex Banach algebra. Designate by $c(A)$ the family of all non-void subsets of A consisting of pairwise commuting elements. Suppose that for each $x_I = (x_i)_{i \in I} \in c(A)$, there corresponds a non-void compact subset $\tilde{\sigma}(x_I) \subset \boldsymbol{C}^I$. We say that the map $\tilde{\sigma}$: $x_I \to \tilde{\sigma}(x_I)$ is a *subspectrum* on A if

1° $\tilde{\sigma}(x_I) \subset \prod_{i \in I} \sigma(x_i) \subset \boldsymbol{C}^I$, where $\sigma(x_i)$ is the usual spectrum of an element $x_i \in A$;

2° For any system $p_j(t_I)$, $j \in J$, of complex polynomials in indeterminates $t_I = (t_i)_{i \in I}$ (each polynomial depending only upon a finite number of indeterminates t_i) we have the following relation, called spectral mapping property:

$$p_J\big(\tilde{\sigma}(x_I)\big) = \tilde{\sigma}\big(p_J(x_I)\big),$$

where p_J on the left-hand is understood as a map from $\boldsymbol{C}^I$ into $\boldsymbol{C}^J$ given by $\alpha_I \to p_J(\alpha_I)$, and on the right side $p_J(x_I)$ is the family $\big(p_j(x_I)\big)_{j \in J} \in c(A)$.

The following Propositions 2.1 and 2.2 are proved in [2].

PROPOSITION 2.1. *Let $\tilde{\sigma}$ be a subspectrum on A, and let $m(A)$ designate the family of all commutative maximal subalgebras of A. Then to each $\mathscr{A} \in m(A)$ there corresponds a non-void compact subset $\Delta(\mathscr{A})$ of the maximal ideal space $\mathfrak{M}(\mathscr{A})$ such that for any $x_I \in c(A)$ with $x_I \subset \mathscr{A} \in m(A)$ it is*

$$\tilde{\sigma}(x_I) = \big\{\big(f(x_i)\big)_{i \in I} \in \boldsymbol{C}^I : f \in \Delta(\mathscr{A})\big\}.$$

If $\tilde{\sigma}_1, \tilde{\sigma}_2$ are two subspectra on A, we write $\tilde{\sigma}_1 \leqslant \tilde{\sigma}_2$ when $\tilde{\sigma}_1(x_I) \subset \tilde{\sigma}_2(x_I)$ for every $x_I \in c(A)$. A subspectrum $\tilde{\sigma}_0$ is called *minimal* if $\tilde{\sigma} \leqslant \tilde{\sigma}_0$ implies $\tilde{\sigma} = \tilde{\sigma}_0$.

PROPOSITION 2.2. *Any Banach algebra possesses minimal subspectra.*

DEFINITION 2.3. A subspectrum $\tilde{\sigma}$ on A is called a *single point subspectrum* (shortly an sp-subspectrum) if for each $x_I \in c(A)$ the set $\tilde{\sigma}(x_I)$ consists of a single point. Clearly, any sp-subspectrum is minimal.

THEOREM 2.4. *Let A be a complex Banach algebra. There is a one-to-one correspondence between sp-subspectra of A and semicharacters of A, given by the relation*

$$\tilde{\sigma}(x_I) = \{(\varphi(x_i))_{i\in I}\} \subset C^I,$$

where $\tilde{\sigma}$ is an sp-subspectrum and φ the corresponding semicharacter.

An easy proof of this theorem follows immediately from Proposition 2.1.

COROLLARY 2.5. *If X is a Hilbert space,* $\dim X > 2$, *or if X is any Banach space with a decomposition of the form* (2), *then the algebra $B(X)$ has no sp-subspectra.*

Since any commutative Banach algebra A possesses an sp-subspectrum, we have the following

Remark 2.6. Generally speaking, a sp-subspectrum defined on a subalgebra cannot be extended to a sp-subspectrum defined on the whole algebra.

3. Semicharacters on groups

In this section we remark shortly that the definition and some results of Section 1 have their analogues in the theory of locally compact groups. We give no proof of the theorem formulated here, since it is the same as in Section 1.

DEFINITION 3.1. Let G be a locally compact group. A *semicharacter on G* is a complex valued function φ on G, with $|\varphi(s)| \equiv 1$, such that φ restricted to any commutative subgroup of G is a character.

THEOREM 3.2. *The group* $\mathrm{Sl}(n, C)$ *of all non-singular $n \times n$ matrices with complex entries and determinant equal to one possesses semicharacters if and only if $n \leqslant 2$, and proper semicharacters if and only if $n = 2$.*

We pose also a problem, analogous to that formulated in Section 1.

PROBLEM. Let G be a connected l.c. group and φ a continuous semicharacter on G. Does it follow that φ is a character?

For disconnected groups the answer is in negative, e.g. for $\mathrm{Gl}(2, C)$ with discrete topology.

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FREDHOLM THEORY IN BANACH ALGEBRAS

M. R. F. SMYTH

Computer Centre, Queen's University, Belfast

1. Introduction

The classical Fredholm theory of bounded linear operators on a Banach space is familiar to many mathematicians. The thesis of this paper is that to every 2-sided ideal in the pre-socle of a general Banach algebra there corresponds a sensible Fredholm theory. It is a consequence of Atkinson's theorem [1] and the fact that the finite rank operators constitute the pre-socle of the Banach algebra of all bounded linear operators on a Banach space, that our theory includes the classical theory as a special case. Progress in the case of semisimple Banach algebras has already been made by Barnes [2] using the socle as his basic ideal, and our work is an extension of this.

Inessential, Riesz and Fredholm elements are defined in § 3 and some of their elementary properties are developed. This is general Fredholm theory (i.e. there is no reference to the pre-socle) nevertheless several of the results are important. In particular, the Fredholm elements form an open multiplicative semigroup and the inessential elements and Fredholm perturbations coincide. In § 4 we define the nullity, defect and index of a Fredholm element, prove the punctured neighbourhood theorem and establish the well-known continuity, stability and multiplicative properties of the index. In § 5 we apply these results to deduce that every Riesz point is a Fredholm point of index zero, and every Fredholm point in the boundary of the spectrum is a Riesz point; results which lead to a Ruston-type characterization of Riesz elements. We also show that if the algebra is commutative, then the sets of Riesz and Fredholm points of any given element are equal, and this enables us to derive spectral mapping theorems for the essential spectra of Wolf and Browder. The index function of a Fredholm element is defined in § 6, and in § 7 this is applied in order to extend important results of Schechter and Stampfli concerning the Weyl spectrum. For an up to data account of other generalizations of Fredholm theory the reader is referred to Chapter VI of [4].