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A DECOUPLING APPROACH TO RESONANCES FOR MULTIPARTICLE QUANTUM SCATTERING SYSTEMS

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1. The multichannel scattering model

Let $\mathfrak{H}$ be a separable Hilbert space, let H be a self-adjoint operator in $\mathfrak{H}$ (the Hamiltonian), and let $\mathfrak{A}$ be an algebra of pairwise commuting self-adjoint operators A with the following properties: There is a linear manifold $\mathfrak{D}$ in $\mathfrak{H}$, dense in $\mathfrak{H}$, such that $\mathfrak{D} \subseteq \operatorname{dom} A$, $\operatorname{ima} A\restriction\mathfrak{D} \subseteq \mathfrak{D}$ for all A and $A\restriction\mathfrak{D}$ is essentially self-adjoint, $1 \in \mathfrak{A}$ and all A with $A \neq \gamma 1$, γ real, are absolutely continuous.

H is "asymptotically tested" by $\mathfrak{A}$. That is, we consider for each $A \neq \gamma 1$ the subspace of vectors $f \in \mathfrak{H}$ with the property "$\operatorname*{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itA)f$ exists". We denote the orthoprojection onto this subspace by $Q_A^{\pm}$ and assume $Q_A^{+} = Q_A^{-} = Q_A$. The operator A defines a *scattering channel* with respect to H if $Q_A \neq 0$. In this case we call A a *channel Hamiltonian*, Q_A the corresponding *channel projection*, $Q_A\mathfrak{H}$ the *channel subspace*, $W_A^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itA)Q_A$ the *channel wave operator*. Let $P_A^{\pm} = W_A^{\pm}(W_A^{\pm})^*$. $P_A^{\pm}$ reduces H and $H\restriction P_A^{\pm}\mathfrak{H}$ is absolutely continuous. One can prove (see, for instance, H. Baumgärtel [2]): If $A \neq B$, then $P_A^{\pm}P_B^{\pm} = 0$. Therefore the set of channels is at most denumerable. Hence we may define the orthoprojection $P_{\mathfrak{A}}^{\pm} = \sum_{A\in\mathfrak{A}} P_A^{\pm}$. The model is called *asymptotically complete*, if $P_{\mathfrak{A}}^{\pm} = P^{ac}(H)$, where $P^{ac}(H)$ denotes the orthoprojection onto the subspace of absolute continuity of H. We denote the sequence of channels by $A_1, A_2, \ldots$ Let $\mathfrak{H}^0$ denote the Hilbert space $Q_{A_1}\mathfrak{H} \times Q_{A_2}\mathfrak{H} \times \ldots$, A^0 the self-adjoint operator $A_1 \times A_2 \times \ldots$ in $\mathfrak{H}^0$ and let $W^{\pm} f = \sum_{\varrho=1}^{\infty} W_{A_\varrho}^{\pm} f_\varrho$, where $f = \{f_1, f_2, \ldots\} \in \mathfrak{H}^0$. We put $S = (W^+)^* W_-$, $T = S-1$. S commutes with A^0 and is unitary if the model is asymptotically complete. Let $Q_{A_\varrho}\mathfrak{H} = \int_{\operatorname{spec} A_\varrho} \mathfrak{K}_\varrho(\lambda)\, d\lambda$ be a resolution of $Q_{A_\varrho}\mathfrak{H}$ in a direct integral with respect to A_ϱ, where the $\mathfrak{K}_\varrho(\lambda)$ are suitable separable Hilbert spaces. Let $\mathfrak{K}(\lambda) = \mathfrak{K}_1(\lambda) \times \mathfrak{K}_2(\lambda) \times \ldots$ Then $\mathfrak{H}^0 \ni f \leftrightarrow f(\lambda) = \{f_1(\lambda),$

$f_2(\lambda), \ldots\}$, $\lambda \in \bigcup_{\varrho=1}^{\infty} \operatorname{spec} A_\varrho$. One obtains $(Sf)(\lambda) = S(\lambda)f(\lambda)$, $(Q_\varrho f)(\lambda) = Q_\varrho(\lambda)f(\lambda)$, where a.e. $S(\lambda)$ is unitary in $\mathfrak{K}(\lambda)$, $Q_\varrho(\lambda)$ is an orthoprojection (which projects $\mathfrak{K}(\lambda)$ onto $\mathfrak{K}_\varrho(\lambda)$ as a subspace of $\mathfrak{K}(\lambda)$). The so-called *partial cross-sections* $\sigma_{\alpha\to\beta}(\lambda)$ are defined by

$$\sigma_{\alpha\to\beta}(\lambda) = \|Q_\beta(\lambda)\, T(\lambda) Q_\alpha(\lambda)\|^2_{2,\mathfrak{K}(\lambda)},$$

where $\|\cdot\|_{2,\mathfrak{K}(\lambda)}$ denotes the Hilbert-Schmidt-norm of operators in $\mathfrak{K}(\lambda)$. These functions relate the model with the experimental facts.

If a family $H = H(\mu)$ of Hamiltonians is given, the channels in general depend on μ. If the channels and the channel projections are independent of μ, we call this property the *channel structure invariance* property of the family $H(\mu)$. If the model corresponding to $H(\mu)$ is asymptotically complete, then the channel structure invariance property is satisfied if and only if for all μ_1, μ_2 complete wave operators for $H(\mu_1)$, $H(\mu_2)$ exist with respect to the absolutely continuous projection.

2. The small trace class perturbation scattering model

Let C_μ be a holomorphic self-adjoint family of bounded non-negative operators on $0 \leqslant \mu < \infty$. Let C_∞ be a bounded self-adjoint non-negative operator with the following properties: $C_\infty = C_\infty^{\mathrm{ac}} \oplus C_\infty^d$, C_∞^{ac} absolutely continuous, C_∞^d compact. Let $\operatorname{spec} C_\infty^{\mathrm{ac}} = [0, c]$, $c > 0$, $\operatorname{spec} C^d = \{\lambda_1, \lambda_2, \ldots\}$, $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_i \to 0$. Beginning with some index ϱ the eigenvalues λ_ϱ are embedded. Further, let $C_\mu - C_\infty = V_\mu \in \gamma_1$ and moreover $\|V_\mu\|_1 \to 0$, $\mu \to \infty$. That is C_μ may be considered for large μ as a small trace class perturbation of C_∞. As is well known, the pair $\{C_\mu, C_\infty\}$ forms a complete scattering system, that is the wave operator $W_\mu^\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itC_\mu)\exp(-itC_\infty)P_\infty^{\mathrm{ac}}$ exists and is complete: $(W_\mu^\pm)^* W_\mu^\pm = P_\infty^{\mathrm{ac}}$, $W_\mu^\pm (W_\mu^\pm)^* = P_\mu^{\mathrm{ac}}$. Further, we have $\operatorname*{s-lim}_{\mu\to\infty} W_\mu^\pm = P_\infty^{\mathrm{ac}}$. Let $P_\infty^{\mathrm{ac}}\mathfrak{H} = \int_0^c \mathfrak{K}(\lambda)\,d\lambda$ be a resolution of $P_\infty^{\mathrm{ac}}\mathfrak{H}$ in a direct integral with respect to C_∞. As usual, we put $S_\mu = (W_\mu^+)^* W_\mu^-$ and denote the corresponding S-matrix by $S_\mu(\lambda)$. Then the scattering amplitude $T_\mu(\lambda) = S_\mu(\lambda) - 1_{\mathfrak{K}(\lambda)}$ is a.e. defined and belongs to the trace class $\gamma_1(\mathfrak{K}(\lambda))$. If the perturbation is switched on, $\mu = \infty \to \mu < \infty$, then the isolated eigenvalues of C_∞ are stable and do not influence the scattering properties of the system. But the embedded eigenvalues are usually unstable, they disappear and may be absorbed by the absolutely continuous spectrum. Then the following problem arises: Determination of the influence of an unstable eigenvalue λ_ϱ upon the scattering amplitude $T_\mu(\lambda)$ in the neighbourhood of λ_ϱ and for large μ.

The assumptions of this model are strong but not too strong. But if we add further conditions, for instance conditions on analytic continuability of certain factorized resolvents, then on the one hand the mentioned influence may be directly

calculated by the so-called virtual pole method, but on the other hand such a model is not realistic for the N-particle system. Only for $N = 2$, that is, in the two-body case, one obtains a realistic model (see, for instance, A. A. Arsen'ev [1]).

3. Realization of the models by N-particle Hamiltonians

Let H be an N-particle Hamiltonian with two-body interactions which is defined by the differential expression

$$-\sum_i (2m_i)^{-1}\Delta_i + \sum_{i<j} V_{ij}(x)$$

in $\mathfrak{H} = L^2(R^{3N-3})$, where the centre of mass is removed. We assume H self-adjoint and bounded below. Let $\mathfrak{A}$ be the class of polynomials with respect to the conjugated coordinates. Then by $\{H, \mathfrak{A}\}$ the channel structure is well-defined and may be calculated by the well-known clustering of the system of the particles. We assume asymptotic completeness (for instance let the conditions of I. Sigal [7] be satisfied). Then we obtain a realization of the model of Section 1. Now with respect to such a system we may formulate the so-called *resonance problem*: It should be possible to deduce mathematically special properties of the partial cross-sections, namely the existence of sharp peaks in the neighbourhood of certain values λ_0 from the model via a suitable Ansatz for H.

The resonance problem may be transformed into the problem of the investigation of the unstable eigenvalues of a certain small trace class perturbation scattering model of Section 2. Namely, as we shall see, there is a realization of the model of Section 2 which is deduced by a certain decoupling process from a starting N-body Hamiltonian by using an additional physical assumption. This result consists of two parts: (i) Description of the decoupling process. (ii) Verification of the properties of the model.

(i) If we assume $V_{ij} = V_{ij}(r)$ and of the form

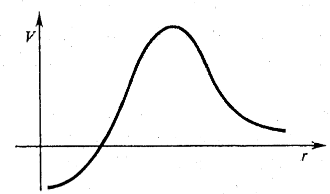

Fig. 1

then we obtain for the interaction in the configuration space the well-known star structure:

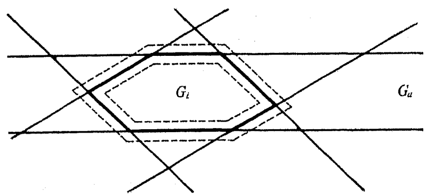

Fig. 2

(Fig. 2 corresponds to a system of 3 one-dimensional particles), that is we obtain a bounded region such that the values of the interaction are very large on the boundary of this inner region. Now we take a δ-neighbourhood of the boundary of the inner region, which we denote by G_δ, and form the new Hamiltonian

$$H_\mu = H+\mu\chi_{G_\delta}(x).$$

This corresponds to the physical assumption of the existence of a special N-body interaction. Now let $\mu \to \infty$. We ask for the construction of the decoupled Hamiltonian H_∞. Perhaps this construction is a certain folk-theorem, but we shall give an operator-theoretic foundation, which is not connected with special assumptions on boundaries (for corresponding constructions without abstract formulation see for instance T. Kako [6]).

Let H be unbounded, self-adjoint, non-negative, acting on $\mathfrak{H}$. Let P be an orthoprojection on $\mathfrak{H}$. H is called *local* with respect to P if $P\mathfrak{H}\cap \operatorname{dom} H$ is dense in $P\mathfrak{H}$ and if the relation $Hf = PHf$, $f \in P\mathfrak{H}\cap\operatorname{dom}H$, is satisfied. Let $H_\mu = H+\mu P$, $\mu \geqslant 0$ and let H be local with respect to $1-P$. Then, as is well known, H_μ is strongly resolvent convergent for $\mu\to\infty$. We denote the corresponding pseudo-resolvent by $R_\infty(z)$. Now we obtain: $\ker R_\infty = P\mathfrak{H}$, that is in $(1-P)\mathfrak{H}$ is $R_\infty(z)$ a true resolvent $(z-H_\infty)^{-1}$ with a certain self-adjoint H_∞. This operator H_∞ is exactly the Friedrichs extension of $H{\restriction}(1-P)\mathfrak{H}\cap\operatorname{dom}H$, which is a symmetric and non-negative operator.

Application of this lemma yields the decoupled Hamiltonian H_∞ in $(1-P)\mathfrak{H}$, which is generated by the original differential expression together with the Dirichlet boundary condition (for the proof of the lemma see H. Baumgärtel and M. Demuth [3]).

(ii) We put $C_\mu = R_\mu^\alpha$, $C_\infty = R_\infty^\alpha$ at some fixed point, where α is some power. The scattering properties are not influenced by this transformation because of the general validity of the invariance principle, which was proved by M. Wollenberg. Now for suitable α we obtain $R_\mu^\alpha - R_\infty^\alpha \in \gamma_1$. This has been proved already by M. Š. Birman [4] in similar cases. There is also a recent paper of P. Deift and B. Simon [5]. In these papers there are no results on $\mu\to\infty$. But it is possible to amplify the strong resolvent convergence to $\|R_\mu^\alpha - R_\infty^\alpha\|_1 \to 0$, $\mu\to\infty$. This has been proved by H. Baumgärtel and M. Demuth [3].

Now we denote by G_i the inner region with respect to G_δ, by G_a the outer region. G_i is compact. We obtain $H_\infty = H_\infty^i \oplus H_\infty^a$ and $R_\infty^\alpha = (R_\infty^i)^\alpha \oplus (R_\infty^a)^\alpha$. R_∞^i and hence also $(R_\infty^i)^\alpha$ is compact. From R_∞^a we consider only the part of absolute continuity. Then we obtain a realization of the model of Section 2.

In a forthcoming paper the partial cross-sections of the model are investigated, if μ is large.

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