PROPOSITION 2.2. *Any Banach algebra possesses minimal subspectra.*

DEFINITION 2.3. A subspectrum $\tilde{\sigma}$ on A is called a *single point subspectrum* (shortly an sp-subspectrum) if for each $x_I \in c(A)$ the set $\tilde{\sigma}(x_I)$ consists of a single point. Clearly, any sp-subspectrum is minimal.

THEOREM 2.4. *Let A be a complex Banach algebra. There is a one-to-one correspondence between sp-subspectra of A and semicharacters of A, given by the relation*

$$\tilde{\sigma}(x_I) = \{(\varphi(x_\iota))_{\iota\in I}\} \subset C^I,$$

where $\tilde{\sigma}$ is an sp-subspectrum and φ the corresponding semicharacter.

An easy proof of this theorem follows immediately from Proposition 2.1.

COROLLARY 2.5. *If X is a Hilbert space,* $\dim X > 2$, *or if X is any Banach space with a decomposition of the form* (2), *then the algebra $B(X)$ has no sp-subspectra.*

Since any commutative Banach algebra A possesses an sp-subspectrum, we have the following

Remark 2.6. Generally speaking, a sp-subspectrum defined on a subalgebra cannot be extended to a sp-subspectrum defined on the whole algebra.

3. Semicharacters on groups

In this section we remark shortly that the definition and some results of Section 1 have their analogues in the theory of locally compact groups. We give no proof of the theorem formulated here, since it is the same as in Section 1.

DEFINITION 3.1. Let G be a locally compact group. A *semicharacter on G* is a complex valued function φ on G, with $|\varphi(s)| \equiv 1$, such that φ restricted to any commutative subgroup of G is a character.

THEOREM 3.2. *The group $\mathrm{Sl}(n, C)$ of all non-singular $n\times n$ matrices with complex entries and determinant equal to one possesses semicharacters if and only if $n \leqslant 2$, and proper semicharacters if and only if $n = 2$.*

We pose also a problem, analogous to that formulated in Section 1.

PROBLEM. Let G be a connected l.c. group and φ a continuous semicharacter on G. Does it follow that φ is a character?

For disconnected groups the answer is in negative, e.g. for $\mathrm{Gl}(2, C)$ with discrete topology.

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FREDHOLM THEORY IN BANACH ALGEBRAS

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1. Introduction

The classical Fredholm theory of bounded linear operators on a Banach space is familiar to many mathematicians. The thesis of this paper is that to every 2-sided ideal in the pre-socle of a general Banach algebra there corresponds a sensible Fredholm theory. It is a consequence of Atkinson's theorem [1] and the fact that the finite rank operators constitute the pre-socle of the Banach algebra of all bounded linear operators on a Banach space, that our theory includes the classical theory as a special case. Progress in the case of semisimple Banach algebras has already been made by Barnes [2] using the socle as his basic ideal, and our work is an extension of this.

Inessential, Riesz and Fredholm elements are defined in § 3 and some of their elementary properties are developed. This is general Fredholm theory (i.e. there is no reference to the pre-socle) nevertheless several of the results are important. In particular, the Fredholm elements form an open multiplicative semigroup and the inessential elements and Fredholm perturbations coincide. In § 4 we define the nullity, defect and index of a Fredholm element, prove the punctured neighbourhood theorem and establish the well-known continuity, stability and multiplicative properties of the index. In § 5 we apply these results to deduce that every Riesz point is a Fredholm point of index zero, and every Fredholm point in the boundary of the spectrum is a Riesz point; results which lead to a Ruston-type characterization of Riesz elements. We also show that if the algebra is commutative, then the sets of Riesz and Fredholm points of any given element are equal, and this enables us to derive spectral mapping theorems for the essential spectra of Wolf and Browder. The index function of a Fredholm element is defined in § 6, and in § 7 this is applied in order to extend important results of Schechter and Stampfli concerning the Weyl spectrum. For an up to data account of other generalizations of Fredholm theory the reader is referred to Chapter VI of [4].

2. Preliminaries

Throughout this paper all algebras and vectors spaces will be over the field of complex numbers $\boldsymbol{C}$. Π will denote the structure space of the algebra A and the radical and socle of A (if it exists) will be denoted by $\operatorname{rad}(A)$ and $\operatorname{soc}(A)$ respectively. If $\Omega \subset \Pi$ and $S \subset A$ we write $k(\Omega)$ and $h(S)$ to denote the kernel of Ω and the hull of S. For $x \in A$ we denote the spectrum and resolvent set of x by $\sigma(x)$ and $\varrho(x)$ respectively, and the left and right annihilators of x by $\operatorname{lan}(x)$ and $\operatorname{ran}(x)$. The circle operation $\circ$ is defined by

$$x \circ y = x+y-xy \quad \text{for all } x, y \in A.$$

An element $x \in A$ is quasi-invertible if there exist $u, v \in A$ such that $u \circ x = x \circ v = 0$, and nearly-nilpotent if λx is quasi-invertible for all $\lambda \in \boldsymbol{C}$.

Considerable use will be made of the quotient algebra $A/\operatorname{rad}(A)$. We denote by x' the image of $x \in A$ under the canonical mapping $A \to A/\operatorname{rad}(A)$, and for $S \subset A$ we write $S' = \{x';\ x \in S\}$. It is well known that A' is semisimple and $\sigma(x) = \sigma(x')$ for all $x \in A$. Also the socle of A' exists and we define $\operatorname{psoc}(A) = \{x \in A;\ x' \in \operatorname{soc}(A')\}$ and call $\operatorname{psoc}(A)$ the pre-socle of A. For the elementary properties of the pre-socle see [8].

Finally suppose that A is semisimple. If J is an ideal of finite order of A (see [2] for the definition and details) then we write $\theta(J)$ to denote the order of J. If $\theta(Ax) = n$ then by [2], 1.1, there exist minimal idempotents $p_1, \ldots, p_n$ such that $x = xp_1 + \ldots + xp_n$ hence $xA \subset xp_1A + \ldots + xp_nA$. If $p = p_1 + \ldots + p_n$ then $Ax = Ap$ so $A(x-xp) = 0$; hence $[x-xp]$ is a left ideal whose square is zero. Therefore $x = xp$. It follows from [3], 30.7, that $\theta(xA) \leqslant n$, and hence by symmetry $\theta(xA) = n$. This common value is called the rank of x and is denoted by $\theta(x)$.

3. General Fredholm theory

In this section A will denote an algebra and F will be any fixed 2-sided ideal of A. The following definitions will be made relative to this fixed ideal F.

DEFINITION 3.1. The members of F are called *finite elements* of A. We write $I = k(h(F))$ and call the members of I *inessential elements* of A. An element $x \in A$ is said to be *quasi-Fredholm* if there exist $u, v \in A$ such that $u \circ x \in F$ and $x \circ v \in F$. The set of quasi-Fredholm elements is denoted by Ψ. If $\boldsymbol{C}x \subset \Psi$ we say that x is a *Riesz element* of A and we denote the set of Riesz elements by R.

If A has an identity we say that x is a *Fredholm element* of A if $1-x \in \Psi$. The set of Fredholm elements is denoted by Φ. $\lambda \in \boldsymbol{C}$ is said to be a *Fredholm point* of x if $\lambda - x \in \Phi$. If λ is a Fredholm point of x which is not an accumulation point of $\sigma(x)$ then we say that λ is a *Riesz point* of x.

As an example take X to be a Banach space and A to be the Banach algebra of all bounded linear operators on X. Take F to be the 2-sided ideal of finite rank operators of A. Then the sets of inessential, Riesz and Fredholm elements of A are precisely the sets of inessential, Riesz and Fredholm operators on X (see [4] for definitions).

The results in the next theorem are standard and we state them without proof.

THEOREM 3.2. *Let J be a 2-sided ideal of A such that $F \subset J \subset \Psi$. Then*

(i) *$x \in \Psi \Leftrightarrow J+x$ is quasi-invertible in A/J.*
(ii) *$x \in R \Leftrightarrow J+x$ is a nearly-nilpotent element of A/J.*
(iii) *$x \in I \Leftrightarrow J+x$ lies in the radical of A/J.*
(iv) *$F \subset J \subset I \subset R \subset \Psi$.*
(v) *$h(F) = h(J) = h(I)$.*
(vi) *A/J is semisimple $\Leftrightarrow J = I$.*
(vii) *Every idempotent of Ψ lies in F.*
(viii) *I is the largest ideal which lies in Ψ.*

COROLLARY 3.3. *Any one of the sets $h(F)$, I, R, Ψ uniquely determines each of the others.*

Proof. By 3.2 (iv) and (viii) any one of the above sets uniquely determines I. By 3.2 (i), (ii) and (v), I uniquely determines each of the others.

The key to the development of Fredholm index theory will prove to be consideration of the semisimple algebra A'. F' is a 2-sided ideal of A' and it will be understood that the inessential, quasi-Fredholm and Riesz elements of A' are always computed relative to F'. The proof of the next result is straightforward and is omitted.

THEOREM 3.4. *The sets of inessential, Riesz and quasi-Fredholm elements of A' are given by I', R', Ψ', respectively.*

Now suppose that A is a Banach algebra with identity. Certain observations can immediately be made. I is a closed 2-sided ideal of A, and $x \in \Phi$ if and only if there exist $u, v \in A$ such that $ux-1 \in F$ and $xv-1 \in F$. Also if F is proper, then A/I is a Banach algebra with identity, and $x \in \Phi$ if and only if $I+x$ is invertible in A/I. The continuity of the canonical mapping $A \to A/I$ and the fact that the group of invertible elements of a Banach algebra is open, lead to the important conclusion that Φ is an open multiplicative semigroup of A. These results are so fundamental to the theory that we shall make frequent implicit use of them.

The following result illustrates why inessential elements are sometimes called *Fredholm perturbations.*

THEOREM 3.5. *Let A be a Banach algebra with identity. Then*

$$I = \{x \in A;\ u+x \in \Phi \text{ for all } u \in \Phi\}.$$

Proof. Let J denote the right-hand side of this equation. Clearly, $I \subset J \subset \Psi$. We shall show that J is an ideal of A and the result will then follow from 3.2 (viii). Obviously, J is a subspace of A so it is sufficient to show that $yx \in J$ for $x \in J$ and $y \in A$. But there exists $\lambda \in \boldsymbol{C}$ such that $(y-\lambda)^{-1} \in A$ and then for $u \in \Phi$ we have

$$u+(y-\lambda)x = (y-\lambda)\{(y-\lambda)^{-1}u+x\} \in \Phi$$

since both $(y-\lambda)$ and $(y-\lambda)^{-1}u$ belong to Φ. Therefore $(y-\lambda)x \in J$ and since J is a subspace, $yx \in J$ as required.

The sets of Fredholm and Riesz points of an element x in a Banach algebra A are both open subsets of C, and by 3.4 they are equal to the corresponding sets for x'. The Riesz points may be characterized as follows.

THEOREM 3.6. *Let A be a Banach algebra with identity. Then λ is a Riesz point of $x \in A$ if and only if one of the following two mutually exclusive conditions is satisfied*:

(i) $\lambda \in \varrho(x)$,

(ii) *λ is isolated in $\sigma(x)$ and the corresponding spectral idempotent lies in F.*

Proof. The result is trivial if $F = A$ so we shall assume that F is proper. Suppose that 0 is a Riesz point of $x \in A$ and that $0 \in \sigma(x)$. Then by definition 0 is isolated in $\sigma(x)$ and $x \in \Phi$. Let p be the spectral idempotent of x at 0. Now there exists $u \in A$ such that $(I+x)(I+u) = (I+u)(I+x) = I+1$ and also $\|px^n\|^{1/n} \to 0$ as $n \to \infty$. Therefore, $\|I+p\| \leqslant \|I+px^n\| \cdot \|I+u^n\| \leqslant \|px^n\| \cdot \|u\|^n \to 0$ as $n \to \infty$ and hence $p \in I$. Now by 3.2 (vii) $p \in F$.

Conversely suppose that 0 is isolated in $\sigma(x)$ and the corresponding spectral idempotent p lies in F. Then $x-xp+p$ is invertible and since $xp-p \in I$ it follows that $1+x$ is invertible in A/I. Hence $x \in \Phi$ and by definition 0 must be a Riesz point of x.

This result shows that Riesz points of the spectrum are a generalization of the concept of poles of finite rank of a bounded linear operator on a Banach space. It should be stressed that they are not normally poles, unless F happens to be a 2-sided ideal of algebraic elements of A. (See [7] for details.)

4. Fredholm index theory

From now on A will be a Banach algebra with identity and F will be a 2-sided ideal in the pre-socle of A. The theory developed within this framework will be called a *Fredholm index theory*. By 3.3 it seems reasonable to say that two Fredholm theories (in the same algebra) are identical if the ideals of inessential elements in each theory are equal. We now calculate the number of Fredholm index theories of A.

THEOREM 4.1. *There exists a one-to-one mapping from the class of Fredholm index theories of A onto the powerset of $\Pi \setminus h(\mathrm{psoc}(A))$.*

Proof. There is a one-to-one correspondence between the powerset of $\Pi \setminus h(\mathrm{psoc}(A))$ and the class of sets $\{S; h(\mathrm{psoc}(A)) \subset S \subset \Pi\}$. For each ideal of finite elements F, $h(F)$ is in this class, so by 3.3 it only remains to show that a given set S in the class is equal to $h(F)$ for some 2-sided ideal $F \subset \mathrm{psoc}(A)$. Write $F = \mathrm{psoc}(A) \cap k(S)$. Clearly $S \subset h(F)$. If $P \in \Pi \setminus S$ then $P \notin h(\mathrm{psoc}(A))$ and an argument similar to that of [8], Theorem 3, shows that there exists $u \in \mathrm{psoc}(A)$ such that $h(u)$ is the complement of $\{P\}$ in Π. Then $S \subset h(u)$ hence $u \in F$ and we conclude that $P \notin h(F)$. Therefore $S = h(F)$ as required.

As an example suppose that A is primitive. Then $\Pi^* = \Pi \setminus \{0\}$ and therefore by [8], Theorem 3, $\Pi \setminus h(\mathrm{psoc}(A)) \subset \{0\}$ so by 4.1 there exists at most one non-trivial Fredholm index theory in such an algebra.

Now suppose that A is semisimple and consider the Fredholm elements relative to the socle of A. These are precisely the Fredholm elements considered by Barnes, and we shall assume that the reader is familiar with their properties as given in [2]. We shall call such elements *B-Fredholm* in order to distinguish them from our rather more general Fredholm elements. The following lemma is a direct consequence of 3.4.

LEMMA 4.2. *If $x \in \Phi$ then x' is a B-Fredholm element of A'.*

DEFINITION 4.3. Let $x \in \Phi$. We define the *nullity* of x to be $\theta(\mathrm{ran}(x'))$, the *defect* of x to be $\theta(\mathrm{lan}(x'))$, and the *index* of x to be $\theta(\mathrm{ran}(x'))-\theta(\mathrm{lan}(x'))$. We denote the nullity, defect and index of x by $n(x)$, $d(x)$ and $i(x)$ respectively.

It follows from 4.2 and [2], 2.3, that the nullity, defect and index of a Fredholm element are all finite. It is important to emphasise that these functions are only defined on Φ. This is because, although apparently well-defined at points not in Φ, the defect may break away from the classical concept of defect at such points. Two see this take T to be the compact linear operator on l^2 defined by $Te_n = n^{-1}e_n$ for the usual orthonormal basis $\{e_n\}_1^\infty$ of l^2. This has apparently nullity and defect both zero, but the classical defect of T is ∞. However if x is a Fredholm element of A with nullity and defect zero then by [2], 2.3, $x'A' = A'x' = A'$ and therefore x' and *a fortiori* x is invertible. We state this formally as follows.

THEOREM 4.4. *x is invertible $\Leftrightarrow x \in \Phi$ and $n(x) = d(x) = 0$.*

We now give the main index theorem. The following results are known as the multiplicative property, the continuity, and the stability of the index respectively. The proof is an easy application of 4.2 and [2], 3.2 and 4.1.

THEOREM 4.5. (i) *$i(xy) = i(x)+i(y)$ for all $x, y \in \Phi$.*

(ii) *i is continuous in $\Phi \to Z$.*

(iii) *$i(x+u) = i(x)$ for all $x \in \Phi$, $u \in I$.*

The remainder of this section is devoted to the proof of the following important result, which is known as the punctured neighbourhood theorem.

THEOREM 4.6. *Suppose that $x \in \Phi$. Then for some $\varepsilon > 0$, $n(x-\lambda)$ is constant for $0 < |\lambda| < \varepsilon$, and this constant is less than or equal to $n(x)$. A similar result holds for defect.*

Proof. We shall only give the proof for nullity, the proof for defect being similar. Furthermore, by 4.2 it is sufficient to prove the result for the case of A a semisimple Banach algebra with identity and x a B-Fredholm element of A. Suppose then that this is the case. There exists $\delta > 0$ such that $|\lambda| < \delta$ implies $x-\lambda$ is B-Fredholm, thus by [2], 2.3, we may write $A(x-\lambda) = A(1-q_\lambda)$ for some idempotents $q_\lambda \in \mathrm{soc}(A)$ whenever $|\lambda| < \delta$. Clearly $n(x-\lambda) = \theta(q_\lambda)$.

Now x^n is B-Fredholm for all n, hence by [2], 2.3, x^nA is closed for all n and therefore $M = \bigcap_1^\infty x^nA$ is a closed right ideal of A. Now $M \cap q_0 A$ is a right ideal of finite order, hence by [2], 1.1, $M\cap q_0A = eA$ for some idempotent $e\in \operatorname{soc}(A)$. Also $e\in M$ thus $M = eM\oplus(1-e)M$ and $(1-e)M$ is closed. Clearly $\theta(e)\leqslant\theta(q_0)$. Now let $\tilde{x}$ denote left multiplication on M by x. Let $y\in M$ and consider $q_0A\cap x^nA$ for $n\in Z^+$. This is a decreasing sequence of right ideals of finite order, hence the sequence is ultimately constant and therefore there exists $m\in Z^+$ such that $q_0A\cap x^mA = q_0A\cap M$. Now there exists $\{u_k\}_1^\infty$ such that $x^{m+k}u_k = y$. Let $v_k = x^mu_1-x^{m+k-1}u_k$ for $k\in Z^+$. Then $xv_k = 0$ so $v_k\in q_0A\cap x^mA\subset M$ and therefore $x^mu_1 = v_k+x^{m+k-1}u_k\in x^{m+k-1}A$ for all $k\in Z^+$. Hence $x^mu_1\in M$ and because $x(x^mu_1) = y$ it now follows that $\tilde{x}M = M$. Also $\tilde{x}$ is zero on $eM = eA$ and one-to-one on $(1-e)M$, therefore by the Closed Graph Theorem there exists a bounded linear operator T on M whose range is $(1-e)M$ such that $\tilde{x}T$ is the identity on M and $T\tilde{x}$ is the identity on $(1-e)M$.

The proof now splits into two parts — in the first we show that for some $\varepsilon>0$ we have $\theta(q_\lambda)\leqslant\theta(e)$ for $0<|\lambda|<\varepsilon$, and in the second we show that $\theta(e)\leqslant\theta(q_\lambda)$ for $|\lambda|<\varepsilon$.

Now for $y\in(1-e)M$ we have $||y|| = ||Txy||\leqslant ||T||\cdot||xy||$ and hence $||xy||\geqslant ||T||^{-1}||y||$. Choose $\varepsilon>0$ such that $\varepsilon<\inf\{\delta,\frac{1}{2}||T||^{-1}\}$. Then $||(x-\lambda)y||\geqslant \big|||xy||-|\lambda|\cdot||y||\big|\geqslant\varepsilon||y||$ for $|\lambda|<\varepsilon$. We therefore have $q_\lambda A\cap(1-e)M = 0$. Now suppose that $w\in q_\lambda A\cap(1-e)A$. Since for all $\lambda\neq 0$ we have $x^nq_\lambda = \lambda^nq_\lambda$ it follows that $q_\lambda\in M$ for $\lambda\neq 0$. Hence $q_\lambda A\subset M$. Now $w = (1-e)w$ and $w\in M$ hence $w\in(1-e)M$ and therefore $w = 0$, that is $q_\lambda A\cap(1-e)A = 0$ for $0<|\lambda|<\varepsilon$. But $A = eA\oplus(1-e)A$ and applying [2], 1.2 we have $\theta(q_\lambda)\leqslant\theta(e)\leqslant\theta(q_0)$ for $0<|\lambda|<\varepsilon$.

For the proof of the second part we return to our bounded linear operator T on M. Now for $|\lambda|<\varepsilon$ and $y\in M$ we have $(x-\lambda)(1-\lambda T)^{-1}y = xy$ hence $(1-\lambda T)^{-1}(eA)\subset q_\lambda A$. But the range of T is $(1-e)M$ hence $eT = 0$ and therefore $e(1-\lambda T)^{-1}$ is the identity on eA. Hence $eA = e\big((1-\lambda T)^{-1}(eA)\big)\subset eq_\lambda A$ and therefore $\theta(e) = \theta(eA)\leqslant\theta(eq_\lambda A) = \theta(eq_\lambda) = \theta(Aeq_\lambda)\leqslant\theta(Aq_\lambda) = \theta(q_\lambda)$ for $|\lambda|<\varepsilon$.

Combining these two parts we have $\theta(q_\lambda) = \theta(e)\leqslant\theta(q_0)$ for $0<|\lambda|<\varepsilon$ which completes the proof of the theorem.

5. Spectral theory

In this section we derive fundamental relationships between Riesz points and Fredholm points. By 4.4 and 4.5 every Riesz point is a Fredholm point of index zero. A partial converse is that every Fredholm point in the boundary of the spectrum is a Riesz point. This follows immediately from the next theorem.

THEOREM 5.1. *If S is a connected open set of Fredholm points of $x\in A$ and $x-\lambda$ is invertible for some $\lambda\in S$, then every point of S is a Riesz point of x and in particular $\sigma(x)\cap S$ is a countable discrete set.*

Proof. Let D be the set of discontinuities of the function $\mu\to n(x-\mu)$ defined on S. Then by 4.6, D is a countable discrete set and since $x-\lambda$ is invertible for some $\lambda\in S$ we have $n(x-\mu) = 0$ for $\mu\in S\setminus D$. Also by 4.5 $i(x-\mu) = 0$ for $\mu\in S$, hence $d(x-\mu) = 0$ for $\mu\in S\setminus D$ and therefore by 4.4, $x-\mu$ is invertible for $\mu\in S\setminus D$. It now follows that each point of S is a Riesz point of x, and $\sigma(x)\cap S = D$.

COROLLARY 5.2. $x\in R\Leftrightarrow$ *every non-zero point of C is a Fredholm point of $x\Leftrightarrow$ every non-zero point of C is a Riesz point of x.*

This result shows that the spectrum of a Riesz element is countable with zero as the only possible accumulation point. In particular the resolvent set of an inessential element is connected; a fact which is of considerable significance when dealing with the connected components of the set of Fredholm elements and the group of invertible elements of A/I. See for example [4], 6.2.5.

THEOREM 5.3. *If A is commutative then the sets of Fredholm points and Riesz points of a given element are equal.*

Proof. Suppose $x\in\Phi$, $0\in\sigma(x)$, and let $\Pi^\#$ be the set of accumulation points of Π in the Gelfand topology. Then there exists $u\in A$ such that $u\hat{x}-1 = t\in F$ and then $\hat{u}\hat{x} = 1+\hat{t}$. But by [8], Theorem 4, $\hat{t}(P) = 0$ for all $P\in\Pi^\#$. Now $\Pi^\#$ is closed hence compact in Π, thus for some $\delta>0$ we have $|\hat{x}(P)|\geqslant 2\delta$ for all $P\in\Pi^\#$. By continuity there exists an open set Ω, $\Pi^\#\subset\Omega\subset\Pi$ such that $|\hat{x}(P)|\geqslant\delta$ for all $P\in\Omega$. But $\Pi\setminus\Omega$ is a closed hence compact set with no accumulation points. Thus $\{P\in\Omega;\ |\hat{x}(P)|<\delta\}$ is a finite set and therefore $\{\lambda\in\sigma(x);\ |\lambda|<\delta\}$ is a finite set. Hence 0 is an isolated point of $\sigma(x)$ and is therefore a Riesz point of x.

We conclude this section with a brief discussion of essential spectra. We shall consider two such types — the *Wolf essential spectrum* or *Fredholm spectrum* defined by $\omega(x) = \{\lambda\in C;\ \lambda$ is not a Fredholm point of $x\}$, and the *Browder essential spectrum* or *Riesz spectrum* defined by $\beta(x) = \{\lambda\in C;\ \lambda$ is not a Riesz point of $x\}$ for all $x\in A$ (see [5]). Clearly $\omega(x) = \sigma(I+x)$ whenever F is proper. We remark that $\omega(x)$ and $\beta(x)$ are compact subsets of C such that $\omega(x)\subset\beta(x)\subset\sigma(x)$ and $\partial\beta(x)\subset\partial\sigma(x)$. From 5.1 we see that $\partial\beta(x)\subset\partial\omega(x)$. If A is commutative then by 5.2 we have $\omega(x) = \beta(x)$, however even if A is non-commutative $\omega(x)$ and $\beta(x)$ may still be linked by the following extension of a method of Gramsch and Lay [5]. Let Z be the bicommutant of any element of A. Then Z is a commutative Banach algebra with identity and $F\cap Z$ is a 2-sided ideal of Z. By [8], Corollary 6, we have $F\cap Z\subset\operatorname{psoc}(Z)$ and we may therefore take $F\cap Z$ to be the set of finite elements of Z. For $x\in Z$ write $\omega_Z(x)$ and $\beta_Z(x)$ to denote the Fredholm and Riesz spectra of x relative to $F\cap Z$. Since all of the spectral idempotents of x must lie in Z it is now easy to see that $\beta(x) = \beta_Z(x) = \omega_Z(x)$ for all $x\in Z$. Applying the spectral mapping theorem in A/I, and using this result with Z equal to the bicommutant of x, we may derive the following spectral mapping theorem for essential spectra.

THEOREM 5.4. *Suppose $x\in A$ and f is analytic on an open set containing $\sigma(x)$. Then $\omega(f(x)) = f(\omega(x))$ and $\beta(f(x)) = f(\beta(x))$.*

6. Index function theory

The main shortcoming of the theory which we have developed is the fact, as pointed out by Pearlman in § 1 of [6], that if $x \in \Phi$ and $i(x) = 0$ then $x+u$ can fail to be invertible for all $u \in I$. The reason for this failure is that the definition of the index is not sharp enough. We propose to remedy this by defining new concepts of nullity, defect and index on Φ. Instead of being integer-valued functions these will be mappings of Φ into the ring of integer-valued functions on Π which have finite support and are zero on $h(\operatorname{psoc}(A))$. Note that by [8], Theorem 3, each point of the support must be an isolated point of Π, and if A is primitive then this support consists of at most one point. The new nullity, defect and index reduce to the corresponding previously defined integer-valued concepts in this case.

LEMMA 6.1. *For all $P \in \Pi$ the primitive Banach algebras A/P and A'/P' are isomorphic under the mapping $P+x \to P'+x'$ for all $x \in A$.*

We remark that by Johnson's Theorem [3], 25.9, A/P and A'/P' are actually norm equivalent. Now if $x \in \Phi$ then by 4.2, x' is B-Fredholm, therefore $P'+x'$ is B-Fredholm. An application of 6.1 now gives

LEMMA 6.2. *If $x \in \Phi$ then $P+x$ is B-Fredholm for all $P \in \Pi$.*

LEMMA 6.3. *Let A be semisimple and x be a B-Fredholm element of A with $xA = (1-p)A$ and $Ax = A(1-q)$ for idempotents p, $q \in \operatorname{soc}(A)$ as in 2.3 of [2]. Then for all $P \in \Pi$ we have*

(i) $n(P+x) = 0 \Leftrightarrow q \in P$.

(ii) $d(P+x) = 0 \Leftrightarrow p \in P$.

(iii) *$P+x$ is invertible* $\Leftrightarrow p,\ q \in P$.

(iv) *$\Pi \setminus h(p)$ and $\Pi \setminus h(q)$ are finite sets.*

Proof. (i) If $q \in P$ then $(A/P)(P+x) = (A/P)$ hence $n(P+x) = 0$. Conversely if $n(P+x) = 0$ then $(A/P)(P+x) = (A/P)$ and therefore $(A/P) = (A/P) \times \times (P+1-q)$, hence there exists $u \in A$ such that $u(1-q)-1 \in P$. Multiplying through on the right by q we see that $q \in P$. (iv) Suppose that $\theta(q) = n$ and let $P_1, \ldots, P_{n+1}$ be distinct elements of Π. Let $q_1, \ldots, q_n$ be a set of minimal idempotents such that $qA \subset q_1A + \ldots + q_nA$. Now by [8], Lemma 2, each q_i, $1 \leqslant i \leqslant n$, lies in at least n of the elements $\{P_j\}_1^{n+1}$, hence for some k, $1 \leqslant k \leqslant n+1$, P_k contains each q_i, $1 \leqslant i \leqslant n$. Then $q \in P_k$ and therefore $\Pi \setminus h(q)$ contains at most n points.

DEFINITION 6.4. Let $x \in \Phi$. We define the *nullity function, defect function, index function* of x which are denoted by $\nu(x)$, $\delta(x)$, $\iota(x)$ respectively and are all functions from Π into Z as follows. $\nu(x)(P) = n(P+x)$ and $\delta(x)(P) = d(P+x)$ for all $P \in \Pi$. $\iota(x) \equiv \nu(x) - \delta(x)$.

Now if $x \in \Phi$ then by 4.2 x' is B-Fredholm and an application of 6.1 gives the following result.

LEMMA 6.5. *The functions $\nu(x')$, $\delta(x')$ and $\iota(x')$ are given by $\nu(x')(P') = \nu(x)(P)$, $\delta(x')(P') = \delta(x)(P)$ and $\iota(x')(P') = \iota(x)(P)$ for all $x \in \Phi$, $P \in \Pi$.*

We now give the five main results of this section. For convenience we shall only give the proofs for the case of A semisimple and leave the extensions to the non-semisimple case to the reader, using the semisimple Banach algebra A' and the lemmas at the beginning of this section.

THEOREM 6.6. *Let $x \in \Phi$. Then*

(i) $n(x) = \sum_{P \in \Pi} \nu(x)(P)$,

(ii) $n(x) = 0 \Leftrightarrow \nu(x) \equiv 0$,

(iii) *$\nu(x)$ has finite support and is zero on $h(\operatorname{psoc}(A))$.*

Similar results hold for the defect function.

Proof. Assume A is semisimple. Then x is a B-Fredholm element of A and by [2], 2.3 we may write $Ax = A(1-q)$ for some idempotent $q \in \operatorname{soc}(A)$. Suppose $\theta(q) = n$. Then by [2], 1.1 we have $qA = q_1A + \ldots + q_nA$ for some orthogonal minimal idempotents $q_1, \ldots, q_n$. Now for $P \in \Pi$, $(A/P)(P+x) = (A/P)(P+1-q)$ hence $\nu(x)(P) = n(P+x) = \theta\big((P+q)(A/P)\big)$. But $(P+q)(A/P) = (P+q_1)(A/P) + \ldots + (P+q_n)(A/P)$ hence it follows that $\nu(x)(P) = \theta\big((P+q)(A/P)\big) = \{q_k \notin P;\ 1 \leqslant k \leqslant n\}$. From [8], Lemma 2, we see that each idempotent $q_1, \ldots, q_n$ is counted precisely once in the summation $\sum_{P \in \Pi} \nu(x)(P)$ and therefore $\sum_{P \in \Pi} \nu(x)(P) = n$. The rest of the proof is easy.

COROLLARY 6.7. *$x \in A$ is invertible $\Leftrightarrow x \in \Phi$ and $\nu(x) \equiv \delta(x) \equiv 0$.*

Note that if the index function is zero then the index is zero but not conversely. Herein lies the importance of the extra sharpness in the definition of the index function. The vital result where the index function scores heavily over the index is the following.

THEOREM 6.8. *If $x \in \Phi$ and $\iota(x) \equiv 0$ then there exists $u \in F$ such that $x + \lambda u$ is invertible for all $\lambda \in C \setminus \{0\}$.*

Proof. If A is semisimple then x is B-Fredholm and by [2], 2.3, there exist idempotents p, $q \in A$ such that $xA = (1-p)A$ and $Ax = A(1-q)$. It is a trivial extension of [6], 1.4, that there exists $u \in pAq$ such that $x + \lambda u$ is invertible for all $\lambda \in C \setminus \{0\}$. Now there exist $v, w \in A$ such that $xv - 1 \in F$ and $x = (1-p)w$, hence $(1-p)wv - 1 \in F$. Multiplying through on the left by p we have $p \in F$ and hence $u \in F$.

The final two results in this section are extensions of the index theorem 4.5 and the punctured neighbourhood theorem 4.6. The proof of 6.9 is trivial.

THEOREM 6.9. (i) *$\iota(xy) \equiv \iota(x) + \iota(y)$ for all $x, y \in \Phi$.*

(ii) *If x and y lie in the same connected component of Φ then $\iota(x) \equiv \iota(y)$.*

(iii) *$\iota(x+u) \equiv \iota(x)$ for all $x \in \Phi$, $u \in I$.*

THEOREM 6.10. *If $x \in \Phi$ then there exists $\varepsilon > 0$ such that $\nu(x-\lambda)$ is independent of λ for $0 < |\lambda| < \varepsilon$, and is bounded above by $\nu(x)$.*

A similar result holds for the defect function.

Proof. Suppose A is semisimple and x is a B-Fredholm element of A. By [2], 2.3, $xA = (1-p)A$ and $Ax = A(1-q)$ for some idempotents $p, q \in \mathrm{soc}(A)$. Let $J = k(h(\{p, q\}))$ which is a closed 2-sided ideal of A. Since $p, q \in J$ we have $(A/J)(J+x) = (A/J)$ and $(J+x)(A/J) = (A/J)$ therefore $J+x$ is invertible in A/J. Hence $J+x-\lambda$ is invertible for $|\lambda| < \varepsilon_0 = \|(J+x)^{-1}\|^{-1}$. But if $P \in h(\{p, q\})$ then $J \subset P$, hence $P+x-\lambda$ is invertible and therefore $\nu(x-\lambda)(P) = 0$ for $|\lambda| < \varepsilon_0$ and $P \in h(\{p, q\})$. But by 6.3 (iv), $\Pi \setminus h(\{p, q\})$ is a finite set $P_1, \ldots, P_n$ say, and by 4.6 there exist $\varepsilon_1, \ldots, \varepsilon_n$ such that $\nu(x-\lambda)(P_i) = n(P_i+x-\lambda)$ is constant $\leqslant n(P_i+x)$ for $0 < |\lambda| < \varepsilon_i$. Write $\varepsilon = \inf\{\varepsilon_0, \ldots, \varepsilon_n\}$. Then $\nu(x-\lambda)$ is independent of λ for $0 < |\lambda| < \varepsilon$, and is bounded above by $\nu(x)$.

7. The Weyl spectrum

In this section we illustrate the use of the index function by extending a result of Stampfli, [9], Theorem 4. Our result is 7.3. For $x \in A$ we define $W(x) = \bigcap_{u \in I} \sigma(x+u)$ and we call $W(x)$ the *Weyl spectrum of* x. This is a compact subset of C which is non-empty if and only if F is proper. Clearly $W(x) = W(x+u)$ for all $u \in I$, and from 6.8 we have the following extension of a well-known theorem of Schechter.

THEOREM 7.1. $W(x) = \bigcap_{u \in F} \sigma(x+u) = \sigma(x) \setminus \{\lambda \in C;\ x-\lambda \in \Phi$ *and* $\iota(x-\lambda) \equiv 0\}$.

This result shows that in terms of the Fredholm spectrum $\omega(x)$, and Riesz spectrum $\beta(x)$, defined in § 5 we have $\omega(x) \subset W(x) \subset \beta(x)$ and $\partial\beta(x) \subset \partial W(x) \subset \partial\omega(x)$.

We shall show that in a B^* algebra there exists $u \in I$ such that $W(x) = \sigma(x+u)$. The proof is in two stages. First we find $u \in I$ such that $\sigma(x+u) \setminus W(x)$ is countable.

THEOREM 7.2. *Given* $x \in A$ *there exists* $u \in \overline{F}$ *such that* $\sigma(x+u) \setminus W(x)$ *is a countable set whose accumulation points lie in* $W(x)$.

Proof. Let $S = \{\lambda \in C;\ x-\lambda \in \Phi$ and $\iota(x-\lambda) \equiv 0\}$. This is an open subset of C and we may write $S = \bigcup S_n$ where each S_n is a connected component of S. Let $\varepsilon_0 = 1$ and construct inductive sequences $\{\lambda_k\}_1^\infty \subset C$, $\{u_k\}_1^\infty \subset A$, $\{\varepsilon_k\}_1^\infty \subset R^+$ as follows. Choose $\lambda_n \in S_n$, then $u_n \in F$ such that $\|u_n\| \leqslant \frac{1}{2}\varepsilon_{n-1}$ and $x + \sum_1^n u_k - \lambda_n$ is invertible (this is possible by 6.8), and finally $\varepsilon_n \leqslant \frac{1}{2}\varepsilon_{n-1}$ such that $x + \sum_1^n u_k - \lambda_n + y$ is invertible for $\|y\| \leqslant \varepsilon_n$. It will be seen that $u = \sum_1^\infty u_k$ is a well-defined element of $\overline{F}$, and $\lambda_n \in \varrho(x+u)$ for all $n \in Z^+$. Now $x+u-\lambda_n$ is invertible and $\lambda_n \in S_n$, hence by 5.1, $S_n \cap \sigma(x+u)$ is a countable set of Riesz points of $x+u$. Hence $\sigma(x+u) \setminus W(x) = \bigcup (S_n \cap \sigma(x+u))$ is a countable set of Riesz points of $x+u$, whose accumulation points lie in $W(x)$.

We remark that this result is best possible in the sense that $\sigma(x+u) \setminus W(x)$ may be countably infinite for all $u \in \overline{F}$. This is illustrated by considering the example preceding [7], 4.6. However if A is a B^* algebra then $I = \overline{F}$ and the result may be sharpened in the following manner.

THEOREM 7.3. *If* A *is a* B^* *algebra and* $x \in A$ *then there exists* $u \in I$ *such that* $\sigma(x+u) = W(x)$.

We shall only give a sketch of the proof. First of all by 7.2 we may assume that $\sigma(x) \setminus W(x)$ is a countable set of Riesz points of x. If this is a finite set the proof is trivial, so we assume that $\sigma(x) \setminus W(x)$ is a countably infinite sequence $\{\lambda_n\}_1^\infty$ such that $\mathrm{dist}(\lambda_n, W(x)) \geqslant \mathrm{dist}(\lambda_{n+1}, W(x))$ for all $n \in Z^+$. There exists a sequence $\{\alpha_n\}_1^\infty \subset W(x)$ such that $|\lambda_n - \alpha_n| < 2\,\mathrm{dist}(\lambda_n, W(x))$ for all n. Note that $\lambda_n - \alpha_n \to 0$ as $n \to \infty$. We define $p_0 = 0$ and let p_n be the spectral idempotent of x at λ_n. Clearly $p_n \in F$ for all n. We write $s_n = \sum_0^n p_k$ and by [7], 6.1, there exists a sequence $\{q_n\}_0^\infty$ of self-adjoint idempotents of A such that $q_n s_n = s_n$ and $s_n q_n = q_n$ for all n. Clearly $q_0 = 0$ and $q_n \in F$ for all n. Following the argument of [7], 6.3, 6.4, 6.5 we see that $u = \sum_1^\infty (\alpha_k - \lambda_k)(q_k - q_{k-1})$ is a well-defined element of I. We shall show that $\sigma(x+u) \subset W(x)$. Suppose on the contrary that $\lambda \in \sigma(x+u) \setminus W(x)$. Then $x+u-\lambda \in \Phi$ and $\iota(x+u-\lambda) \equiv 0$, hence $i(x+u-\lambda) = 0$. It follows that $d(x+u-\lambda) \neq 0$, hence by [2], 2.3, there exists a non-zero element $w \in A$ such that $w(x+u-\lambda) = 0$. Let T be the image of $x+u-\lambda$ under the left regular representation of A. Now for each n, $q_n A = s_n A$, hence $q_n A$ is invariant under T and we claim that $T|q_n A$ is one-to-one. This is trivially true for $n = 0$. As an inductive hypothesis assume it is true for $0, 1, \ldots, n-1$. Suppose $y \in q_n A$ and $Ty = 0$. Then

$$xy + \sum_1^n (\alpha_k - \lambda_k)(q_k - q_{k-1})y = \lambda y.$$

Multiplication through this equation on the left by p_n gives

$$xp_n y + (\alpha_n - \lambda_n)p_n y = \lambda p_n y,$$

hence if $p_n y \neq 0$ it will be an eigenvector of x corresponding to the eigenvalue $\lambda - \alpha_n + \lambda_n \neq \lambda_n$ since $\lambda \notin W(x)$. But this is impossible since all such eigenvectors lie in $(1-p_n)A$, and hence $p_n y = 0$. Now

$$y = q_n y = s_n q_n y = s_n y = s_{n-1} y = q_{n-1} s_{n-1} y \in q_{n-1} A$$

hence by the induction hypothesis $y = 0$ and therefore $T|q_n A$ is one-to-one for all n. Therefore $q_n A q_n$ (a finite-dimensional space by [7], 3.1 (i)) is invariant under T and $T|q_n A q_n$ is one-to-one, hence $T|q_n A q_n$ is an invertible operator. It follows that $q_n \in (x+u-\lambda)A$ and therefore $wq_n = 0$ for all n. Hence $w(x-\lambda) = 0$, therefore $\lambda = \lambda_n$ for some n and we have $w \in Ap_n$. Hence $w = ws_n = wq_n s_n = 0$ and this contradiction completes the proof.

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A SURVEY ON REPRESENTATIONS OF THE UNITARY GROUP $U(\infty)$

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CONTENTS

This paper surveys some results concerning the representation problem for the unitary group $U(\infty)$. It is based on the results of the papers [7], [15], [16], [17], [18], [19], [21], [22].

For groups which are not locally compact, the representation theory, as in the case of canonical commutation and anticommutation relations of mathematical physics, deals with special classes of representations and with a global study based on the use of some associated C^*-algebras.

The methods developed are, applicable, as well, to the related groups $O(\infty)$, $\mathrm{Sp}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{SU}(\infty)$ and, moreover, some of the results presented below are sufficiently general to include also these other groups.

A survey on infinite-dimensional spin groups was given by R. J. Plymen [12].

0. Notations

Let H be a complex separable infinite-dimensional Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $K(H)$ the ideal of compact operators, $C_1(H)$ the Banach space of nuclear operators endowed with the norm $\|X\|_1 = \mathrm{Tr}(|X|)$ and $C_2(H)$ the Hilbert space of Hilbert–Schmidt operators endowed with the norm $\|X\|_2 = \mathrm{Tr}(X^*X)^{1/2}$.

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