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A SURVEY ON REPRESENTATIONS OF THE UNITARY GROUP $U(\infty)$

ŞERBAN STRĂTILĂ and DAN VOICULESCU*

Department of Mathematics, The National Institute for Scientific and Technical Creation, Bucharest, România

CONTENTS

This paper surveys some results concerning the representation problem for the unitary group $U(\infty)$. It is based on the results of the papers [7], [15], [16], [17], [18], [19], [21], [22].

For groups which are not locally compact, the representation theory, as in the case of canonical commutation and anticommutation relations of mathematical physics, deals with special classes of representations and with a global study based on the use of some associated C^*-algebras.

The methods developed are, applicable, as well, to the related groups $O(\infty)$, $\mathrm{Sp}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{SU}(\infty)$ and, moreover, some of the results presented below are sufficiently general to include also these other groups.

A survey on infinite-dimensional spin groups was given by R. J. Plymen [12].

0. Notations

Let H be a complex separable infinite-dimensional Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $K(H)$ the ideal of compact operators, $C_1(H)$ the Banach space of nuclear operators endowed with the norm $||X||_1 = \mathrm{Tr}(|X|)$ and $C_2(H)$ the Hilbert space of Hilbert–Schmidt operators endowed with the norm $||X||_2 = \mathrm{Tr}(X^*X)^{1/2}$.

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Let $U(H)$ be the group of unitary operators endowed with the strong-operator topology, $U_0(H) = \{V \in U(H);\ V-I \in K(H)\}$ with the norm topology, $U_1(H) = \{V \in U(H);\ V-I \in C_1(H)\}$ with the metric $d_1(V', V'') = \|V'-V''\|_1$ and $U_2(H) = \{V \in U(H);\ V-I \in C_2(H)\}$ with the metric $d_2(V', V'') = \|V'-V''\|_2$.

We consider also the topological group $U(\infty)$ defined as the direct limit of the classical unitary groups $U(n)$ with respect to the inclusions

$$U(n) \ni V \mapsto \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} \in U(n+1).$$

There are different realizations of $U(\infty)$ as a subgroup of $U(H)$. If $\{e_n\}_{n\in N}$ is any orthonormal basis of H and H_n denotes the linear span of $\{e_1, \ldots, e_n\}$, then we can identify $U(\infty)$ with

$$\{V \in U(H);\ V|H\ominus H_n = I|H\ominus H_n \text{ for some } n \in N\}.$$

Then $U(\infty) \subset U_1(H) \subset U_0(H) \subset U(H)$ and $U(\infty)$ is dense in all these groups relative to their respective topologies.

Similarly, starting with the classical groups $O(n)$, $\mathrm{Sp}(n)$, $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$, one can define the direct limit groups $O(\infty)$, $\mathrm{Sp}(\infty)$, $\mathrm{SO}(\infty)$ or $\mathrm{SU}(\infty)$, respectively. Also, the group $S(\infty)$ of finite permutations of N can be viewed as the direct limit of the symmetric groups $S(n)$ of all permutations of $\{1, \ldots, n\}$.

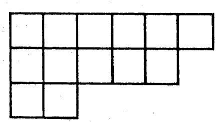
By a representation of a topological group we shall always mean a continuous unitary representation on a Hilbert space.

1. Segal–Kirillov representations

The classical theorem of H. Weyl [23] shows that all irreducible representations of $U(n)$ are realized in spaces of tensors of determined symmetry types classified by decreasing n-tuples of integers $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n$ called "signatures".

In the infinite-dimensional case a similar result holds for the group $U_0(H)$.

1.1. For a positive signature $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n \geqslant 0$ consider the "Young diagram"



the rows of which have lengths $m_1, \ldots, m_n$, respectively, and insert in squares the numbers $1, 2, \ldots, m$, where

$$m = m_1 + \ldots + m_n,$$

filling first the first column, then the second one and so on. Let P and Q be the subgroups of the symmetric group $S(m)$ consisting of those permutations which preserve the rows of the Young diagram, and, respectively, its columns (horizontal and vertical permutations). Let $\varepsilon(\sigma)$ denote the sign of $\sigma \in S(m)$.

Consider the representation $\tilde{\varrho}$ of $U(H)$ on $H^m = H \otimes \ldots \otimes H$ (m times), given by

$$\tilde{\varrho}(V)\Big(\bigotimes_{j=1}^{m} \xi_j\Big) = \bigotimes_{j=1}^{m} V\xi_j; \qquad V \in U(H),$$

and the representation π of $S(m)$ on H^m given by

$$\pi(\sigma)\Big(\bigotimes_{j=1}^{m} \xi_j\Big) = \bigotimes_{j=1}^{m} \xi_{\sigma^{-1}(j)}; \qquad \sigma \in S(m),$$

and define the linear map $R\colon H^m \to H^m$ by

$$R = \sum_{(p,q)\in P\times Q} \varepsilon(q)\pi(qp).$$

Then $R(H^m)$ is an invariant subspace for the $\tilde{\varrho}(V)$, $V \in U(H)$, and the *restriction ϱ of $\tilde{\varrho}$ to $R(H^m)$ is an irreducible representation of $U(H)$* ([23], [7], [15]).

1.2. The same construction applied to H_k instead of H with respect to the signature $m_1 \geqslant \ldots \geqslant m_k \geqslant 0$ ($m_j = 0$ for $j > n$), yields an irreducible representation ϱ_k of $U(H_k)$ and it is apparent that $\varrho|U(\infty)$ is the *natural direct limit of the ϱ_k's.*

1.3. The representations ϱ of $U(H)$ described in 1.1, were first considered by I. E. Segal ([15]), who proved that *these are the only irreducible representations of $U(H)$ which, when restricted to any $U(H_n)$, decompose only in irreducible representations of $U(H_n)$ corresponding to positive signatures.* In Segal's terminology these are called "physical representations".

1.4. *A slight modification of the construction in* 1.1 *is possible in order to associate an irreducible representation of $U_0(H)$ with an arbitrary* (not necessarily positive) *signature* and A. A. Kirillov ([7]) showed that *any irreducible representation of $U_0(H)$ is obtained in this way.* However, the general construction involves mixed tensors (tensor products like $H\otimes \ldots \otimes H \otimes \bar{H} \otimes \ldots \otimes \bar{H}$) and the argument is no longer a straightforward extension of the finite-dimensional case (see [7], Lemma 1).

1.5. A. A. Kirillov ([7]) also shows that *every representation of $U_0(H)$ is a discrete direct sum of irreducible representations.* A similar statement is proved by I. E. Segal ([15]) for physical representations of $U(H)$.

1.6. For $U(\infty)$, however, there are many other irreducible representations, for instance arbitrary direct limits of irreducible representations of the $U(H_n)$'s.

Let, for each $n \in N$, ϱ_n be an irreducible representation of $U(H_n)$ and assume that $\varrho_n \prec \varrho_{n+1}$, i.e., $\varrho_{n+1}|U(H_n)$ contains ϱ_n. If ϱ_n corresponds to the signature $m_1^{(n)} \geqslant \ldots \geqslant m_n^{(n)}$, then

$$\varrho_n \prec \varrho_{n+1} \Leftrightarrow m_{j-1}^{(n+1)} \geqslant m_{j-1}^{(n)} \geqslant m_j^{(n+1)} \qquad (1 \leqslant j \leqslant n+1),$$

and in this case the multiplicity $[\varrho_{n+1} : \varrho_n]$ of ϱ_n in ϱ_{n+1} is exactly one ([23]).

Since $\varrho_n \prec \varrho_{n+1}$, there are isometric imbeddings

$$i_n\colon H_{\varrho_n} \to H_{\varrho_{n+1}}$$

such that $(\varrho_{n+1}|U(H_n))\circ i_n = i_n \circ \varrho_n$ and, moreover, since $[\varrho_{n+1} : \varrho_n] = 1$, the i_n's are unique up to a scalar factor of modulus 1. On the completion H_ϱ of the direct limit of the H_{ϱ_n}'s along the i_n's there is a natural representation ϱ of $U(\infty)$, which is independent of the choice of the i_n's.

The representation $\varrho \sim (\varrho_1 \prec \varrho_2 \prec \ldots)$ *is irreducible and two such representations* $\varrho \sim (\varrho_1 \prec \varrho_2 \prec \ldots)$ *and* $\varrho' \sim (\varrho_1' \prec \varrho_2' \prec \ldots)$ *are equivalent if and only if* ϱ_n *is equivalent to* ϱ_n' *for all sufficiently large* n*'s* (Ş. Strătilă, D. Voiculescu, [17], III.2).

1.7. Later on (§ 5) we shall associate a C^*-algebra $A(U(\infty))$ with the factor representations of $U(\infty)$ and (§ 6) we shall characterize its primitive ideal space. Since $A(U(\infty))$ is not of type I, there appear all the pathologies known for this case. Let us mention that already A. A. Kirillov ([7]) pointed out that $U(\infty)$ is not of type I.

1.8. Results similar to those presented above for $U(\infty)$ are valid for the groups $O(\infty)$, $\mathrm{Sp}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{SU}(\infty)$ ([7], [17]).

2. Infinite tensor product representations

In the situation considered in 1.1, a theorem of H. Weyl ([23]; for infinite-dimensional H see A. A. Kirillov [7]) asserts that the commutant of $\tilde{\varrho}(U(H))$ is the linear span of $\pi(S(m))$. A similar result holds for infinite tensor products. However, the representations of $U_1(H)$ arising in this way are factorial of type II_∞, as we shall see.

2.1. Consider an arbitrary orhonormal system α in H,

$$\alpha = (a_1, a_2, \ldots, a_n, \ldots),$$

and define the Hilbert space $\mathscr{H}^\alpha$ as the von Neumann infinite tensor product of a sequence of copies of H along the sequence α ([11]). There are natural representations ϱ^α of $U_1(H)$ and π^α of $S(\infty)$ on $\mathscr{H}^\alpha$ such that

$$\varrho^\alpha(V)\Big(\bigotimes_{j=1}^{\infty} \xi_j\Big) = \bigotimes_{j=1}^{\infty} V\xi_j; \qquad V \in U_1(H),$$

$$\pi^\alpha(\sigma)\Big(\bigotimes_{j=1}^{\infty} \xi_j\Big) = \bigotimes_{j=1}^{\infty} \xi_{\sigma^{-1}(j)}; \qquad \sigma \in S(\infty),$$

for all decomposable vectors $\bigotimes_{j=1}^{\infty} \xi_j \in \mathscr{H}^\alpha$.

Note that ϱ^α is the representation of $U_1(H)$ associated with the function of positive type

$$\varphi^\alpha(V) = \prod_{j=1}^{\infty} (Va_j|a_j); \qquad V \in U_1(H).$$

2.2. Theorem (Ş. Strătilă, D. Voiculescu, [17], V). *The representation* ϱ^α *of* $U_1(H)$ *on* $\mathscr{H}^\alpha$ *is a factor representation of type* II_∞ *and the commutant of* $\varrho^\alpha(U_1(H))$ *is the von Neumann algebra generated by* $\pi^\alpha(S(\infty))$.

2.3. The proof of this theorem goes as follows (cf. [17]).

First one considers the case $\alpha \subset \{e_n\}_{n\in N}$ where $\{e_n\}_{n\in N}$ is the orthonormal basis by means of which one realizes $U(\infty) \subset U(H)$ and one defines $\varrho^\alpha(V)$ only for $V \in U(\infty)$, which is clearly legitimate. In this case one proves, by an approximation argument using the commutation theorem of H. Weyl, that the commutant of $\varrho^\alpha(U(\infty))$ is generated by $\pi^\alpha(S(\infty))$. It is well known that the left regular representation of $S(\infty)$ is factorial of type II_1 and from this one infers that $\pi^\alpha(S(\infty))$ generates a type II_1 factor, hence ϱ^α is a type II factor representation of $U(\infty)$. By constructing an infinite family of mutually orthogonal and equivalent projections in $\varrho^\alpha(U(\infty))''$, one shows that ϱ^α is actually of type II_∞. Moreover, by a direct computation one shows that the function of positive type φ^α is uniformly continuous with respect to the metric of $U_1(H)$ and hence ϱ^α extends to a representation of $U_1(H)$.

The general case now follows owing to the fact that every unitary $W \in U(H)$ defines an automorphism $V \mapsto W^*VW$ of $U_1(H)$.

2.4. Now, given two orthonormal systems $\alpha = \{a_n\}$, $\beta = \{b_n\}$ in H, it is natural to ask for necessary and sufficient conditions in order that the representations ϱ^α and ϱ^β be equivalent.

A reasonable conjecture might be that this is the case if and only if there exist a permutation σ of N, an operator $U \in U_1(H)$ (or, maybe, $U \in U_2(H)$?) and $\theta_n \in C$, $|\theta_n| = 1$ such that $Ub_n = \theta_n a_{\sigma(n)}$ $(n \in N)$. We were able to prove only the following fact:

Proposition (Ş. Strătilă, D. Voiculescu, [17], V). *If* ϱ^α *and* ϱ^β *are equivalent then there are finite sets* $F_\alpha \subset N$, $F_\beta \subset N$ *and a bijective map* $\sigma\colon N\setminus F_\beta \to N\setminus F_\alpha$ *such that*

$$\lim_n \|b_n - \theta_n a_{\sigma(n)}\| = 0 \quad \textit{for suitable } \theta_n \in C,\ |\theta_n| = 1.$$

Let us also mention that if $a_n = e_{k_n}$, $b_n = e_{j_n}$, where $\{e_n\}_{n\in N}$ is an orthonormal basis of H, and $\{k_n\}$, $\{j_n\}$ are strictly increasing sequences of positive integers, then ([17], V.1.7)

$$\varrho^\alpha \simeq \varrho^\beta \Leftrightarrow \exists n_0 \in N \textit{ such that } k_n = j_n\ \forall n \geqslant n_0.$$

3. The characters

For $U(n)$ the determination of all irreducible representations is equivalent to the determination of all its characters, that is, of all indecomposable central functions χ of positive type on $U(n)$ with $\chi(1) = 1$. The explicit formula of characters of $U(n)$ is due to H. Weyl ([23]).

For $U(\infty)$, the determination of all its characters means the classification not of irreducible but of finite factor representations (types I_n and II_1). As we shall see, in the infinite-dimensional case there are similarities with the case of commutative groups due to the fact that $U(\infty)$ is stable under the direct sum operation.

3.1. More generally, let G be a group and let Γ be the set of classes of conjugate elements in G. For $g \in G$ let $\hat{g} \in \Gamma$ denote its conjugacy class and for a central function χ on G let χ denote the corresponding function on Γ. Any homomorphism $\varphi\colon G \times G \to G$ defines an operation "$\oplus$" on Γ:

$$\hat{g}_1 \oplus \hat{g}_2 = \varphi(g_1, g_2)^\wedge.$$

In what follows *we shall assume the existence of a homomorphism φ such that the operation "$\oplus$" on Γ is commutative, associative and with neutral element* $\hat{e}$ (where $e \in G$ is the neutral element of G). Then there are homomorphisms $\varphi_n\colon G^n \to G$ such that $\varphi_n(g_1, \ldots, g_n)^\wedge = \hat{g}_1 \oplus \ldots \oplus \hat{g}_n$, and $\hat{g} \mapsto (\hat{g})^* = (g^{-1})^\wedge$ is an automorphism of Γ.

3.2. THEOREM (D. Voiculescu, [21]). *Let χ be a central function of positive type on G with $\chi(e) = 1$. Then χ is the character of a finite factor representation of G if and only if*

$$\tilde{\chi}(\gamma_1 \oplus \gamma_2) = \tilde{\chi}(\gamma_1)\tilde{\chi}(\gamma_2); \qquad \gamma_1, \gamma_2 \in \Gamma.$$

The simple proof of this theorem will be given in 3.8. In some particular cases this result was found, with rather complicated proofs, by E. Thoma ([19], [20]) and D. Voiculescu ([22]).

The following corollaries are obvious:

3.3. COROLLARY (D. Voiculescu, [21]). *The tensor product of two finite factor representations of G is still a finite factor representation.*

3.4. COROLLARY (D. Voiculescu, [21]). *Let G, G' be two groups satisfying the hypotheses in* 3.1, *and $\omega\colon G' \to G$ a homomorphism such that $\varphi \circ (\omega \times \omega) = \omega \circ \varphi'$. If ϱ is a finite factor representation of G, then $\varrho \circ \omega$ is a finite factor representation of G'.*

Below we consider some examples.

3.5. If G is commutative, then $\Gamma = G$ and we can take φ as the group operation in G. Thus, Theorem 3.2 implies the well known characterization of characters of commutative groups.

3.6. If $G = U(\infty)$, then the conjugacy class $\hat{V}$ of $V \in U(n)$ is determined by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of V together with their multiplicities. We shall therefore write:

$$\hat{V} = (\lambda_1, \ldots, \lambda_n, 1, 1, \ldots).$$

The group $U(\infty)$ can be as well realized on $H \oplus H$ with respect to the orthonormal basis $\{e_1, e_2, \ldots\} \oplus \{e_1, e_2, \ldots\}$. If $V_1, V_2 \in U(\infty) \subset U(H)$, then $V_1 \oplus V_2 \in U(\infty) \subset U(H \oplus H)$. The map

$$\varphi\colon U(\infty) \times U(\infty) \ni (V_1, V_2) \mapsto V_1 \oplus V_2 \in U(\infty)$$

is a homomorphism which induces on Γ the operation

$$(\lambda_1, \ldots, \lambda_n, 1, \ldots) \oplus (\mu_1, \ldots, \mu_m, 1, \ldots) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m, 1, \ldots)$$

satisfying all the requirements in 3.1.

Thus, by Theorem 3.2, a continuous central function χ of positive type on $U(\infty)$ with $\chi(1) = 1$, is a continuous character of $U(\infty)$ if and only if

$$\tilde{\chi}(\lambda_1, \ldots, \lambda_n, 1, \ldots) = \chi(\lambda_1) \ldots \chi(\lambda_n)$$

for all $(\lambda_1, \ldots, \lambda_n, 1, 1, \ldots) \in \Gamma$. Thus $\tilde{\chi}$, and hence χ, is determined by $p = \chi|U(1)$:

$$\chi(V) = \det p(V); \qquad V \in U(\infty). \tag{1}$$

Since p is a continuous function on $U(1)$, we have a Fourier expansion

$$p(z) = \sum_{n \in \mathbf{Z}} c_n z^n, \tag{2}$$

where $c_n \geqslant 0$ and $\sum_{n \in \mathbf{Z}} c_n = 1$, because p is of positive type and $p(1) = 1$. Moreover, $\chi|U(n)$ is of positive type, which is equivalent to the fact that the coefficients in its development with respect to the characters of $U(n)$ are all positive. These coefficients can be computed and we are led to the following condition (cf. [22]):

$$\det\big((c_{m_i + (j-i)})_{1 \leqslant i, j \leqslant n}\big) \geqslant 0 \tag{3}$$

for all integers $m_1 \geqslant \ldots \geqslant m_n$.

The series (2) with $c_n = 0$ for $n < 0$, $c_0 = 1$ and satisfying the conditions (3) were studied by E. Thoma ([19]) in connection with the characters of the group $S(\infty)$. In this case, the result of E. Thoma shows that $p(z)$ is of the form

$$p(z) = z^m p_0(z); \qquad p_0(z) = \prod_{i=1}^{\infty} \frac{1 - a_i}{1 - a_i z} \cdot \prod_{j=1}^{\infty} \frac{1 + b_j z}{1 + b_j} \cdot e^{\lambda(z-1)}, \tag{4}$$

with $m \in \mathbf{Z}$, $m \geqslant 0$; $0 \leqslant a_i < 1$, $\sum_i a_i < +\infty$; $b_j \geqslant 0$, $\sum_j b_j < +\infty$; $\lambda \geqslant 0$.

The formulas (1) *and* (4) *determine the characters of all finite factor representations of $U(\infty)$ which, when restricted to the $U(n)$'s, decompose only in irreducible representations of $U(n)$ with positive signatures.*

3.7. Similar characterizations hold for the groups $O(\infty)$, $\mathrm{Sp}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{SU}(\infty)$ or for the group $\mathrm{GL}(\infty, k)$ over a finite field k considered by E. Thoma ([20]). Owing to Corollary 3.4, one obtains characters of $O(\infty)$, $\mathrm{Sp}(\infty)$, etc., by restricting the above determined characters of $U(\infty)$.

In case $G = S(\infty)$, the conjugacy class $\hat{\sigma}$ of $\sigma \in S(\infty)$ is determined by the decomposition of σ into cycles. If σ_1, σ_2 are two finite permutations of $\mathbf{N}$, then they determine a finite permutation $\sigma_1 \cup_d \sigma_2$ of the disjoint union $\mathbf{N} \cup_d \mathbf{N}$ identified with $\mathbf{N}$, and this procedure yields the direct sum operation on the corresponding Γ. Actually, all the characters of $S(\infty)$ were determined by E. Thoma ([19]).

3.8. *Proof of Theorem* 3.2.

Necessity of the multiplicativity condition. Let ϱ be a finite factor representation of G and assume $\chi(g) = \operatorname{Tr}\varrho(g)$, where Tr is the normalized trace, $\operatorname{Tr} I = 1$. Write

$$\varrho_1(g) = \varrho(\varphi(g, e)), \quad \varrho_2(g) = \varrho(\varphi(e, g))$$

and let F, A, B denote the von Neumann algebras generated by $\varrho(G)$, $\varrho_1(G)$, $\varrho_2(G)$, respectively. The restriction of the trace of F to A and B yields faithful traces on A and B. Since $\hat{g}\oplus\hat{e} = \hat{e}\oplus\hat{g} = \hat{g}$, we have $\operatorname{Tr}(\varrho_1(g)) = \operatorname{Tr}(\varrho_2(g)) = \operatorname{Tr}(\varrho(g))$. We hence infer that F, A, B are isomorphic, in particular A and B are also factors. Since A and B are commuting subfactors of F, it follows that

$$\begin{aligned}\tilde{\chi}(\hat{g}_1\oplus\hat{g}_2) &= \chi(\varphi(g_1, g_2)) = \operatorname{Tr}(\varrho_1(g_1)\varrho_2(g_2)) \\ &= \operatorname{Tr}(\varrho_1(g_1))\operatorname{Tr}(\varrho_2(g_2)) = \tilde{\chi}(\hat{g}_1)\tilde{\chi}(\hat{g}_2).\end{aligned}$$

Sufficiency of the multiplicativity condition. Consider the set K of central functions $\chi\colon G \to \boldsymbol{C}$ of positive type such that $\chi(e) = 1$. We show that for $\chi \in K$, $\gamma_i \in \Gamma$ and $a_i \in \boldsymbol{C}$ $(1 \leqslant i \leqslant n)$, we have

$$(5) \qquad \sum_{1\leqslant i,j\leqslant n} a_i a_j \tilde{\chi}(\gamma_i\oplus\gamma_j^*) \geqslant 0.$$

Indeed, for $m \in \boldsymbol{N}$ let

$$c_{km+p} = a_{k+1}/m \qquad (0 \leqslant k \leqslant n-1,\ 1 \leqslant p \leqslant m)$$

and $g_s \in G$ $(1 \leqslant s \leqslant mn)$ be such that

$$\hat{g}_{km+p} = \gamma_{k+1} \qquad (0 \leqslant k \leqslant n-1,\ 1 \leqslant p \leqslant m).$$

Then defining

$$g'_{km+p} = \varphi_{mn}(e, \ldots, e, g_{km+p}, e, \ldots, e)$$

(with g_{km+p} on the $(km+p)$-th place), we have

$$\big(g'_{k_1m+p_1}(g'_{k_2m+p_2})^{-1}\big)^{\wedge} = \begin{cases}\gamma_{k_1}\oplus\gamma_{k_2}^* & \text{if} \quad (k_1, p_1) \neq (k_2, p_2),\\ \hat{e} & \text{if} \quad (k_1, p_1) = (k_2, p_2).\end{cases}$$

Since χ is of positive type, this gives

$$\begin{aligned}0 &\leqslant \sum_{1\leqslant i,j\leqslant mn} c_i\bar{c}_j\chi\big(g'_i(g'_j)^{-1}\big) \\ &= \sum_{1\leqslant p,q\leqslant n} a_p\bar{a}_q\tilde{\chi}(\gamma_p\oplus\gamma_q^*) + \frac{1}{m}\sum_{k=1}^{n}|a_i|^2\big(1-\tilde{\chi}(\gamma_i\oplus\gamma_i^*)\big).\end{aligned}$$

Letting $m \to +\infty$, we obtain (5).

Since Γ is a semigroup with an involutive automorphism $\gamma \mapsto \gamma^*$, $l^1(\Gamma)$ has the structure of an involutive Banach algebra and the set P of functions $f \in l^\infty(\Gamma)$ such that $f(\hat{e}) = 1$ and

$$\sum_{1\leqslant i,j\leqslant n} a_i a_j f(\gamma_i\oplus\gamma_j^*) \geqslant 0$$

for all $n \in \boldsymbol{N}$, $a_1, \ldots, a_n \in \boldsymbol{C}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$, is the set of states of $l^1(\Gamma)$. Thus, relation (5) together with the boundedness of functions of positive type on G, shows that $\chi \mapsto \tilde{\chi}$ is an injective affine map $K \to P$.

Now, if $\tilde{\chi}$ is multiplicative on Γ, then the corresponding representation of $l^1(\Gamma)$ is one-dimensional and hence irreducible. Using 2.5.4, 2.5.5 in [3], we see that $\tilde{\chi}$ is an extreme point of K, and hence χ is then *a fortiori* an extreme point of K, that is the character of a finite factor representation.

3.9. In the above proof of necessity we saw that every finite representation of G generates a factor, which contains two commuting subfactors isomorphic to itself. This implies that for G as above, a finite factor representation is either of type II_1 or of type I_1 (cf. [22]).

3.10. Using a desintegration procedure, D. Voiculescu ([22]) showed that *every central function of positive type on $U(\infty)$ can be extended by continuity to $U_1(H)$.*

4. KMS-functions of positive type

In this section we introduce KMS-functions of positive type on topological groups endowed with a one-parameter automorphism group, by a straightforward analogy to KMS-states on C^*-dynamical systems. This will be applied to a certain class of functions of positive type on $U_1(H)$ derived from the character formula 3.6.(4) by replacing a scalar by a positive operator. This notion proves useful, since in general we do not dispose of any corresponding C^*-dynamical system, and, moreover, it leads to easy computations.

4.1. Let G be a topological group and $\boldsymbol{R} \ni t \mapsto \alpha_t \in \operatorname{Aut}(G)$ a one-parameter automorphism group such that for every $g \in G$ the map $\boldsymbol{R} \ni t \mapsto \alpha_t(g) \in G$ is continuous.

A continuous function $\theta\colon G \to \boldsymbol{C}$ of positive type is called *KMS* with respect to $(\alpha_t)_{t\in\boldsymbol{R}}$ if for every $g, h \in G$ there is a bounded continuous function $F_{g,h}$ defined on the strip $S = \{z \in \boldsymbol{C};\, 0 \leqslant \operatorname{Re} z \leqslant 1\}$ with complex values, which is analytic in the interior of S and such that, for all $t \in \boldsymbol{R}$,

$$F_{g,h}(it) = \theta\big(g\alpha_t(h)\big), \quad F_{g,h}(1+it) = \theta\big(\alpha_t(h)g\big).$$

Using the Kaplansky density theorem and the Phragmén–Lindelöf principle, it is easy to prove the following result:

PROPOSITION. *Let θ be a continuous function of positive type on G, KMS with respect to the continuous one-parameter automorphism group $(\alpha_t)_{t\in\boldsymbol{R}}$ of G. Let ϱ be the cyclic representation of G, with cyclic vector η, associated to θ. Then the representation ϱ is in standard form, that is, η is also a separating vector for the von Neumann algebra $\varrho(G)''$ and, denoting by $(\sigma_t^{\omega_\eta})_{t\in\boldsymbol{R}}$ the modular automorphism group of $\varrho(G)''$ associated to the vector state ω_η, we have*

$$\sigma_t^{\omega_\eta}\big(\varrho(g)\big) = \varrho\big(\alpha_t(g)\big); \quad t \in \boldsymbol{R},\ g \in G.$$

4.2. Recall that the characters of $U(\infty)$ corresponding to positive signatures are given by the formulas 3.6 (1) and (4). For a fixed p as in 3.6 (4), let

$$R = (\sup_i a_i)^{-1} > 1.$$

Then $p(z)$ is an analytic function in the open disk $\{z \in C;\ |z| < R\}$ and $p_0(z) \neq 0$ on some neighbourhood of the segment $[0, +\infty)$.

For every $B \in L(H)$, $\|B\| < R$, $B \geqslant 0$, one defines a function $\theta = \theta_{p,B}$ on $U_1(H)$ by

$$\theta_{p,B}(V) = \det(V^m)\det\big(p_0(BV)p_0(B)^{-1}\big); \qquad V \in U_1(H).$$

Moreover, if $\operatorname{Ker} B = 0$, we have a continuous one-parameter group $\{\alpha_t\}_{t\in R}$ of automorphisms of $U_1(H)$ defined by

$$\alpha_t(V) = B^{it} V B^{-it}; \qquad V \in U_1(H).$$

The main result presented in this section is the following theorem:

4.3. THEOREM (Ş. Strătilă, D. Voiculescu, [18], § 4). *For every p as in* 3.6 (4) *and every $B \in L(H)$, $\|B\| < R$, $B \geqslant 0$, $\theta_{p,B}$ is a continuous function of positive type on $U_1(H)$. If, moreover,* $\operatorname{Ker} B = 0$, *then $\theta_{p,B}$ is* KMS *with respect to the automorphism group $\{\alpha_t\}_{t\in R}$.*

4.4. The proof of the fact that $\theta = \theta_{p,B}$ is of positive type consists in approximating the operator B by diagonable operators and thus reducing the problem to the finite-dimensional case.

If $H = H_n$ is finite-dimensional, then

$$\tau_n\colon U(H_n) \ni V \mapsto \det p(V)$$

is a central function of positive type on $U(H_n)$ and

$$\tau_n(V) = \sum_\mu c_\mu \chi_\mu(V); \qquad V \in U(H_n),$$

where μ runs over all the positive signatures $m_1 \geqslant \ldots \geqslant m_n \geqslant 0$ and $c_\mu \geqslant 0$. Then, for $V \in U(H_n)$ we have

$$\det\big(p(B)\big)\theta(V) = \tau_n(BV) = \sum_\mu c_\mu \chi_\mu(BV) = \sum_\mu c_\mu \operatorname{Tr}\big(\varrho_\mu(B)\varrho_\mu(V)\big),$$

which shows that θ is of positive type on $U(H_n)$.

4.5. To see that $\theta = \theta_{p,B}$ is KMS with respect to $\{\alpha_t\}_{t\in R}$, consider $V, W \in U_1(H)$ such that $V-I$, $W-I$ are of finite rank and define

$$F_{V,W}(z) = \det(V^m W^m)\det\big(p_0(VB^z WB^{1-z})p_0(B)^{-1}\big).$$

Then $F_{V,W}$ is a well-defined bounded continuous function on the strip $S = \{z \in C;\ 0 \leqslant \operatorname{Re} z \leqslant 1\}$, analytic in the interior of S and

$$F_{V,W}(it) = \theta\big(V\alpha_t(W)\big), \quad F_{V,W}(1+it) = \theta\big(\alpha_t(W)V\big).$$

5. The associated C^*-algebra and applications

In this section we associate a C^*-algebra to a direct limit of compact groups, which reflects the factor representations of the direct limit group. This C^*-algebra turns out to be approximately finite-dimensional (AF-algebra) and for such algebras we give a diagonalization that reduces their study to dynamical systems of a particular nature. The ideals of the AF-algebra and some classes of representations can be easily handled by using the associated dynamical system.

5.1. Let G be the direct limit of compact separable groups

$$\{e\} = G_0 \subset G_1 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$$

such that G_n is Haar negligible in G_{n+1}. We denote by $A(G)$ the envelopping C^*-algebra of the involutive normed algebra defined as the direct limit of $\sum_{j=1}^n L^1(G_j) \subset M(G_n)$ along the isometric imbeddings induced by $M(G_n) \subset M(G_{n+1})$. Then:

THEOREM (Ş. Strătilă, D. Voiculescu, [17], II.1). *$A(G)$ is a C^*-algebra whose factor representations are in bijection with the factor representations of G and those of the G_n's. This bijection preserves the type and the equivalence of representations.*

5.2. The C^*-algebra $A = A(G)$ is an AF-algebra, that is, there is an ascending sequence $\{A_n\}_{n\geqslant 0}$ of finite-dimensional C^*-subalgebras in A with

$$A = \overline{\bigcup_n A_n}. \tag{1}$$

For an arbitrary AF-algebra (1) we have a diagonalization method in the sense of the following theorem.

THEOREM (Ş. Strătilă, D. Voiculescu, [17], I.1). *For an* AF-*algebra A there exists*

(a) *a maximal abelian *-subalgebra C in A;*
(b) *a conditional expectation $P\colon A \to C$;*
(c) *a subgroup U of the unitary group of A;*

such that

(i) *$u^*Cu = C$ for all $u \in U$;*
(ii) *$P(u^*xu) = u^*P(x)u$ for all $u \in U$, $x \in A$;*
(iii) $A = \text{c.l.m.}(UC) = \text{c.l.m.}(CU)$.

Moreover, let Ω be the Gelfand spectrum of C and Γ be the group of homeomorphisms of Ω induced by U.

Consider the Hilbert space $l^2(\Omega)$ with orthonormal basis $\{t;\ t \in \Omega\}$ and denote by $(\cdot\,|\,\cdot)$ the scalar product. Each $f \in C(\Omega)$ defines a "multiplication operator" T_f on $l^2(\Omega)$ by

$$T_f(h) = fh; \qquad h \in l^2(\Omega)$$

and each element $\gamma \in \Gamma$ defines a "permutation operator" V_γ on $l^2(\Omega)$ by

$$V_\gamma(h)(t) = h\big(\gamma^{-1}(t)\big); \qquad t \in \Omega,\ h \in l^2(\Omega).$$

Let
$$A(\Omega,\Gamma)$$
be the C^*-algebra generated in $L(l^2(\Omega))$ by the operators T_f and V_γ ($f\in C(\Omega)$, $\gamma\in\Gamma$).

Then there exists a *-isomorphism
$$A\simeq A(\Omega,\Gamma)$$
such that
$$P(x)(t)=(xt|t);\quad t\in\Omega,\ x\in A.$$

5.3. The diagonalization of A presented in 5.2 is not at all canonical; among other things it depends on the expression (1) of A as an AF-algebra.

However, as shown subsequently by W. Krieger ([9]), any two dynamical systems (Ω,Γ), (Ω',Γ') arising from the same A by such a diagonalization are isomorphic.

5.4. In the case $A=A(G)$, the dynamical system $(\Omega(G),\Gamma(G))$ can be described as follows:

The points of Ω are the symbols
$$t=(\varrho_0(t)\to\ldots\to\varrho_{n-1}(t)\xrightarrow{k_n(t)}\varrho_n(t)\to\ldots),$$
where $1\leqslant n<n_0(t)\leqslant+\infty$, $\varrho_n(t)\in\hat{G}_n$ and $1\leqslant k_n(t)\leqslant[\varrho_n(t):\varrho_{n-1}(t)]\neq 0$. If $t\in\Omega$ and $\omega\subset\Omega$, then $t\in\overline{\omega}$ if and only if either

(a) $t\in\omega$

or

(b) $n_0(t)=+\infty$ and for every $m\in N$ there is $s\in\omega$ with
$$\varrho_n(s)=\varrho_n(t),\quad k_n(s)=k_n(t);\quad \forall n\leqslant m$$
or

(c) $n_0(t)<+\infty$ and the set
$$\{\varrho_{n_0(t)+1}(s);\ s\in\omega,\ \varrho_n(s)=\varrho_n(t),\ k_n(s)=k_n(t),\ \forall n\leqslant n_0(t)\}$$
is infinite.

For a permutation σ of the set
$$\{t\in\Omega;\ n_0(t)=m,\ \varrho_m(t)=\varrho_m\},$$
($m\in N$, $\varrho_m\in\hat{G}_m$), let $\gamma=\gamma(m,\varrho_m,\sigma)$ be the transformation of Ω such that
$$\gamma(t)=t \text{ if either } n_0(t)<m \text{ or } \varrho_m(t)\neq\varrho_m$$
and
$$\gamma(t)=\left(\sigma\left(\varrho_0(t)\xrightarrow{k_1(t)}\ldots\xrightarrow{k_m(t)}\varrho_m(t)\right)\xrightarrow{k_{m+1}(t)}\varrho_{m+1}(t)\to\ldots\right)$$
in the opposite case. The transformation group $\Gamma(G)$ is generated by these $\gamma(m,\varrho_m,\sigma)$.

For details see [17], II.2.

5.5. As regards the ideals of an AF-algebra A, we record the following result:

THEOREM (Ş. Strătilă, D. Voiculescu, [17], I.2). *The primitive ideals of A are in bijection with the closures of the orbits of Γ in Ω.*

5.6. Let μ be a Γ-quasi-invariant probability measure on Ω. Then μ can be regarded as a state of the commutative C^*-algebra $C\cong C(\Omega)$ and therefore
$$\varphi_\mu=\mu\circ P$$
is a state of A. Let π_μ be the cyclic representation of A associated to φ_μ. Then π_μ is a standard representation and, in fact, π_μ coincides with the representation given by the Krieger construction ([8]) applied to the dynamical system (Ω,Γ,μ).

π_μ is a factor representation if and only if μ is Γ-ergodic.

π_μ is a finite representation if and only if μ is equivalent to some Γ-invariant probability measure on Ω. Moreover, every finite representation of A is quasi-equivalent to some π_μ.

π_μ is semi-finite if and only if the transformation group Γ is μ-measurable, i.e., there exists a Γ-invariant sigma-finite positive measure on Ω, equivalent to μ.

For details and more precise characterizations of the type of π_μ see [17], I.3.

5.7. For μ as in 5.6, there is a unique *-representation ϱ_μ of $A(\Omega,\Gamma)$ on $L^2(\Omega,\mu)$ such that
$$\varrho_\mu(T_f)h=fh;\quad h\in L^2(\Omega,\mu),\ f\in C(\Omega)$$
and
$$(\varrho_\mu(V_\gamma)h)(t)=(d\mu^\gamma/d\mu)^{1/2}(t)h(\gamma^{-1}(t));\quad t\in\Omega,\ h\in L^2(\Omega,\mu),\ \gamma\in\Gamma.$$
Then
$$\varrho_\mu \text{ is irreducible} \Leftrightarrow \mu \text{ is ergodic}$$
and
$$\varrho_{\mu_1} \text{ is equivalent to } \varrho_{\mu_2} \Leftrightarrow \mu_1 \text{ is equivalent to } \mu_2$$
(see [17], I.3).

6. The primitive ideal space of $A(U(\infty))$

For the group $U(n)$, the determination of characters, the determination of irreducible representations and the determination of primitive ideals of $C^*(U(n))$ amount to the same thing. Of course, this is no longer true for a non-type I group, in particular for $U(\infty)$.

Here we give a complete description of the primitive ideal space of the C^*-algebra $A(U(\infty))$ associated to $U(\infty)$ as in § 5.

6.1. Let Ω and Γ be a compact space and its group of transformations provided by the diagonalization of the AF-algebra $A(U(\infty))$. By 5.4, the points of Ω are of the form
$$t=(\varrho_1\prec\varrho_2\prec\ldots\prec\varrho_n\prec\ldots),$$
where ϱ_n is an irreducible representation of $U(n)$ (note that if $\varrho_n\prec\varrho_{n+1}$, then

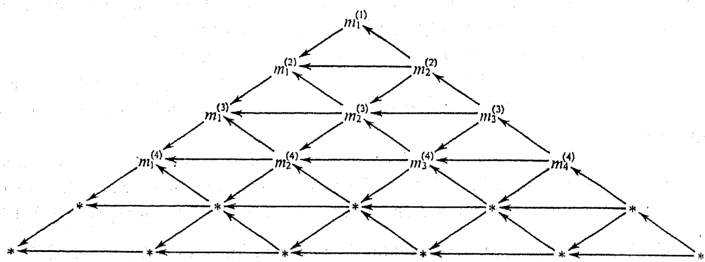
$[\varrho_{n+1}:\varrho_n] = 1)$. According to the description with signatures of the ϱ_n's and of the relations $\varrho_n \prec \varrho_{n+1}$ (see 1.6), it follows that the points of Ω are the symbols

$$t = \{(m_j^{(n)}(t))_{1\leqslant j\leqslant n}\}_{1\leqslant n<n_0(t)},$$

where

$$1 \leqslant n_0(t) \leqslant +\infty, \quad m_j^{(n)}(t) \in \mathbf{Z} \quad \text{and} \quad m_{j-1}^{(n)}(t) \geqslant m_{j-1}^{(n-1)}(t) \geqslant m_j^{(n)}(t).$$

Therefore, a point of Ω looks like the picture below, where $a \to b$ means $a \leqslant b$:



The description of the topology on Ω and the description of the transformation group Γ follow obviously from Section 5.4. Roughly speaking, a point $t \in \Omega$ with $n_0(t) = +\infty$ is adherent to a set $\omega \subset \Omega$ if for every "horizontal line" in the picture of t one can find a point in ω with the same "beginning" until that line. Also, the generators of Γ change these beginnings among themselves, leaving fixed the rest of the picture.

6.2. We recall from 5.5 that the primitive ideals of $A(U(\infty))$ correspond in a canonical way to the closures of the orbits of Γ. In the next lemma we determine these sets. First, some notations. For $t \in \Omega$ and $1 \leqslant j < n_0(t)$ we define

$$L_j(t) = \sup\{m_j^{(n)}(t);\ j \leqslant n < n_0(t)\} \in \mathbf{Z}\cup\{+\infty\},$$
$$M_j(t) = \inf\{m_{n-j+1}^{(n)}(t);\ j \leqslant n < n_0(t)\} \in \mathbf{Z}\cup\{-\infty\}.$$

These definitions can be easily visualised on the picture of t.

It is clear that $\{L_j(t)\}_j$ is decreasing, $\{M_k(t)\}_k$ is increasing and $L_j(t) \geqslant M_k(t)$ for all j, k.

LEMMA. *Consider $t_0 \in \Omega$, $L_j = L_j(t_0)$, $M_j = M_j(t_0)$ and denote by $\omega = \overline{\Gamma(t_0)}$ the closure of the t_0-orbit of Γ in Ω. Then:*

1. *If $n_0(t_0) < +\infty$, then*

$$\omega = \{t \in \Omega;\ n_0(t) = n_0(t_0),\ m_j^{(n_0(t)-1)}(t) = L_j;\ 1 \leqslant j < n_0(t)\}.$$

2. *If $n_0(t_0) = +\infty$, then*

$$\omega\cap\{t \in \Omega;\ n_0(t) = +\infty\}$$
$$= \{t \in \Omega;\ n_0(t) = +\infty,\ L_j \geqslant m_j^{(n)}(t) \geqslant M_{n-j+1};\ 1 \leqslant j \leqslant n < +\infty\},$$

$$\omega\cap\{t \in \Omega;\ n_0(t) < +\infty\}$$
$$= \begin{cases} \emptyset & \text{if } L_1 - M_1 < +\infty, \\ \{t \in \Omega;\ n_0(t) < +\infty,\ L_j \geqslant m_j^{(n)}(t) \geqslant M_{n-j+1};\ 1 \leqslant j \leqslant n < n_0(t)\}. & \text{if } L_1 - M_1 = +\infty. \end{cases}$$

6.3. The next lemma answers a natural converse question.

LEMMA. *For any given $L_j \in \mathbf{Z}\cup\{+\infty\}$, $M_j \in \mathbf{Z}\cup\{-\infty\}$ $(j \in \mathbf{N})$, such that*

$$+\infty \geqslant L_1 \geqslant L_2 \geqslant \ldots \geqslant L_j \geqslant \ldots \geqslant M_j \geqslant \ldots \geqslant M_2 \geqslant M_1 \geqslant -\infty$$

there exists a point $t_0 \in \Omega$ with $n_0(t_0) = +\infty$ such that

$$L_j = L_j(t_0), \quad M_j = M_j(t_0); \quad j \in \mathbf{N}.$$

6.4. Using the above remarks and lemmas, one obtains the following result:

THEOREM (Ş. Strătilă, D. Voiculescu, [17], III.1). *The primitive spectrum of the C^*-algebra $A(U(\infty))$ can be identified with the set of all the symbols*

$$\xi = \{L_j(\xi), M_j(\xi)\}_{1\leqslant j<n_0(\xi)},$$

where either $n_0(\xi) = +\infty$ and, for all $1 \leqslant j < +\infty$, we have

$$\mathbf{Z}\cup\{+\infty\} \ni L_j(\xi) \geqslant L_{j+1}(\xi) \geqslant M_{j+1}(\xi) \geqslant M_j(\xi) \in \mathbf{Z}\cup\{-\infty\},$$

or $n_0(\xi) \in \mathbf{N}$ and, for all $1 \leqslant j < n_0(\xi)$, we have

$$\mathbf{Z} \ni M_{n_0(\xi)-j}(\xi) = L_j(\xi) \geqslant L_{j+1}(\xi) = M_{n_0(\xi)-j-1}(\xi).$$

Namely, if ϱ is a factor representation of $U(\infty)$ (or of some $U(k)$), then the kernel of ϱ corresponds to the symbol

$$L_j = \sup\{\sup\{m_j^{(n)};\ n \geqslant j\}\},$$
$$M_j = \inf\{\inf\{m_{n-j+1}^{(n)};\ n \geqslant j\}\},$$

where the first sup and the first inf are taken over all signatures $(m_1^{(n)}, \ldots, m_n^{(n)}) \in \widehat{U(n)}$ which appear in $\varrho|U(n)$.

The points $\xi \in \mathrm{Prim}(A(U(\infty)))$ with $n_0(\xi) = +\infty$ correspond to factor representations of $U(\infty)$, while the points ξ with $n_0(\xi) = n_0 \in \mathbf{N}$ correspond to factor representations of $U(n_0-1)$.

The topology of the space $\mathrm{Prim}(A(U(\infty)))$ can also be described (see [17], III.1.5). For instance, the one point set $\{\xi_\infty\} \subset \mathrm{Prim}(A(U(\infty)))$, where $L_j(\xi_\infty) = +\infty$, $M_j(\xi_\infty) = -\infty$ for all j, is everywhere dense.

For $\xi \in \mathrm{Prim}(A(U(\infty)))$, $\{L_j(\xi)\}_j$ will be called the *upper signature* of ξ and $\{M_j(\xi)\}_j$ will be called the *lower signature* of ξ.

6.5. In 1.6 we have associated with every point $t = (\varrho_1 \prec \ldots \prec \varrho_n \prec \ldots)$ of Ω an irreducible representation ϱ_t of $U(\infty)$ which is the direct limit of the ϱ_n's.

On the other hand, let μ be a completely atomic Γ-quasi-invariant probability measure concentrated on the Γ-orbit $\Gamma(t)$. Then, for all $\gamma \in \Gamma$ we have

$$\mu(\{\gamma(t)\}) > 0$$

and, for all Borel sets $B \subset \Omega$, we have

$$\mu(B) = \sum_{s \in \overline{\Gamma(t)} \cap B} \mu(\{s\}).$$

Clearly, μ is Γ-ergodic and therefore the representation ϱ_μ is irreducible (see 5.7). Moreover, $\operatorname{Ker}\varrho_\mu$ corresponds to $\overline{\Gamma(t)}$ and the representations ϱ_μ and ϱ_t are equivalent (Ş. Strătilă, D. Voiculescu, [17], III.2).

Thus, *every primitive ideal of $A(U(\infty))$ which corresponds to a factor representation of $U(\infty)$ is the kernel of an irreducible representation of $U(\infty)$, which is a direct limit of irreducible representations of the $U(n)$'s.*

6.6. The following result shows that the representations of $U(\infty)$ corresponding to bounded signatures can be extended to *norm*-continuous representations of $U_1(H)$ (while, usually, only the strong continuity is required).

Proposition (Ş. Strătilă, D. Voiculescu, [18], 2.8). *Let σ be a continuous representation of $U(\infty)$ such that, for any $n \in N$, $\sigma|U(H_n)$ contains only irreducible representations of signatures $(m_1 \geqslant \ldots \geqslant m_n)$ with $|m_j| \leqslant M < +\infty$. Then*

$$\|\sigma(V') - \sigma(V'')\| \leqslant M\|V' - V''\|_1; \quad V', V'' \in U(\infty).$$

6.7. Consider a primitive ideal J of $A(U(\infty))$ corresponding to a *bounded* upper and lower signature, and let $A/J = A(U(\infty))/J$. Using the above proposition one obtains the following

Corollary (Ş. Strătilă, D. Voiculescu, [18], 2.9). *There is a canonical norm-continuous representation*

$$\varrho_J\colon U_1(H) \to A/J$$

*and, for every unitary $W \in U(H)$ there exists a unique *-automorphism α_W of A/J such that*

$$\alpha_W(\varrho_J(V)) = \varrho_J(WVW^*); \quad V \in U_1(H).$$

The mapping

$$U(H) \ni W \mapsto \alpha_W \in \operatorname{Aut}(A/J)$$

is a representation, continuous with respect to the strong-operator topology on $U(H)$ and the point-norm topology on $\operatorname{Aut}(A/J)$.

Since for arbitrary signatures a similar result does not hold, we had to replace in § 4 the C^*-algebra KMS condition by a group-theoretic KMS condition.

6.8. For the finite-dimensional group $U(n)$ it is known that the signatures classify the symmetry types of tensors over H_n. For instance, the signatures of the form $(1, 1, \ldots, 1, 0, \ldots, 0)$ correspond to antisymmetric tensors while the signatures of the form $(m, 0, \ldots, 0)$ correspond to symmetric tensors.

For the infinite-dimensional case, we can say that the upper and lower signatures correspond to symmetry types of tensors over H, or that the primitive ideal spectrum of $A(U(\infty))$ is described in terms of symmetry types of tensors over H. In fact, we shall see later that the representations of $U(\infty)$ corresponding to upper signature $L_j = 1$ $(j \in N)$ and lower signature $M_j = 0$ $(j \in N)$ are realized in spaces of antisymmetric tensors over H.

7. Representations in antisymmetric tensors

In this section we consider a particular class of KMS-functions of positive type on $U_1(H)$, among those presented in § 4, for which we give a complete classification according to type and quasi-equivalence, and also we show how the corresponding representations can be realized in spaces of antisymmetric tensors over H. All these representations, when restricted to $U(\infty)$, correspond to the upper signature $L_j = 1$ and to the lower signature $M_j = 0$ $(j \in N)$.

Also, these representations are related to the restrictions of the gauge-invariant generalized free states ([13]) of the CAR-algebra to the subalgebra of gauge-invariant elements. In fact, the main result presented in this section can be viewed as the classification according to type and quasi-equivalence of these restrictions.

7.1. Let $A \in L(H)$, $0 \leqslant A \leqslant I$, and define

$$\psi_A(V) = \det((I-A)+AV); \quad V \in U_1(H).$$

Then ψ_A is a continuous function of positive type on $U_1(H)$. Actually, ψ_A corresponds, as in § 4, to the character given by $p(z) = (1+z)/2$ and to $B = A(I-A)^{-1}$. Let ϱ_A be the associated cyclic representation of $U(\infty)$. We shall also consider

$$W_A = A^{1/2} + i(I-A)^{1/2} \in U(H).$$

For $T \in L(H)$ let $\sigma(T)$ and $\sigma_{\mathrm{ess}}(T)$ be the spectrum and the essential spectrum of T, respectively.

For two projections $P, Q \in L(H)$ such that $P-Q$ is a compact operator, we denote by $\operatorname{cd}(P, Q)$ the relative codimension of Q in P, i.e., the index of the Fredholm operator $QP\colon PH \to QH$, or, equivalently,

$$\operatorname{cd}(P, Q) = \dim(P - s(PQP)) - \dim(Q - s(QPQ)),$$

where $s(T)$ denotes the support projection of T.

The main result is the following theorem.

7.2. Theorem (Ş. Strătilă, D. Voiculescu, [18], 3.1). *Let $A, B \in L(H)$, $0 \leqslant A \leqslant I$, $0 \leqslant B \leqslant I$. Then:*

1. *ψ_A is of type* I *$\Leftrightarrow A(I-A) \in C_1(H)$. In this case ϱ_A is a direct sum of irreducible representations.*

2. *ψ_A is factorial and of type* I *$\Leftrightarrow A$ is a projection. In this case ϱ_A is irreducible.*

3. *ψ_A is factorial but not of type* I *$\Leftrightarrow A(I-A) \notin C_1(H)$. In this case:*

(a) *ψ_A is of type* II_1 *$\Leftrightarrow A - pI \in C_2(H)$ for some $p \in (0, 1)$;*

(b) *ψ_A is of type* II_∞ *$\Leftrightarrow A(I-A)(A-pI)^2 \in C_1(H)$ for some $p \in (0, 1)$ and $\{0, 1\} \cap \sigma_{\mathrm{ess}}(A) \neq \emptyset$;*

(c) ψ_A *is of type* III $\Leftrightarrow A(I-A)(A-pI)^2 \notin C_1(H)$ *for all* $p \in (0,1)$.

4. *If A, B are projections, then* $\psi_A \sim \psi_B \Leftrightarrow A-B \in C_2(H)$ *and* $\mathrm{cd}(A,B) = 0 \Leftrightarrow$ $\Leftrightarrow$ *there exists* $V \in U_2(H)$ *such that* $VAV^* = B$.

5. *If* $A(I-A) \notin C_1(H)$, $B(I-B) \notin C_1(H)$, *then* $\psi_A \sim \psi_B \Leftrightarrow W_A - W_B \in C_2(H)$.

In the above statement the sign "$\sim$" stands for the quasi-equivalence of the associated cyclic representations.

In what follows we sketch the main ideas of the proof.

7.3. Let $\Omega = \Omega(U(\infty))$, $\Gamma = \Gamma(U(\infty))$ and $\omega \subset \Omega$ the Γ-invariant subset of Ω corresponding to the primitive ideal J of $A(U(\infty))$ with upper signature $L_j = 1$ and lower signature $M_j = 0$ ($j \in N$). Let G denote the restriction of Γ to ω. Then the C^*-algebra $A(U(\infty))/J$ is $*$-isomorphic to the AF-algebra $A(\omega, G)$ constructed as in 5.2 from the couple (ω, G). By 6.7, we may consider $U_1(H) \subset A(\omega, G)$ and then the function of positive type ψ_A extends to a state φ_A of $A(\omega, G)$, the type problem and the quasi-equivalence problem being thus transferred to φ_A.

7.4. After identifications, we get $\omega = \{0,1\}^{\aleph_0}$ and G consists of transformations $\gamma_{n,\sigma}$,

$$\gamma_{n,\sigma}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots) = (\sigma(\alpha_1, \ldots, \alpha_n), \alpha_{n+1}, \ldots),$$

where $n \in N$, and σ is a bijection of the set $\{0,1\}^n$ which preserves the sum of the components:

$$\alpha, \beta \in \{0,1\}^n, \quad \sigma(\alpha) = \beta \Rightarrow \alpha_1 + \ldots + \alpha_n = \beta_1 + \ldots + \beta_n.$$

7.5. If A is diagonable, with eigenvalues $\{\lambda_n\}_{n\in N}$, and $A(I-A)$ is injective, then we can define the measure $\mu = \mu_A$ on ω as the product of the measures μ_n on $\{0,1\}$ defined by $\mu_n(\{0\}) = p_n = 1-\lambda_n$, $\mu_n(\{1\}) = q_n = \lambda_n$. Note that in this case μ is a Γ-quasi-invariant probability measure.

Moreover, in this case the state φ_A on $A(\omega, G)$ is of the form

$$\varphi_A = \mu_A \circ P$$

where $P\colon A(\omega, G) \to C(\omega)$ is the conditional expectation. Therefore (see 5.6), the factor and type problems for φ_A are reduced to the G-ergodicity of μ and to the μ-measurability of G, respectively. The corresponding results are:

THEOREM (Ş. Strătilă, D. Voiculescu, [17], IV.4). *μ is G-ergodic if and only if* $\sum_{n=1}^{\infty} p_n(1-p_n) = +\infty$.

THEOREM (Ş. Strătilă, D. Voiculescu, [18], 1.2). *G is μ-measurable if and only if* $\sum_{n=1}^{\infty} p_n(1-p_n)(p_n-p)^2 < +\infty$ *for some* $p \in (0,1)$.

7.6. On the other hand, let G' be the group of transformations on ω consisting of *all* finite permutations of the coordinates of a point $\alpha \in \{0,1\}^{\aleph_0}$, i.e., the direct sum of the permutation groups on each $\{0,1\}$. The AF-algebra $A(\omega, G')$ is $*$-isomorphic to the so-called CAR-algebra, associated to the canonical anticommutation relations.

Since $G \subset G'$, we have $A(\omega, G) \subset A(\omega, G')$, namely $A(\omega, G)$ identifies with the subalgebra of "gauge-invariant" elements of the CAR-algebra. Moreover, the states φ_A are exactly the restrictions of the "gauge-invariant generalized free states" of the CAR-algebra (see [13]).

By use of the methods and the results of R. T. Powers and E. Størmer ([13]) for the classification of gauge-invariant generalized free states of the CAR-algebra, the problem for φ_A is also reducible to the case of A diagonale (see [18] § 3).

Let us mention that every φ_A is quasi-equivalent to a similar state with A diagonable.

7.7. Note that Theorem 7.2 solves also the classification problem, according to type and quasi-equivalence, for the restrictions of gauge-invariant generalized free states of the CAR-algebra, to the gauge-invariant subalgebra. This is different from the classification of non-restricted states given in [13].

7.8. For A diagonable, say $Ae_n = \lambda_n e_n$, $0 < \lambda_n < 1$, $\sum_n \lambda_n(1-\lambda_n) < +\infty$, the representation ϱ_A can be realized as follows on a space of antisymmetric tensors.

Consider $X_n = \bigoplus_{k=0}^{n} (\Lambda^k H_n \otimes \Lambda^k H_n)$; then, with the convention $(a\otimes b)\wedge(c\otimes d) = (a\wedge c)\otimes(b\wedge d)$,

$$J_n\colon X_{n-1} \ni \xi \mapsto \xi \wedge \big((1-\lambda_n)^{1/2} 1 \otimes 1 + \lambda_n^{1/2} e_n \otimes e_n\big) \in X_n,$$

and we have the representation

$$\varrho_n = (\text{natural representation}) \otimes 1$$

of $U(n)$ on X_n.

Then ϱ_A is unitarily equivalent to the direct limit of the ϱ_n's along the J_n's (see [17], IV).

7.9. For an arbitrary $A \in L(H)$, $0 \leqslant A \leqslant I$, with $\mathrm{Ker}\, A = \mathrm{Ker}(I-A) = 0$, let $B = A(I-A)^{-1}$. By 6.7, the automorphism group $\{B^{it} \cdot B^{-it}\}_t$ of $U_1(H)$ extends to a $*$-automorphism group $\{\alpha_t\}_t$ of $A(\omega, G)$. For the states φ_A one can prove something more than was asserted in 4.3, namely φ_A *satisfies the* C^*-KMS-*conditions with respect to* $\{\alpha_t\}_{t\in R}$ (see [18], 4.7).

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LINEAR PREDICTOR FOR STATIONARY PROCESSES IN COMPLETE CORRELATED ACTIONS

ION SUCIU and ILIE VALUŞESCU

Department of Mathematics, The National Institute for Scientific and Technical Creation, Bucharest, România

1. Introduction

In this paper we shall continue the study of prediction theory of a stationary process considered as time evolution in a correlated action, which was began in [4]. As in the precedent paper, we shall follow the line of Wiener and Masani prediction schema for (finite) multivariate stationary process [7], [8].

The notion of completion of a correlated action, which we shall introduce in Section 2, will allow us to give a precise meaning to the predictible part of the process and, consequently, to formulate more precisely the prediction problems (Section 3). Since some results from [4] are used here in a slightly different context, we prefer to outline their proofs. In Section 4, under the supplementary condition of boundedness imposed on the spectral distribution of the process, similar to Wiener–Masani boundedness condition [8], we shall determine the predictible part of the process by means of a linear (infinite) Wiener filter. The solution of prediction problems are given in terms of Taylor coefficients of the maximal outer function which factorizes the spectral distribution of the process (see [3]).

The reader will certainly note that we permanently use the ideas from the Sz.-Nagy and C. Foiaş model for contraction [6] to give an operator or functional model for prediction. Our model is based on an operator valued positive definite map (on the integers), which corresponds to an infinite variate (discrete) stationary process.

2. Complete correlated actions

The notion of *correlated action* was introduced in [4] as the triplet $\{\mathscr{E}, \mathscr{H}, \Gamma\}$, where $\mathscr{E}$ is a Hilbert space (the space of the parameters), $\mathscr{H}$ is a right $\mathscr{L}(\mathscr{E})$-module (the state space), and $\Gamma\colon \mathscr{H}\times\mathscr{H} \to \mathscr{L}(\mathscr{E})$ is an $\mathscr{L}(\mathscr{E})$-valued map (the correlation) with the properties:

(i) $\Gamma[h, h] \geqslant 0$, $\Gamma[h, h] = 0 \Rightarrow h = 0$.