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BOUNDEDNESS IN DILATION THEORY

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Let S be an involution semigroup (or a *-semigroup). We do not assume that S has a unit. Let H be a complex Hilbert space. Denote by $B(H)$ the algebra of all bounded linear operators on H. A function $\varphi\colon S \to B(H)$ is said to be *positive definite* (in short PD) if for all $s_1, \ldots, s_n \in S$ and $f_1, \ldots, f_n \in H$,

$$\sum_{i,j} (\varphi(s_i^* s_j) f_j, f_i) \geqslant 0.$$

An involution preserving semigroup homomorphism $\Phi\colon S \to B(K)$, K another Hilbert space, is called a *dilation* of φ if

$$\varphi(s) = V^* \Phi(s) V, \quad s \in S,$$

where V is a bounded linear operator from H to K.

Recall the Sz.-Nagy general dilation theorem [10]: a PD φ has a dilation if and only if it satisfies the following *boundedness condition*:

$$\text{(BC)} \qquad \sum_{i,j} (\varphi(s_i^* u^* u s_j) f_j, f_i) \leqslant c(u) \sum_{i,j} (\varphi(s_i^* s_j) f_j, f_i)$$

for $c(u)$ independent of $s_1, \ldots, s_n$ in S and $f_1, \ldots, f_n$ in H; S has a unit.

We wish to say a few words about the role played by (BC) in the above. Suppose for a moment that we do not know whether $\varphi(s)$'s are bounded operators or not. Since positive definiteness and (BC) still keep a sense, we easily find out that if a PD φ satisfies (BC), then all $\Phi(s)$'s are bounded operators and also $\varphi(s)$'s must necessarily be bounded provided so is $\varphi(1)$. On the other hand, if φ has a dilation Φ and $\Phi(s)$'s are bounded operators, then φ must satisfy (BC) and again $\varphi(s)$'s are bounded operators. This explains how (BC) is responsible for the boundedness of $\varphi(s)$'s as well as of $\Phi(s)$'s and justifies the term we have used for it.

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Now we list some known instances of dilatable functions looking upon *how* (BC) pertains to them.

(a) Let T be a contraction in H. Define $S =$ the group of integers and $\varphi(n) = T^n$ if $n \geqslant 0$ and $(T^*)^{-n}$ for $n < 0$. Then φ is PD [10] and, because (BC) is redundant ($n^* = -n$), it is dilatable. More generally, every PD definite function on a group has a dilation.

(b) Suppose E: $\mathscr{B} \to B(H)$, $\mathscr{B}$ being a σ-field of sets, is a semispectral measure, that is, for every f $(E(\cdot)f, f)$ is a positive scalar measure of total mass $\|f\|^2$. $\mathscr{B}$ can be regarded as a *-semigroup with intersection as a semigroup multiplication and the identity mapping as involution. Then [10] E is a PD function on B. This means that $\sum_{i,j} (E(\sigma_i \cap \sigma \cap \sigma_j)f_j, f_i)$ is a positive measure in σ, and this implies immediately (BC). Thus we come to the Naĭmark dilation theorem.

(c) Suppose A is a C^*-algebra with a unit, which with the algebra multiplication forms a *-semigroup. The Stinespring theorem [4] asserts that a completely positive linear map φ: $A \to B(H)$ can be dilated to a *-representation of A. We can look at this theorem as a dilation theorem for the PD function φ on A. Here the boundedness condition (BC) is due to some intimate properties of C^*-algebras. More generally, let A be a Banach *-algebra with a unit. Also in this case (BC) is ensured (cf. [5]) by an appropriate inequality for positive linear functionals on A ([2], Lemma 6(iv), p. 197).

These instances can suggest that positive definiteness must force (BC). We shall show by an example (postponed to the end of this paper) that this is *not* the case.

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In later discussion we shall need handier forms of the boundedness condition than (BC). We itemize carefully all the possibilities which help us to see what is going on.

Proposition 1. *Suppose φ is a PD function on S (no unit in S is required). Then the following conditions are equivalent:*

(1) *φ satisfies* (BC),

(2) $(\varphi(s^*u^*us)f, f) \leqslant c(n)\,(\varphi(s^*s)f, f)$ *for $f \in H$ and $s \in S$,*

(3) $\|\varphi(s^*u^*us)\| \leqslant C(s)\,\alpha(u)$, *where φ is a submultiplicative function of u,*

(4) $\liminf_{k\to\infty} \Big(\sum_{i,j} \big(\varphi(s_i^*(u^*u)^{2^k}s_j)f_j, f_i\big)\Big)^{2^{-k}}$ *is finite and does not depend on f_i's and s_i's.*

We put the proof in the following order: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Only two of these implications need some short comments.

The best choice of $c(u)$ in (2), when combined with a double use of (2), implies that $\alpha = c$ in (3) is submultiplicative. Next, (1) follows from (4) via the following inequality:

$$\sum_{i,j} (\varphi(s_i^*u^*us_j)f_j, f_i) \leqslant \Big(\sum_{i,j} (\varphi(s_i^*s_j)f_j, f_i)\Big)^{1-2^{-k}} \Big(\sum_{i,j} \big(\varphi(s_i^*(u^*u)^{2^k}s_j)f_j, f_i\big)\Big)^{2^{-k}}$$

(for more detail we refer to [6]).

The next step is to assume a little bit more about φ.

Proposition 2. *Suppose φ: $S \to B(H)$ satisfies*

$$(*) \qquad \sum_{i,j} (\varphi(s_i^*s_j)f_j, f_i) \geqslant c\Big\|\sum_i \varphi(s_i^*)f_i\Big\|^2$$

with $c > 0$ independent of $s_1, \ldots, s_n$ in S and $f_1, \ldots, f_n$ in H. Then the following conditions are equivalent:

(1), (2), (3),

(3′) $\|\varphi(us)\| \leqslant C'(s)\,\alpha'(u)$, $u, s \in S$, *α' is submultiplicative,*

and (4).

For the proof of implication (3) $\Rightarrow$ (3′) use (*) and set $C'(s) = c^{-1}C(s)^{1/2}$.

It is easy to see (cf. [5]) that if S has a unit, every PD φ satisfies (*). In this case (3′) yields

(3″) $\|\varphi(u)\| \leqslant C'(1)\,\alpha'(u)$, $u \in S$, *α' submultiplicative.*

Since (3″) implies (4), we have

Proposition 3. *Suppose S has a unit and φ: $S \to B(H)$ is PD. The following conditions are equivalent:* (1), (2), (3), (3′), (3″) *and* (4).

Remark. We can by-pass (4) and prove the implication (3″) $\Rightarrow$ (1) in the following way: Define $l^1(S, \alpha)$ as the set of all complex functions ξ on S such that $\sum_{s\in S} |\xi(s)|\,\alpha(s) < +\infty$. This forms a Banach *-algebra (cf. [2], ex. 23, p. 8); involution on $l^1(S, \alpha)$ comes naturally from that on S. Define $\hat{\varphi}$: $l^1(S, \alpha) \to B(H)$ by $\hat{\varphi}(\xi) = \sum_s \xi(s)\varphi(s)$. Thus $\hat{\varphi}$ is linear and PD and consequently ([2], Lemma 6(iv), p. 197; cf. also case (c)) it must satisfy (BC). Then φ satisfies (BC), too (take $\xi = \delta_{s_i}$ to check this). Unfortunately, this way is not elementary (Ford's square root lemma is involved) but ours (via (4)) is.

Some more comment. The condition (4) and its equivalence to (BC) have been proposed in [6]. Moreover, examples of applications (cf. cases (d) and (f) below) have been considered in [6] and [7]. Condition (3″) and its applications (see case (e) below) have been isolated in [8]. Then Masani ([3], cf. also [9]) has produced (2) using the very same method as that in [6].

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Now we are able to continue our discussion.

(d) Suppose φ is a PD function on a *-semigroup, which is norm bounded, that is, $\sup\|\varphi(s)\| < +\infty$.

Then it satisfies (BC) owing to the implication (3) $\Rightarrow$ (1). Thus it is dilatable (Arveson, see [6]).

(e) Here our concern is rather with Proposition 3 than in dilation (read: normal extension) itself. More precisely, let $A \in B(H)$ be such that $\sum_{i,j} (A^i f_j, A^j f_i) \geqslant 0$

for $f_1, \ldots, f_n$ in H. Bram has proved, using the Heinz–Kato inequality, that such an A must satisfy

$$\sum_{i,j} (A^{i+1}f_j, A^{j+1}f_i) \leqslant \|A\|^2 \sum_{i,j} (A^i f_j, A^j f_i)$$

(in other words: A is *subnormal*). We can prove this elementarily using just the implication $(3'') \Rightarrow (1)$ (for more detail see [8]).

(f) Now the problem is in *extension* of a PD function φ on S to a dilatable one on the unitization of S (recall: the dilation theorem requires S to have a unit). It S has no unit, adjoin 1, set $1^* = 1$ and define the resulting *-semigroup by S_1. If S_1 already has a unit, set $S_1 = S$. Call S_1 the unitization of S. We have shown [5] that $\varphi\colon S \to B(H)$ can be extended to a PD function, say φ_1, on S_1, if and only if it satisfies $(*)$ and $\varphi(s^*) = \varphi(s)^*, s \in S$. $\varphi_1(1)$ can be defined as $\varphi(1) = A$ with $A > c$ the identity operator in H (here c is as in $(*)$). We also have shown in [7], using (4), that φ_1 satisfies (BC) if so does φ. Here we do the same using (2). The only thing we have to check is that

$$(\varphi_1(u^*u)f, f) \leqslant b_1(u)(\varphi_1(1)f, f), \quad u \neq 1.$$

Indeed,

$$\begin{aligned}(\varphi_1(u^*u)f, f) &= (\varphi(u^*u)f, f) \leqslant \|\varphi(u^*u)\|(f, f)\\ &\leqslant c^{-1}\|\varphi(u^*u)\|(Af, f) = b_1(u)(\varphi_1(1)f, f)\end{aligned}$$

where $b_1(u) = \max\{b(u), c^{-1}\|\varphi(u^*u)\|\}$.

(g) Here we present an opportunity to apply (4). Suppose S is an *inverse-semigroup*, that is, for every $s \in S$ there is precisely one s^* such that $ss^*s = s$ and $s^*ss^* = s^*$ (we refer to [1] for this matter). Then $(s^*s)^n = s^*s$ for every non-negative integer n and this shows that a PD function on an inverse-semigroup, like in a group case, always satisfies (4) and consequently (BC) with $c(u) \equiv 1$.

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EXAMPLE. The example(1) we are going to present shows that there is a PD function, which does not satisfy (BC). Let S be the algebra of all real or complex polynomials in a real variable. Take the usual involution (conjugation in complex case). Then S becomes a *-semigroup (even more, a *-algebra). Let w be in S and define

$$\varphi(w) = \int_0^{+\infty} w(t)\exp(-t)\,dt.$$

Take $w_0(t) = t$. Then we have

$$\varphi(w_0^n) = n!$$

(1) This example is due to Professor C. Ryll-Nardzewski. The author would like to thank Professor Ryll-Nardzewski for letting him present the example here.

for $n = 0, 1, \ldots$ Suppose φ satisfies (BC). Then it must satisfy $(3'')$ with some submultiplicative α. Thus we have

$$n! = |\varphi(w_0^n)| \leqslant C\alpha(w_0^n) \leqslant C\alpha(w_0)^n,$$

and consequently $\alpha(w_0) \geqslant C^{1/n}(n!)^{1/n}$ for $n = 0, 1, \ldots$, which is impossible.

Notice that φ is in fact a linear positive functional of a *-algebra.

Added in the proof, December 1981. The theme of the present paper has been resettled in a recent one by J. Stochel and the author "Boundedness of linear and related nonlinear maps" to appear in Expositiones Math.

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