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SUMMARY OF THREE LECTURES

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Functions acting on weighted l^p-algebras

Let $\mathscr{A}$ be a commutative semi-simple complex completely regular Banach algebra with identity and hermitean involution. Assume that a function $F\colon (-1,1)\to \boldsymbol{R}$ with $F(0)=0$ and

$$(*)\qquad \lim_{t\to 0}\left|\frac{F(t)}{t}\right| = \infty$$

operates on $\mathscr{A}$. Then $\mathscr{A}$ is the algebra of all continuous functions on $\hat{\mathscr{A}}$, the spectrum of $\mathscr{A}$.

We show that the above condition $(*)$ may not be weakened to $\overline{\lim_{t\to 0}}\left|\dfrac{F(t)}{t}\right| = \infty$. The examples demonstrating this are *weighted l^p-algebras* for $1\leqslant p<\infty$.

Let S be a set and let $e\colon S\to [1,\infty)$ be a function. Then

$$\mathscr{A} = \Big\{x\colon S\to \boldsymbol{C}\Big|\Big(\sum_{s\in S}|x(s)|^p e(s)\Big)^{1/p} = |x| < \infty\Big\}$$

is an involutive Banach algebra with norm $|\ |$ and pointwise operations. One easily sees that $\hat{\mathscr{A}} = S$. Let $\mathscr{L}_0$ be the class of all functions F with $F(0)=0$, which are Lipschitz continuous at 0. Then $\mathscr{L}_0$ operates on $\mathscr{A}$.

Without changing $\mathscr{A}$ as a set, we may replace e by a weight function with values in $\boldsymbol{N}$ only.

THEOREM 1. *Only $\mathscr{L}_0$ operates on $\mathscr{A}$, if $e^{-1}(\{n\})$ is infinite for some $n\in \boldsymbol{N}$.*

Let $e_1, e_2, \ldots$ be the values of e in increasing order and assume that the value e_k is attained m_k times. Then we have:

THEOREM 2. *Only $\mathscr{L}_0$ operates on $\mathscr{A}$, if $e_{k+1}/e_k\leqslant K$ for some $K>0$.*

THEOREM 3. *Assume $e_{k+1}/e_k\geqslant K>1$, $m_k e_k e_{k+1}^{-1}\leqslant K'$ and $\underline{\lim}\, m_k e_k e_{k+1}^{-1} = 0$ for some K, K'. Then there exists a continuous but not Lipschitz continuous function operating on $\mathscr{A}$.*

THEOREM 4. *Assume* $e_k e_{k+1}^{-1} \to 0$ *and* $m_k e_k e_{k+1}^{-1} \geqslant \varepsilon > 0$ *for all* k. *Then only* $\mathcal{L}_0$ *operates on* $\mathcal{A}$.

Hence, assuming the weight e satisfies the conditions of Theorem 3, then $\tilde{\mathcal{A}}$, the algebra $\mathcal{A}$ with unit adjoined, provides an example for the statement above.

Local completeness in operator algebras

The results presented in this lecture were obtained jointly with J. Cuntz.

A local property of an involutive algebra $\mathcal{A}$ is a property determined by or on its commutative star subalgebras. In recent years a number of papers have been devoted to the study of global properties of Banach star algebras which are implied by corresponding local properties. As examples we mention the connection between hermiticity and symmetry of involutive Banach algebras by Shirali and Ford, diverse characterizations of $\mathcal{C}^*$-algebras or the characterization of W^*-algebras by Pedersen. An example of a global property which is implied by the corresponding local property is also the continuity of a linear functional on $\mathcal{C}^*$-algebras [Cuntz].

The present paper is another attempt to weaken the axioms of $\mathcal{C}^*$-algebras. At the same time we obtain some insight into the structure of (certain) $\mathcal{C}^*$-algebras.

DEFINITION 1. A complex normed algebra $\mathcal{A}$ with involution is a *local* $\mathcal{C}^*$-*algebra*, if all of its closed commutative star subalgebras are $\mathcal{C}^*$-algebras with the given norm and involution.

Let $|\cdot|$ be the given norm on $\mathcal{A}$. Then $\|\cdot\|$ defined by $\|x\| = \max\{|x|, |x^*|\}$ is an algebra-norm on $\mathcal{A}$ which makes the involution isometric. The involution can, in a natural way, be extended to an involution on the $\|\cdot\|$-completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$. It is easy to see that $\tilde{\mathcal{A}}$ is a $\mathcal{C}^*$-algebra with a norm equivalent to $\|\ \|$.

Since $\mathcal{A}$ is dense in $\tilde{\mathcal{A}}$ we define:

DEFINITION. A $\mathcal{C}^*$-algebra $\tilde{\mathcal{A}}$ is called *interwoven* if the only local $\mathcal{C}^*$-algebra $\mathcal{A}$ dense in $\tilde{\mathcal{A}}$ is $\tilde{\mathcal{A}}$ itself.

THEOREM 1. *The algebra* $\mathcal{K}(\mathcal{H})$ *of all compact operators on some Hilbert space* $\mathcal{H}$ *is interwoven.*

With the aid of Theorem 1 one can then show

THEOREM 2. *Let* $\mathcal{C}$ *be a* $\mathcal{C}^*$-*algebra with unit* 1 *that contains a projection* E *such that* $E \sim (1-E) \sim 1$ *in* $\mathcal{C}$. *Then* $\mathcal{C}$ *is interwoven.*

COROLLARY. (a) *Every* $\mathcal{C}^*$-*algebra of the form* $\mathcal{C} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{C}'$, *where* $\mathcal{C}'$ *is a* $\mathcal{C}^*$-*algebra with unit and* $\mathcal{H}$ *an infinite-dimensional Hilbert space, is interwoven.*

(b) *Every* AW^*-*algebra* $\mathcal{C}$ *without a type* II_1 *direct summand is interwoven.*

The proof of Theorem 2 is rather involved and in order to prove the general conjecture

CONJECTURE. *All* $\mathcal{C}^*$-*algebras are interwoven*

one would need some new methods.

Stability of deficiency indices

The results presented in this lecture were obtained jointly with H. Focke.

The aim of this lecture is to extend several criteria about the self-adjointness of symmetric operators to results about the stability of the deficiency index under perturbation. In fact most results are even stated for the larger class of closed densely defined dissipative operators on a Banach space.

THEOREM 1. *Let* X *be a Banach space and let* A *be a closed dissipative densely defined operator with domain* $\mathcal{D}_A \subset X$. *Let* B *be dissipative with* $\mathcal{D}_A \subset \mathcal{D}_B \subset X$ *and*

$$|Bx| \leqslant a|Ax| + b|x| \quad \forall x \in \mathcal{D}_A \text{ with } a < 1. \tag{1}$$

Then $A+B$ *is a closed dissipative operator and for the range* $\mathcal{R}_{A-\lambda}$ *of* $A-\lambda$ *we have*

$$\operatorname{codim} \mathcal{R}_{A-\lambda} = \operatorname{codim} \mathcal{R}_{A+B-\lambda}.$$

COROLLARY 1. *Let* A *be a closed symmetric operator on the Hilbert space* X *and let* B *be symmetric with* $\mathcal{D}_A \subset \mathcal{D}_B$ *and* (1). *Then*

$$\operatorname{def.} A = (\operatorname{codim} \mathcal{R}_{A+i}, \operatorname{codim} \mathcal{R}_{A-i}) = \operatorname{def.} A+B.$$

COROLLARY 2. *Let* X *be a* B *space and let* A *be a closed dissipative densely defined operator with* $\mathcal{D}_A \subset \mathcal{D}_B$ *and*

$$A+tB \text{ is closed for all } t \in [0, 1]. \tag{2}$$

Then $\operatorname{codim} \mathcal{R}_{A-\lambda} = \operatorname{codim} \mathcal{R}_{A+B-\lambda}$.

Of course a result like Corollary 2 also holds for symmetric operators. Corollaries 1 and 2 (for symmetric operators) extend well known self-adjointness results of Rellich, Kato and Wüst.

THEOREM 2. *Let* A *be a closed dissipative densely defined operator on the* B *space* X. *Let* B *be dissipative with* $\mathcal{D}_A \subset \mathcal{D}_B$ *and*

$$|Bx| \leqslant |Ax| + b|x| \quad \forall x \in \mathcal{D}_A. \tag{3}$$

Assume B' *has dense domain in* X'. *Then for all* $\lambda > 0$ *one has*

$$\operatorname{codim} \mathcal{R}_{\overline{A+B}-\lambda} \leqslant \operatorname{codim} \mathcal{R}_{A-\lambda}.$$

COROLLARY. *Let* A *be a closed symmetric operator on the Hilbert space* X *and let* B *be symmetric with* $\mathcal{D}_A \subset \mathcal{D}_B$ *and* (3). *Then* $\operatorname{def.} \overline{A+B} \leqslant \operatorname{def.} A$.

THEOREM 3. *Let* A_n *be closed dissipative operators on the Banach space* X. *Let* $\mathcal{D}$ *be a common domain of the* A_n *and let* A *be a closed operator with core* $\mathcal{D}$ *such that*

(i) $A_n' x' \to A' x'$ *for all* $x' \in \mathcal{D}'$ *dense in* X',

(ii) $\|(A_n - A)x\| \leqslant a\|A_n x\| + b\|x\|$ *for all* $x \in \mathcal{D}$.

Moreover, assume $\operatorname{codim} \mathcal{R}_{A_n-\lambda} = m \leqslant \infty$ *for* $\lambda > 0$. *Then* A *is dissipative with* $\operatorname{codim} \mathcal{R}_{A-\lambda} \leqslant m$.

COROLLARY. *Let A_n be closed symmetric operators on the Hilbert space X with common domain $\mathcal{D}$. Let A be a closed operator with core $\mathcal{D}$ such that*

(i) $A_n x \to Ax$ *for all* $x \in \mathcal{D}$,

(ii) $\|(A_n - A)x\| \leqslant a\|A_n x\| + b\|x\|$ *for all* $x \in \mathcal{D}$.

Moreover, assume def. $A_n =$ def. A_{n+1}. *Then* def. $A \leqslant$ def. A_n.

These results have useful application to differential operators. For simplicity we apply these results only for Schrödinger operators on R^n, $n \geqslant 2$.

THEOREM 4. (a) *Let* $T = -\Delta + q_1 + q_2 + q_3$ *be a Schrödinger operator on* $\mathcal{L}^2(R^n)$, $n \geqslant 2$, *with* $\mathcal{D}_T = \mathcal{C}_0^\infty(R^n_+)$, $R^n_+ \setminus \{0\}$ *and:*

(i) q_1 *is spherically symmetric and* $q_1 \in \mathcal{L}^2_{\mathrm{loc}}(R^n_+)$,

(ii) $q_1(r) \geqslant -Kr^2$ *for large* r,

(iii) $\lim_{r \to 0+} r^2 q_1(r) = c$ *exists,*

(iv) $\alpha(x) \geqslant q_2(x) \geqslant -K|x|^2$ *for some* $K > 0$ *and* α *continuous,*

(v) $q_3 \in \mathcal{L}^{n/2}_{\mathrm{loc}}(R^n_+)$ *satisfies Stummel conditions.*

Then one has

$$\mathrm{def.}\, T = 0 \quad \text{if} \quad 4c > 3-(n-1)(n-3),$$

$$\mathrm{def.}\, T = 1 + \sum_{l=1}^{s} \frac{(2l+n-2)(n+l-3)!}{l!\,(n-2)!} = d(n, s) \ \text{if}$$

$$c(n, s) = 3-(n-1)(n-3)-4s(s+n-2) > 4c > c(n, s+1) \ \text{for}\ s = 0, 1, 2, \ldots$$

(b) *If instead of* (iii) q_1 *satisfies* $q_1(r) = \dfrac{f(r)}{r^2}$ *with* f *monotonic and* $\lim_{r \to 0} f(r) = -\infty$, *then* def. $T = \infty$.

Since Schrödinger operators with real potentials always have equal deficiency indices, we have written def. $T = l$ instead of def. $T = (l, l)$.

In the proof one treats q_2 by means of a series of cutoffs and uses Theorem 3. The potential is treated as an additive perturbation (Theorem 1) and the remaining operator $-\Delta + q_1$ is investigated by means of polar decomposition.

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THE SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR IN NONSEPARABLE HILBERT SPACES

(A shortened version of the lectures given at the Banach Center, November 1977)

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The Schrödinger operator $L = -\Delta + q(x)$, $x \in R^n$ is usually considered in $L^2(R^n)$. There are many reasons for this; one of the more important is that the mathematical models of quantum physics use $L^2(R^n)$ as the basic space. Why, then, are we going in our lectures to consider the operator L in other, nonseparable spaces? Our idea is that if one investigates the spectral properties of L in a given Hilbert space, say, in $L^2(R^n)$, then it can turn out useful to operate at the same time with spectral representations of L in other functional Hilbert spaces. We then have at our disposal several different points of view on the spectral properties of the operator L, and we may thus obtain better results in the basic space. From this point of view the nonseparable functional spaces $\mathfrak{H}(R^n)$ are very useful because: 1. the essential spectra of L in $L^2(R^n)$ and $\mathfrak{H}(R^n)$ are in many cases equal, 2. the spectral resolutions of L in $L^2(R^n)$ and $\mathfrak{H}(R^n)$ are quite different, in particular, eigenfunctions of L which are not square-integrable belong to $\mathfrak{H}(R^n)$.

This method was applied to the investigation of the spectral properties of L in the papers [4]–[9] and [3]. In the last few years in the papers [11]–[13] the nonseparable Hilbert spaces of almost periodic functions were used in the investigation of the spectral properties of the almost periodic elliptic differential and pseudo-differential operators.

I. The nonseparable functional Hilbert spaces

We will use the following norms:

(1) $$|\!|\!| f |\!|\!|^2 = \limsup_{T \to \infty} \frac{1}{T^n} \int\limits_{|x|<T} |f|^2 dx,$$

(2) $$\|f\|^2 = \lim_{T \to \infty} \frac{1}{T^n} \int\limits_{|x|<T} |f|^2 dx,$$