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ALGEBRAS OF GENERALIZED ANALYTIC FUNCTIONS

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1

Let Γ be an additive and dense subgroup of the real axis $\boldsymbol{R}$. We shall consider that Γ is equipped with the discrete topology. Let $G = \hat{\Gamma}$ be the compact group of characters of Γ, i.e. the group of multiplicative functions χ on Γ with $|\chi| = 1$. According to the Pontrjagin duality theorem, the character group of G equipped with the sup-norm topology is isomorphic and homeomorphic to Γ, and we shall always identify $\Gamma = \{a\}_{a\in\Gamma}$ with the group $\hat{G} = \{\chi_a\}_{a\in\Gamma}$.

By the *generalized complex plane* we mean in the following the space $\boldsymbol{C}_G = G\times[0, \infty)/G\times\{0\}$ with factor-topology. By $*$ we denote the image of $G\times\{0\}$ in $\boldsymbol{C}_G$. We extend the characters χ_a of G on $\boldsymbol{C}_G$ as $\tilde{\chi}_a(\lambda, g) = \lambda^a\chi_a(g)$, if $\lambda \neq 0$, $a \neq 0$; $\tilde{\chi}_a(*) = 0$, $a \neq 0$; and $\tilde{\chi}_0(\lambda, g) \equiv 1$ if $\chi_0 \equiv 1$. Let K be a compact set in $\boldsymbol{C}_G$. By $P_G(K)$ $[R_G(K)]$, we denote the closure in $C(K)$ (with sup-norm) of the [ratios of the] finite linear combinations of the extensions $\tilde{\chi}_a$, $a \geqslant 0$, of the characters χ_a, and by $A_G(K)$ the closure in $C(K)$ of the ratios of functions from $P_G(K)$. Note that in the case when K is the set $\bar{\Delta}_G = G\times[0, 1]/G\times\{0\} \subset \boldsymbol{C}_G$, the algebra $P_G(K)$ coincides with the algebra A_G of so called generalized analytic functions on G, introduced by Arens and Singer [1]. We use the term *generalized analytic functions* to denote the elements of $A_G(K)$. It is easy to see, that if Γ consists of rational numbers, then $R_G(K) = A_G(K)$. In this paper we show that algebras $P_G(K)$, $R_G(K)$ and $A_G(K)$, when $K \subset \boldsymbol{C}_G$, preserve most of the properties of classical algebras $P(K)$, $R(K)$ and $A(K)$, $K \subset \boldsymbol{C}^n$.

2

Let K be a bounded set in $\boldsymbol{C}_G$. The (generalized) *polynomial hull* $\hat{K}$ of K we call as usual the set of these $(\lambda, g) \in \boldsymbol{C}_G$, for which

$$|P(\lambda, g)| \leqslant \sup_K |P(\lambda, g)|,$$

where P is any finite linear combination of functions $\tilde{\chi}_a$, $a \geqslant 0$. A bounded set $K \subset \boldsymbol{C}_G$ is called *polynomially convex*, if $\hat{K} = K$.

THEOREM 1. *Let K be a compact subset of C_G. The spectrum* $\operatorname{sp} P_G(K)$ *of algebra* $P_G(K)$ *coincides with the polynomial hull* $\hat{K}$ *of* K.

Proof. Because $\|P_n\|_{\hat{K}} \leqslant \|P_n\|_K$ for every finite linear combination of functions $\tilde{\chi}_a$, $a \geqslant 0$, $\hat{K} \subset \operatorname{sp} P_G(K)$. Let $\varphi \in \operatorname{sp} P_G(K)$. φ coincides with φ_* — the linear multiplicative functional corresponding to the point $* \in C_G$, if $\varphi(\tilde{\chi}_a) = 0$ for all $a > 0$, $a \in \Gamma$ (note that $\varphi(\tilde{\chi}_0) = \varphi(1) = 1$). Otherwise $\varphi(\tilde{\chi}_a) \neq 0$ for all a from $\Gamma_+ = \Gamma \cap [0, \infty)$. As a matter of fact, let $\varphi(\tilde{\chi}_b) \neq 0$ and let $c \in \Gamma_+$, $c \neq b$. Let $m \geqslant 0$ be such an integer, for which $mb - c \geqslant 0$, or equivalently, for which $\chi_{mb-c} \in \Gamma_+$. Since $\chi_{mb-c} = \chi_b^m \overline{\chi}_c$, $\chi_c \chi_{mb-c} = (\chi_b)^m$, from where $\tilde{\chi}_c \tilde{\chi}_{mb-c} = (\tilde{\chi}_b)^m$. Thus $[\varphi(\tilde{\chi}_b)]^m = \varphi(\tilde{\chi}_c) \cdot \varphi(\chi_{mb-c})$. Hence $\varphi(\tilde{\chi}_c)$ is equal to zero simultaneously with $\varphi(\tilde{\chi}_b)$. The function $\theta(a) = \log|\varphi(\tilde{\chi}_a)|$ is nonpositive on Γ_+ and satisfies the equality $\theta(a+b) = \theta(a)+\theta(b)$. As a monotone and additive function on a dense subset in $\boldsymbol{R}$, θ is extendable on the whole real axis $\boldsymbol{R}$ as a continuous linear function. Thus on Γ $\theta(x) = tx$ for some $t \leqslant 0$, i.e. $|\varphi(\tilde{\chi}_a)| = e^{ta}$. Consequently $|\varphi(\tilde{\chi}_a)| = \lambda_0^a$, where $\lambda_0 = e^t \leqslant 1$, and $a \in \Gamma_+$. Let us define

$$\hat{g}_0(\chi_a) = \begin{cases} \lambda_0^{-a}\varphi(\tilde{\chi}_a), & a \in \Gamma_+, \\ \lambda_0^a \overline{\varphi(\tilde{\chi}_{-a})}, & a \in -\Gamma_+. \end{cases}$$

As a character on the group Γ, $\hat{g}_0$ corresponds to some element $g_0 \in G$, such that $\hat{g}_0(\chi_a) = \chi_a(g_0)$ for all $a \in \Gamma$. Now

$$\varphi(\tilde{\chi}_a) = \lambda_0^a \hat{g}_0(\chi_a) = \lambda_0^a \chi_a(g_0) = \tilde{\chi}_a(\lambda_0, g_0), \quad \text{where} \quad 0 < \lambda_0 \leqslant 1.$$

Hence on $\{\tilde{\chi}_a\}_a$ the functional φ coincides with "the value at the point $(\lambda_0, g_0) \in C_G$". If P is a finite linear combination of $\tilde{\chi}_a$, $a > 0$, then $|P(\lambda_0, g_0)| = |\varphi(P)| \leqslant \|P\|_K$, from where $(\lambda_0, g_0) \in \hat{K}$. The theorem is proved.

A simple example of polynomially convex set is the set

$$\overline{\Delta}_G(\alpha) = \{(\lambda, g) \in C_G \mid 0 \leqslant \lambda \leqslant \alpha, \alpha\text{—some real number}, g \in G\}.$$

Hence the polynomial hull of any set $K \subset \overline{\Delta}_G(\alpha)$, such that $G \times \{\alpha\} \subset K$, coincides with $\overline{\Delta}_G(\alpha)$.

3

We call the *rational hull* $r(K)$ of a bounded set $K \subset C_G$, the set of these $(\lambda, g) \in C_G$, for which

$$|R(\lambda, g)| \leqslant \sup_K |R(\lambda, g)|,$$

where $R \subset C(K)$ is any *generalized rational function* (= ratio of finite linear combinations of functions $\tilde{\chi}_a$, $a \geqslant 0$). A bounded set $K \subset C_G$ is called *rationally convex*, if it coincides with its rational hull.

THEOREM 2. *If K is a compact subset of C_G, then* $\operatorname{sp} R_G(K)$ *coincides with the rational hull* $r(K)$ *of* K.

The proof is the same as the proof of Theorem 1.

PROPOSITION 1. *Let Γ consists only of rational numbers and let K be a compact set in C_G. The point $(\lambda, g) \in C_G$ belongs to the rational hull $r(K)$ of K iff $P(\lambda, g) \in P(K)$ for every finite linear combination P of $\tilde{\chi}_{n/m}$, $n/m \geqslant 0$.*

Proof. Let P be a finite linear combination of $\tilde{\chi}_a$, $a \geqslant 0$, for which $P(\lambda, g) \notin P(K)$. Then $1/\big(P - P(\lambda, g)\big)$ is a generalized rational function on K, and obviously the rational hull $r(K)$ of K does not contain the point (λ, g). If (λ_0, g_0) does not belong to the rational hull of K, there exists a generalized rational function R, such that $|R(\lambda_0, g_0)| > \|R\|_K$. Because $\Gamma_+ \subset \operatorname{Rat}[0, \infty)$, there exists a rational function $r(z)$ on $\tilde{\chi}_{1/m}(K) \subset \boldsymbol{C}$ for an integer $m > 0$, such that $r\big(\tilde{\chi}_{1/m}(\lambda_0, g_0)\big) > \|r(\tilde{\chi}_{1/m})\|_{\hat{K}}$. Now

$$1\big/\big(r - r(\tilde{\chi}_{1/m}(\lambda_0, g_0))\big) = p \circ \tilde{\chi}_{1/m} / q \circ \tilde{\chi}_{1/m},$$

where p and q are polynomials without common zeros. Hence the (generalized) polynomial $q(\tilde{\chi}_{1/m})$ does not take the value $0 = q\big(\tilde{\chi}_{1/m}(\lambda_0, g_0)\big)$ on K, and consequently $q\big(\tilde{\chi}_{1/m}(\lambda_0, g_0)\big) \notin q\big(\tilde{\chi}_{1/m}(K)\big)$.

In particular, Proposition 1 shows that if K_0 is a compact subset of Rat_+, then the set $K = \{(\lambda, g) \in C_G \mid g \in G, \lambda \in K_0\}$ is rationally convex. Hence, we have the following

COROLLARY 1. *If K_0 is a compact subset of* Rat_+, *and* $K = \{(\lambda, g) \in C_G \mid \lambda \in K_0, g \in G\}$, *then* $\operatorname{sp} R_G(K) = K$.

THEOREM 3. *Let $K \subset \overline{\Delta}_G$. Then* $\operatorname{br}(K)$ *is a boundary for the algebra* $R_G(K)$.

Proof. It is known [2], that if $(\lambda, g) \in \overline{\Delta}_G$, there exists an imbedding j of $C' = \{z \mid \operatorname{Im} z \geqslant 0\}$ to a dense subset in $\overline{\Delta}_G$ through (λ, g), such that for every $f \in A_G$, $f \circ j^{-1}$ is an analytic function on C'. Let (λ, g) be an inner point of $r(K)$, belonging to the Šilov boundary $\partial R_G(K)$ of algebra $R_G(K)$, let $U \ni (\lambda, g)$ be a neighbourhood of (λ, g) and let $f \in R_G(K)$ be such that $\max\limits_U |f| = 1$, $|f| \underset{K|U}{<} 1$. If $g = f \circ j^{-1}$, where j is the imbedding of $\{\operatorname{Im} z \geqslant 0\}$ in Δ_G through (λ, g), then g is an analytic function, reaching its $\max|\ |$ in an inner point of $j^{-1}(K)$. Hence $g \equiv$ const on the connected component of $j^{-1}(r(K))$, containing the point (λ, g). Consequently f reach its $\max|\ |$ on $\operatorname{br}(K)$, i.e. $\operatorname{br}(K) \supset \partial R_G(K)$.

COROLLARY 2. *Let $\Gamma \subset$ Rat and $K = \{(\lambda, g) \in C_G \mid 0 < \alpha \leqslant \lambda \leqslant \beta, g \in G\}$. The Šilov boundary $\partial R_G(K)$ of algebra $R_G(K)$ coincides with the topological boundary of K.*

Proof. Here $K = r(K)$. According to the previous theorem, $\partial R_G(K) \subset \operatorname{br}(K) = \mathrm{b}K$. Now $G \times \{\beta\} \subset \partial R_G(K)$, because $A_G(K) \subset R_G(K)$, and $\partial A_G(K) = G \times \{\beta\}$; similarly $G \times \{\alpha\} \subset \partial R_G(K)$, so that $\partial R_G(K) = (G \times \{\alpha\}) \cup (G \times \{\beta\}) = \operatorname{br}(K) = \mathrm{b}K$.

THEOREM 4. *Let Γ_+ contains only rational numbers. If K is a compact set in C_G, such that for every $a \in \Gamma_+$ $dxdy(\tilde{\chi}_a(K)) = 0$, then $R_G(K) = C(K)$.*

Proof. Let $\varepsilon > 0$, $f \in C(K)$, and let $g = h \circ \tilde{\chi}_{1/m}$, $h \in C\big(\tilde{\chi}_{1/m}(K)\big)$, $m \in \boldsymbol{Z}_+$, are such that $\max\limits_K |f - g| < \varepsilon/2$. By the Hartogs–Rosentahl's theorem (see [2]),

h is uniformly approximable on $\tilde{\chi}_{1/m}(K)$ by rational functions of z, because $dx\,dy(\tilde{\chi}_{1/m}(K)) = 0$. If $h' \in R(\tilde{\chi}_{1/m}(K))$, such that $\max\limits_{\tilde{\chi}_{1/m}(K)} |h-h'| < \varepsilon/2$, then $\max\limits_{K} |f-h' \circ \tilde{\chi}_{1/m}| < \varepsilon$, so that $f \in R_G(K)$.

In particular every set $K \subset G$ is rationally convex. Every such set is also polynomially convex (note that if $g_0 \in G \setminus K$, the function

$$h(\lambda, g) = \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \left(1+\chi_{p_n}(g_0)\tilde{\chi}_{p_n}(\lambda, g)\right)$$

admits the following conditions:

$$h \in A_G, \quad h(\lambda_0, g_0) = 1 \quad \text{and} \quad |h| \underset{G\setminus\{g_0\}}{<} 1).$$

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A MULTI-DIMENSIONAL SPECTRAL THEORY IN C^*-ALGEBRAS

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1. Preliminaries

The aim of this work is to introduce a notion of joint spectrum for finite commuting systems of elements of (not necessarily commutative) C^*-algebras and to present some significant relevant results. Our definition originates in a characterization of the joint spectrum (in the sense of J. L. Taylor [9]), which can be stated for commuting systems of operators in Hilbert spaces [11]. The present notion of joint spectrum is intrinsically connected to a given C^*-algebra but its general properties can be easily derived from suitable representations on Hilbert spaces. We intend to carry out the whole programme of J. L. Taylor concerning the properties of the joint spectrum and the construction of the analytic functional calculus [9], [10]. Since we work with more restrictive conditions, there occur considerable simplifications of most of the proofs, in comparison with those of Taylor's. However, the analytic functional calculus will be given by a canonical formula, extending the classical formula of Martinelli for analytic functions in several variables [6], [7]. The results related to the commuting systems of operators on Hilbert spaces, which form the basic part of this work, are exposed from [11], [14] and [13]. The terminology and facts concerning C^*-algebras can be found in [4] and [8].

From now on $s = (s_1, \ldots, s_n)$ will denote a fixed system of indeterminates (nevertheless, the index n may vary). Let $\Lambda[s]$ be the exterior algebra over the complex field generated by $s_1, \ldots, s_n$. For any integer p, $0 \leqslant p \leqslant n$, we denote by $\Lambda^p[s]$ the space of all homogeneous exterior p-forms in $s_1, \ldots, s_n$. Of course, $\Lambda^0[s]$ is identified with the complex field $\boldsymbol{C}$. Every space $\Lambda^p[s]$ has a natural Hilbert space structure, in which the elements

$$s_{j_1} \wedge \ldots \wedge s_{j_p} \quad (1 \leqslant j_1 < \ldots < j_p \leqslant n)$$

form an orthogonal basis. The Hilbert space structure of $\Lambda[s]$ is induced by the formula

$$\Lambda(s) = \bigoplus_{p=0}^{n} \Lambda^p[s].$$