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INVARIANT DISTANCES AND INVARIANT DIFFERENTIAL METRICS IN LOCALLY CONVEX SPACES*

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The following theorem has been proven in [16]: (1)

Let (M, Ξ, μ) be a measure space. If $\dim_C L^1(M, \Xi, \mu) > 1$, every bi-holomorphic automorphisms of the open unit ball B of $L^1(M, \Xi, \mu)$ is the restriction to B of a linear isometry of $L^1(M, \Xi, \mu)$ onto itself.

The main tool in the proof of the theorem is the Carathéodory distance in a domain of a complex Banach space. The first part of [16] collects a few properties of Carathéodory's and Kobayashi's pseudo-distances on a domain in a locally convex, complex vector space. Further properties of these pseudo-distances and of their infinitesimal analogues formed the main subject of three lectures delivered by the author at the Stefan Banach International Mathematical Center, in Warsaw, during December 1977.

These notes are an expanded version of some of the topics discussed in those lectures.

I. The Carathéodory and Kobayashi pseudodistances

1. Let $\Delta = \{\zeta \in \boldsymbol{C}: |\zeta| < 1\}$ be the open unit disc in $\boldsymbol{C}$.

The Poincaré metric $ds^2 = (1-|\zeta|^2)^{-2}|d\zeta|^2$ has constant Gaussian curvature, equal to -4. Its geodesics are either the diameters of Δ or the intersections of Δ with orthogonal circles to the unit circle $\partial\Delta$. The distance $\omega(\zeta_1, \zeta_2)$ between two points ζ_1, ζ_2 in Δ, with respect to the Poincaré metric, is

$$\omega(\zeta_1, \zeta_2) = \frac{1}{2}\log\frac{1+\left|\dfrac{\zeta_2-\zeta_1}{1-\bar{\zeta}_1\zeta_2}\right|}{1-\left|\dfrac{\zeta_2-\zeta_1}{1-\bar{\zeta}_1\zeta_2}\right|} = \sum_{\nu=0}^{+\infty}\frac{1}{2\nu+1}\left|\frac{\zeta_2-\zeta_1}{1-\bar{\zeta}_1\zeta_2}\right|^{2\nu+1}. \tag{1.1}$$

According to the classical Schwarz–Pick lemma:

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(1) Cf. also [17], [2] for further detailes and extensions.

Every holomorphic map $f\colon \Delta \to \Delta$ is distance-decreasing:

$$\omega\big(f(\zeta_1), f(\zeta_2)\big) \leqslant \omega(\zeta_1, \zeta_2) \qquad (\zeta_1, \zeta_2 \in \Delta).$$

If $\omega\big(f(\zeta_1), f(\zeta_2)\big) = \omega(\zeta_1, \zeta_2)$ at two distinct points ζ_1, ζ_2, then equality holds for all pairs of points in Δ, and f is a holomorphic automorphism of Δ.

Furthermore

$$\frac{|f'(\zeta)|}{1-|f(\zeta)|^2} \leqslant \frac{1}{1-|\zeta|^2} \quad \textit{for all } \zeta \in \Delta. \tag{1.2}$$

If equality holds at some $\zeta \in \Delta$, then it holds everywhere on Δ and f is a holomorphic automorphism of Δ.

Lemma 1.1. *For any $\zeta_0 \in \Delta$, the function $\zeta \mapsto \log \omega(\zeta_0, \zeta)$ is subharmonic on Δ.*

Proof. The following proof, due to Carlos Berenstein, is simpler than the one given in [16]. Since the group $\mathrm{Aut}(\Delta)$ of all holomorphic automorphisms of Δ acts transitively on Δ, there is no restriction in assuming $\zeta_0 = 0$. Let $\varphi(r) = \log\log \dfrac{1+r}{1-r}$ $(0 \leqslant r < 1)$. Then, for $0 < |\zeta| = r < 1$,

$$4\frac{\partial^2}{\partial\zeta\,\partial\bar\zeta}\log\omega(0, \zeta) = \varphi''(r) + \frac{1}{r}\varphi'(r)$$

$$= \frac{2}{(1-r^2)^2}\left(\log\frac{1+r}{1-r}\right)^{-2}\left(\frac{1+r^2}{r}\log\left(\frac{1+r}{1-r}\right) - 2\right).$$

Being

$$\frac{1+r^2}{r}\log\frac{1+r}{1-r} = 2(1+r^2)\sum_{n=0}^{+\infty}\frac{r^{2n}}{2n+1},$$

then

$$\frac{\partial^2}{\partial\zeta\,\partial\bar\zeta}\log\omega(0, \zeta) > 0 \quad \text{for} \quad \zeta \in \Delta\setminus\{0\}.$$

Since $\omega(0, 0) = 0$ and $\zeta \mapsto \omega(0, \zeta)$ is positive and continuous on Δ, then $\zeta \mapsto \log\omega(0, \zeta)$ is subharmonic on Δ. ∎

Let $\mathscr{E}$ and $\mathscr{E}_1$ be two complex, locally convex, Hausdorff vector spaces, and let D be a domain in $\mathscr{E}$. A function $f\colon D \to \mathscr{E}_1$ is said to be *holomorphic* if

(1) f is continuous, and

(2) f is Gateaux analytic.

The latter condition means that, for every choice of x, y in $\mathscr{E}$ and for every continuous linear form λ on $\mathscr{E}_1$, the scalar-valued function $\zeta \mapsto \lambda\big(f(x+\zeta y)\big)$ is holomorphic on the open set $V_{x,y} = \{\zeta \in \mathbf{C}\colon x+\zeta y \in D\} \subset \mathbf{C}$. Condition (2) and local boundedness imply conditions (1), (2) if $\mathscr{E}_1$ is normed. (For this and further results cf. e.g. [8], [9].)

Let D_1 be a domain in $\mathscr{E}_1$. We denote by $\mathrm{Hol}(D, D_1)$ the set of all holomorphic maps $f\colon D \to \mathscr{E}_1$ for which $f(D) \subset D_1$. We denote by $\mathrm{Aut}(D)$ the subset of $\mathrm{Hol}(D, D)$ consisting of all bi-holomorphic automorphisms of D.

The Kobayashi pseudo-distance is defined on D as follows. An "analytic chain" in D joining two points $x, y \in D$ consists of a finite number, ν, of functions $f_j \in \mathrm{Hol}(\Delta, D)$ and of ν pairs of points $\zeta_j', \zeta_j'' \in \Delta$ $(j = 1, \ldots, \nu)$, such that

$$f_1(\zeta_1') = x, \quad f_j(\zeta_j'') = f_{j+1}(\zeta_{j+1}') \ (j = 1, \ldots, \nu-1), \quad f_\nu(\zeta_\nu'') = y.$$

The sum $\sum\limits_{j=1}^{\nu}\omega(\zeta_j', \zeta_j'')$ is called the "length" of the analytic chain. The domain D being connected, analytic chains joining x and y in D always exist. The *Kobayashi pseudo-distance* $\mathfrak{k}_D(x, y)$ of x and y is, by definition,

$$\mathfrak{k}_D(x, y) = \inf\sum_{j=1}^{\nu}\omega(\zeta_j', \zeta_j''),$$

where the infimum is taken over all analytic chains joining x and y in D. Note that $\mathfrak{k}_D$ is indeed a pseudo-distance and that

$$\mathfrak{k}_{D_1}\big(F(x), F(y)\big) \leqslant \mathfrak{k}_D(x, y) \tag{1.3}$$

for all $F \in \mathrm{Hol}(D, D_1)$, $x, y \in D$.

A simple application of the Schwarz–Pick lemma yields

$$\mathfrak{k}_\Delta(\zeta_1, \zeta_2) = \omega(\zeta_1, \zeta_2) \qquad (\zeta_1, \zeta_2 \in \Delta).$$

Thus, by (1.3),

$$\omega\big(f(x), f(y)\big) \leqslant \mathfrak{k}_D(x, y)$$

for all $x, y \in D$ and every $f \in \mathrm{Hol}(D, \Delta)$. Hence, letting

$$\mathfrak{c}_D(x, y) = \sup\{\omega\big(f(x), f(y)\big)\colon f \in \mathrm{Hol}(D, \Delta)\},$$

we have

$$\mathfrak{c}_D(x, y) \leqslant \mathfrak{k}_D(x, y).$$

$\mathfrak{c}_D(x, y)$ is the *Carathéodory pseudo-distance* of x and y in D. This pseudo-distance is such that

$$\mathfrak{c}_{D_1}\big(F(x), F(y)\big) \leqslant \mathfrak{c}_D(x, y) \tag{1.4}$$

for all $x, y \in D$ and every $F \in \mathrm{Hol}(D, D_1)$.

2. Let p be a continuous semi-norm on $\mathscr{E}$. For $x_0 \in \mathscr{E}$, $r > 0$, let

$$B_p(x_0, r) = \{x \in \mathscr{E}\colon p(x-x_0) < r\}.$$

The following identities have been established in [16].

$$\mathfrak{c}_{B_p(x_0,r)}(x_0, x) = \mathfrak{k}_{B_p(x_0,r)}(x_0, x) = \omega\left(0, \frac{p(x-x_0)}{r}\right) \qquad (x \in B_p(x_0, r)). \tag{2.1}$$

They imply that for any domain $D \subset \mathscr{E}$ and any $x_0 \in D$, the functions $x \mapsto \mathfrak{c}_D(x_0, x)$, $x \mapsto \mathfrak{k}_D(x_0, x)$ are continuous.

Since for x_0, y_0, x, y in D

$$|\mathfrak{c}_D(x_0, y_0)-\mathfrak{c}_D(x, y)| \leqslant \mathfrak{c}_D(x_0, x)+\mathfrak{c}_D(y_0, y),$$

$$|\mathfrak{k}_D(x_0, y_0)-\mathfrak{k}_D(x, y)| \leqslant \mathfrak{k}_D(x_0, x)+\mathfrak{k}_D(y_0, y),$$

then we conclude with

LEMMA 2.1. *The functions $\mathfrak{c}_D$ and $\mathfrak{k}_D$ are continuous functions on $D\times D$.*

PROPOSITION 2.2. *Let $\mathscr{E}$ be a complex locally convex, locally bounded Hausdorff vector space and let D be a bounded domain in $\mathscr{E}$. Then both $\mathfrak{c}_D$ and $\mathfrak{k}_D$ define on D the relative topology.*

Proof. Let $x_0 \in D$ and, for $s > 0$, let $S(x_0, s) = \{x \in D: \mathfrak{c}_D(x_0, x) < s\}$. We will prove that, for any open neighbourhood V of x_0 in D there is some $s > 0$ such that $S(x_0, s) \subset V$. Let p be a continuous seminorm on $\mathscr{E}$ and let $r > 0$ be such that

$$B_p(x_0, r) \subset V.$$

The domain D being bounded, there is some $R > 0$ for which

$$D \subset B_p(x_0, R).$$

Since for $x \in B_p(x_0, R)$

$$\omega\left(0, \frac{p(x-x_0)}{R}\right) = \sum_{n=0}^{+\infty} \frac{1}{2n+1}\left(\frac{p(x-x_0)}{R}\right)^{2n+1} \geqslant \frac{p(x-x_0)}{R},$$

then (1.4) and (2.1) imply

$$\mathfrak{c}_D(x_0, x) \geqslant \mathfrak{c}_{B_p(x_0,R)}(x_0, x) \geqslant \frac{p(x-x_0)}{R} \quad \text{for all } x \in D.$$

Let $s = r/R$. For $x \in S(x_0, s)$,

$$p(x-x_0) \leqslant R\mathfrak{c}_D(x_0-x) < Rs = r,$$

i.e.

$$S(x_0, s) \subset B_p(x_0, r) \subset V.$$

An identical argument holds for $\mathfrak{k}_D$. ■

COROLLARY 2.3. *Under the same hypotheses of Proposition 2.2, both the pseudo-distances $\mathfrak{c}_D$ and $\mathfrak{k}_D$ are distances on the bounded domain D.*

On the opposite extreme, for $D = E$, both $\mathfrak{c}_D$ and $\mathfrak{k}_D$ degenerate completely. However, if $\mathscr{E}$ has finite dimension, and if D is such that $\mathfrak{k}_D$ is a distance, then the relative topology of D in $\mathscr{E}$ is equivalent to the topology defined by $\mathfrak{k}_D$ [1].

The following proposition generalizes to bounded domains in locally convex, Hausdorff, complex vector spaces a result due to J. P. Vigué ([18], p. 279) for bounded domains in complex Banach spaces. It can be established by similar arguments.

PROPOSITION 2.4. *Let D be a bounded domain in a complex, locally convex, locally bounded, Hausdorff vector space $\mathscr{E}$. If D is homogeneous (i.e. if* $\operatorname{Aut}(D)$ *acts transitively on D) and if $\mathscr{E}$ is sequentially complete, then both $\mathfrak{c}_D$ and $\mathfrak{k}_D$ are complete distances.*

A function $\varphi: D \to [-\infty, +\infty)$ is said to be plurisubharmonic on D if φ is upper semi-continuous on D, and if, for every choice of x, y in $\mathscr{E}$, the function $\zeta \mapsto \varphi(x+\zeta y)$ is subharmonic on $V_{x,y} = \{\zeta \in \boldsymbol{C}: x+\zeta y \in D\}$.

The following theorem is a consequence of Lemmas 1.1 and 2.1.

THEOREM 2.5 [16]. *For any $x_0 \in D$, the function $x \mapsto \log \mathfrak{c}_D(x_0, x)$ is a plurisubharmonic function on D.*

If $\mathfrak{k}_D$ is a distance, then it is an *inner* distance (in the sense of Rinow [12]). This fact was proved by H. L. Royden [13] in the case of a finite dimensional complex manifold. A simple proof, given by S. Kobayashi in [6], extends to the case of (complex spaces and of) domains D in $\mathscr{E}$.

3. In the construction of complex geodesic curves, which will be introduced in § 4, a maximum principle for holomorphic functions will be useful. This principle was established by E. Thorp and R. Whitley in [15] for Banach-valued holomorphic functions. The original proof was considerably simplified by L. A. Harris [4], and Harris' arguments can be easily adapted to locally convex spaces.

Let $f: \Delta \to \boldsymbol{C}$ be a holomorphic function such that $f(\Delta) \subset \Delta$. By the Schwarz–Pick lemma

$$\left|\frac{f(z)-f(0)}{1-\overline{f(0)}\,f(z)}\right| \leqslant |z| \quad (z \in \Delta).$$

Since

$$\begin{aligned}|1-f(0)f(z)| &\leqslant 1-|f(0)|^2+|f(0)|\,|f(z)-f(0)|\\ &\leqslant 2(1-|f(0)|)+|f(0)|\,|f(z)-f(0)|,\end{aligned}$$

then

$$|f(z)-f(0)| \leqslant 2|z|\big(1-|f(0)|\big)+|z|\,|f(0)|\,|f(z)-f(0)|,$$

i.e.

$$2|z|\,|f(0)|+\big(1-|z|\,|f(0)|\big)|f(z)-f(0)| \leqslant 2|z| \quad (z \in \Delta).$$

Being

$$1-|z| \leqslant 1-|z|\,|f(0)| \quad \text{for } |z| < 1,$$

then

(3.1) $$2|z|\,|f(0)|+(1-|z|)|f(z)-f(0)| \leqslant 2|z| \quad \text{for all } z \in \Delta.$$

LEMMA 3.1. *Let p be a continuous semi-norm on the complex, locally convex, Hausdorff vector space $\mathscr{E}$, let $B_p = B_p(0, 1) = \{x \in \mathscr{E}: p(x) < 1\}$ and let $f \in \operatorname{Hol}(\Delta, \mathscr{E})$ be such that $f(\Delta) \subset \overline{B}_p$. Then*

(3.2) $$p\big(f(0)+\zeta\,(f(z)-f(0))\big) \leqslant 1$$

for all $z \in \Delta\setminus\{0\}$ and all $\zeta \in \boldsymbol{C}$, with $|\zeta| \leqslant \dfrac{1-|z|}{2|z|}$.

Proof. For $\mathscr{E} = C$, the conclusion follows from (3.1). In the general case, suppose that (3.2) does not hold at some $z \in \Delta \setminus \{0\}$ and some $|\zeta| \leqslant (1-|z|)/2|z|$. By the Hahn–Banach theorem, there is a continuous linear form λ on $\mathscr{E}$ such that: $|\lambda(x)| \leqslant p(x)$ for all $x \in \mathscr{E}$, and

$$\lambda\big(f(0)+\zeta\,(f(z)-f(0))\big) = p\big(f(0)+\zeta\,(f(z)-f(0))\big) > 1.$$

Since $\lambda \circ f \in \mathrm{Hol}(\Delta, C)$, and $\lambda \circ f(\Delta) \subset \bar{\Delta}$, this inequality contradicts (3.1).(²) ∎

Let K be a subset of $\mathscr{E}$. A point $x \in K$ is called a *complex extreme point* of K if $y = 0$ is the only vector in $\mathscr{E}$ such that $x+\zeta y \in K$ for all $\zeta \in \Delta$.

PROPOSITION 3.2. *Let C be a balanced convex open neighbourhood of* 0 *in* $\mathscr{E}$, *and let $f \in \mathrm{Hol}(\Delta, \mathscr{E})$ be such that $f(\Delta) \subset \bar{C}$. If, for some $z_0 \in \Delta$, $f(z_0)$ is a complex extreme point of $\bar{C}$, then f is constant on Δ. Viceversa, if x is a boundary point of C which is not a complex extreme point of $\bar{C}$, then there is a non-constant function $f \in \mathrm{Hol}(\Delta, \mathscr{E})$ such that $f(\Delta) \subset \bar{C}$ and $f(0) = x$.*

Proof. The first part follows from Lemma 3.1, choosing as p the support function of C. The second part is a trivial consequence of the definition of a complex extreme point. ∎

A direct application of Proposition 3.2 yields

THEOREM 3.3 (*Strong maximum principle*). *Let $\mathscr{E}$ and $\mathscr{E}_1$ be complex locally convex Hausdorff vector spaces. Let D and C_1 be respectively a domain in $\mathscr{E}$ and an open, convex balanced neighbourhood of* 0 *in $\mathscr{E}_1$. If all boundary points of C_1 are complex extreme points of $\bar{C}_1$, every function $f \in \mathrm{Hol}(D, \mathscr{E}_1)$ such that $f(D) \subset \bar{C}_1$, $f(D) \not\subset C_1$, is constant.*

4. Let D be a domain in $\mathscr{E}$. For any $f \in \mathrm{Hol}(\Delta, D)$

$$\omega(\zeta, \zeta_0) \geqslant \mathfrak{k}_D\big(f(\zeta), f(\zeta_0)\big) \geqslant \mathfrak{c}_D\big(f(\zeta), f(\zeta_0)\big) \quad (\zeta, \zeta_0 \in \Delta). \tag{4.1}$$

A subset $\Gamma \subset D$ is called a *complex geodesic curve* in D at a point $x_0 \in \Gamma$ if there is some $f \in \mathrm{Hol}(\Delta, D)$ such that

(i) $f(\Delta) = \Gamma$, and therefore $x_0 = f(\zeta_0)$ for some $\zeta_0 \in \Delta$;

(ii) $\mathfrak{c}_D\big(f(\zeta), x_0\big) = \omega(\zeta, \zeta_0)$ for all $\zeta \in \Delta$.

All inequalities in (4.1) become equalities, so that (ii) holds with $\mathfrak{k}_D$ replacing $\mathfrak{c}_D$. The point $\zeta_0 \in \Delta$ such that $f(\zeta_0) = x_0$ is unique. By composing f with a suitable Möbius transformation one can always assume $\zeta_0 = 0$.

THEOREM 4.1. *Let D be an open convex balanced neighbourhood of* 0, *and let p be the support function of D. For any $x \in D$ such that $p(x) > 0$, let $L = \{\zeta x \colon \zeta \in C\}$. Then*:

(1) *$L \cap D$ is a complex geodesic curve in D at all its points.*

(2) *If $L \cap \bar{D}$ contains a complex extreme point of $\bar{D}$, $L \cap D$ is the unique complex geodesic curve at* 0 *containing x.*

(²) Note that if $\lambda \circ f(\Delta) \not\subset \Delta$, then $\lambda \circ f$ is constant $|\lambda \circ f| = 1$.

Proof. Let $f \in \mathrm{Hol}(\Delta, \mathscr{E})$ be defined by $f(\zeta) = \dfrac{\zeta}{p(x)}\,x$. For $\zeta_1, \zeta_2 \in \Delta$ let $y_j = f(\zeta_j)$ $(j = 1, 2)$. If λ is a continuous linear form of $\mathscr{E}$, such that $|\lambda(y)| \leqslant p(y)$ for all $y \in \mathscr{E}$, and $p(x) = \lambda(x)$, then $\lambda \circ f \in \mathrm{Hol}(\Delta, \Delta)$, and thus

$$\omega(\zeta_1, \zeta_2) \geqslant \mathfrak{c}_D\big(f(\zeta_1), f(\zeta_2)\big) \geqslant \omega\big(\lambda \circ f(\zeta_1), \lambda \circ f(\zeta_2)\big)$$
$$= \omega\left(\frac{\zeta_1}{p(x)}\,p(x), \frac{\zeta_2}{p(x)}\,p(x)\right) = \omega(\zeta_1, \zeta_2).$$

Hence

$$\mathfrak{c}_D\big(f(\zeta_1), f(\zeta_2)\big) = \omega(\zeta_1, \zeta_2) \quad \text{for all } \zeta_1, \zeta_2 \in \Delta.$$

Let $\Gamma \subset D$ be a complex geodesic curve at 0, and let $h \in \mathrm{Hol}(\Delta, D)$ be such that $h(\Delta) \ni x$, $h(0) = 0$. Thus

$$\omega(0, \zeta) = \mathfrak{c}_D\big(0, h(\zeta)\big) \quad \text{for all } \zeta \in \Delta,$$

i.e.

$$\log\frac{1+|\zeta|}{1-|\zeta|} = \log\frac{1+p\big(h(\zeta)\big)}{1-p\big(h(\zeta)\big)} \qquad (\zeta \in \Delta),$$

or

$$|\zeta| = p\big(h(\zeta)\big) \quad \text{for all } \zeta \in \Delta.$$

Since $h(0) = 0$, the function $\zeta \mapsto \dfrac{1}{\zeta}h(\zeta)$ is holomorphic on Δ, and

$$p\left(\frac{1}{\zeta}h(\zeta)\right) = 1 \quad \text{for all } \zeta \in \Delta.$$

If $L \cap \bar{D}$ contains some complex extreme point of $\bar{D}$, every boundary point of $L \cap \bar{D}$ is a complex extreme point of $\bar{D}$ and, by Theorem 3.3, the function $\zeta \mapsto \dfrac{1}{\zeta}h(\zeta)$ is constant, i.e. $h(\zeta) = \zeta h'(0)$ with $p\big(h'(0)\big) = 1$. ∎

II. Carathéodory and Kobayashi differential metrics

5. For $\zeta \in \Delta$, $\tau \in C$, the length, $\langle\tau\rangle_\zeta$, of the vector τ, with respect to the Poincaré metric at ζ, is

$$\langle\tau\rangle_\zeta = \frac{|\tau|}{1-|\zeta|^2}.$$

Let D be a domain in a locally convex Hausdorff complex vector space $\mathscr{E}$.

LEMMA 5.1. *For all $v \in \mathscr{E}$, $x \in D$, $\zeta_0 \in \Delta$ there exist $h \in \mathrm{Hol}(\Delta, D)$ and $\tau \in C$ such that $h(\zeta_0) = x$, $dh(\zeta_0)\tau = v$.*

Proof. For any open neighbourhood V of x in D, there is a continuous seminorm p on $\mathscr{E}$ such that $B_p(x, 1) \subset V$. If $p(v) > 0$, the function $k \in \mathrm{Hol}(C, \mathscr{E})$ defined by

$$k(\zeta) = x + \frac{\zeta}{p(v)}\,v$$

maps Δ into V, and therefore $k_{|\Delta} \in \mathrm{Hol}(\Delta, \mathscr{E})$. Furthermore $k(0) = x$, $dk(0)p(v) = p(v)k'(0) = v$. Thus the function $h \in \mathrm{Hol}(\Delta, D)$ defined by

$$h(\zeta) = k\left(\frac{\zeta-\zeta_0}{1-\bar{\zeta}_0\zeta}\right),$$

and the vector $\tau = p(v)(1-|\zeta_0|^2)$ fulfil all the requirements of the lemma, in the case $p(v) > 0$.

If $p(v) = 0$, we define h on Δ by

$$h(\zeta) = x + \frac{\zeta-\zeta_0}{1-\bar{\zeta}_0\zeta}v$$

and we take $\tau = 1-|\zeta_0|^2$. ∎

The *Kobayashi differential metric* is the function $\varkappa\colon D\times\mathscr{E} \to \boldsymbol{R}_+$ defined at $x \in D$, $v \in \mathscr{E}$ by

$$\varkappa(x; v) = \inf\langle\tau\rangle_\zeta,$$

where the infimum is taken over all $\tau \in \boldsymbol{C}$, $\zeta \in \Delta$ and all functions $h \in \mathrm{Hol}(\Delta, D)$ such that

$$h(\zeta) = x, \quad dh(\zeta)\tau = v. \tag{5.1}$$

Since the group $\mathrm{Aut}(\Delta)$ is transitive on Δ, and its elements are isometries for the Poincaré metric, then one can choose $\zeta \in \Delta$ arbitrarily and keep it fixed in the above definition. In particular, choosing $\zeta = 0$, we have

$$\varkappa(x; v) = \inf\{|\tau|\colon \tau \in \boldsymbol{C},\ h \in \mathrm{Hol}(\Delta, D), h(0) = x, dh(0)\tau = v\}.$$

Let $h \in \mathrm{Hol}(\Delta, D)$ be as in (5.1), and let $g \in \mathrm{Hol}(D, \Delta)$. Then

$$dg(x)v = dg\big(h(\zeta_0)\big)dh(\zeta_0)\tau = d(g\cdot h)(\zeta_0)\tau.$$

Since $g\cdot h \in \mathrm{Hol}(\Delta, \Delta)$, by (1.2),

$$\frac{|dg(x)v|}{1-|g(x)|^2} \leqslant \frac{|\tau|}{1-|\zeta|^2}.$$

This inequality holds for any $h \in \mathrm{Hol}(\Delta, D)$ satisfying (5.1). Hence

$$\frac{|dg(x)v|}{1-|g(x)|^2} \leqslant \varkappa(x; v).$$

Thus, the number $\gamma(x; v) \in \boldsymbol{R}_+$, defined by

$$\gamma(x; v) = \sup\{\langle dg(x)v\rangle_{g(x)}\colon g \in \mathrm{Hol}(D, \Delta)\},$$

satisfies the inequality

$$\gamma(x; v) \leqslant \varkappa(x; v) \quad (x \in D, v \in \mathscr{E}), \tag{5.2}$$

and therefore is finite. The function $\gamma\colon D\times\mathscr{E} \to \boldsymbol{R}_+$ is called the *Carathéodory differential metric* on D. For any $a \in \boldsymbol{C}$

$$\varkappa(x; av) = |a|\varkappa(x; v), \tag{5.3}$$

$$\gamma(x; av) = |a|\gamma(x; v). \tag{5.4}$$

Furthermore, for $v_1, v_2 \in \mathscr{E}$,

$$\gamma(x; v_1+v_2) \leqslant \gamma(x; v_1)+\gamma(x; v_2). \tag{5.5}$$

Let D_1 be a domain in a locally convex, Hausdorff, complex vector space $\mathscr{E}_1$. The following proposition is an immediate consequence of the definitions.

Proposition 5.2. *For all $F \in \mathrm{Hol}(D, D_1)$, $x \in D$, $v \in \mathscr{E}$,*

$$\varkappa_{D_1}\big(F(x); dF(x)v\big) \leqslant \varkappa_D(x; v), \quad \gamma_{D_1}\big(F(x); dF(x)v\big) \leqslant \gamma_D(x; v).$$

In particular, if $F \in \mathrm{Aut}(D)$, then

$$\varkappa_D\big(F(x); dF(x)v\big) = \varkappa(x; v), \quad \gamma_D\big(F(x); dF(x)v\big) = \gamma_D(x; v).$$

Let $\mathscr{E} = \boldsymbol{C}$, $D = \Delta$. Taking $g\colon \zeta \mapsto \zeta$ in the definition of γ_Δ, we have

$$\gamma_\Delta(\zeta; \tau) \geqslant \frac{|\tau|}{1-|\zeta|^2}.$$

Being $dg(\zeta)\tau = \tau$, we have also

$$\varkappa_\Delta(\zeta; \tau) \leqslant \frac{|\tau|}{1-|\zeta|^2}.$$

Thus, by (5.2)

$$\gamma_\Delta(\zeta; \tau) = \varkappa_\Delta(\zeta; \tau) = \frac{|\tau|}{1-|\zeta|^2} = \langle\tau\rangle_\zeta \quad (\zeta \in \Delta,\ \tau \in \boldsymbol{C}). \tag{5.6}$$

More in general, for $\Delta_R = \{\zeta \in \boldsymbol{C}\colon |\zeta| < R\}$,

$$\gamma_{\Delta_R}(\zeta; \tau) = \varkappa_{\Delta_R}(\zeta; \tau) = \frac{R|\tau|}{R^2-|\zeta|^2} \quad (\zeta \in \Delta_R,\ \tau \in \boldsymbol{C}). \tag{5.6'}$$

Lemma 5.3. *Let $B_p = B_p(0, 1)$ be the open unit ball for a continuous semi-norm p on $\mathscr{E}$. Then*

$$\gamma_{B_p}(0; v) = \varkappa_{B_p}(0; v) = p(v) \quad \textit{for all } v \in \mathscr{E}.$$

Proof. Suppose first that $p(v) > 0$, and let λ be a continuous linear form on $\mathscr{E}$ such that $|\lambda(y)| \leqslant p(y)$ for all $y \in \mathscr{E}$, and $\lambda(v) = p(v)$. Thus $\lambda \in \mathrm{Hol}(B_p, \Delta)$. If $h \in \mathrm{Hol}(\Delta, B_p)$ is defined by $h(\zeta) = \dfrac{\zeta}{p(v)}v$, then the lemma follows from the sequence of inequalities:

$$\begin{aligned} p(v) &= \gamma_\Delta\big(0; p(v)\big) = \gamma_\Delta\big(\lambda(0); \lambda(v)\big) = \gamma_\Delta\big(\lambda(0); d\lambda(0)v\big) \\ &\leqslant \gamma_{B_p}(0; v) \leqslant \varkappa_{B_p}(0; v) = \varkappa_{B_p}\big(h(0); dh(0)p(v)\big) \leqslant \varkappa_\Delta\big(0; p(v)\big) = p(v). \end{aligned}$$

If $p(v) = 0$, let $t > 1$, and let $h_t \in \mathrm{Hol}(\Delta, B_p)$ be defined by $h_t(\zeta) = t\zeta v$. Then

$$\gamma_{B_p}(0; v) \leqslant \varkappa_{B_p}(0; v) = \varkappa_{B_p}\left(h_t(0); dh_t(0)\frac{1}{t}\right) \leqslant \varkappa_\Delta\left(0; \frac{1}{t}\right) = \frac{1}{t}.$$

Letting $t \to \infty$, the conclusion follows. ∎

For any $R > 0$, and $B_p(0, R) = \{x \in \mathscr{E}: p(x) < R\}$ we have then

$$\gamma_{B_p(0,R)}(0; v) = \varkappa_{B_p(0,R)}(0; v) = \frac{p(v)}{R}. \tag{5.7}$$

LEMMA 5.4. *Let p_1 and p_2 be continuous semi-norms on two complex, locally convex, Hausdorff vector spaces $\mathscr{E}_1$ and $\mathscr{E}_2$. Then for $R_1 > 0$, $R_2 > 0$ $v_1 \in \mathscr{E}_1$, $v_2 \in \mathscr{E}_2$,*

$$\gamma_{B_{p_1}(0,R_1)\times B_{p_2}(0,R_2)}\big((0,0); (v_1, v_2)\big) = \max\{\gamma_{B_{p_1}(0,R_1)}(0; v_1), \gamma_{B_{p_2}(0,R_2)}(0; v_2)\},$$
$$\varkappa_{B_{p_1}(0,R_1)\times B_{p_2}(0,R_2)}\big((0,0); (v_1, v_2)\big) = \max\{\varkappa_{B_{p_1}(0,R_1)}(0; v_1), \varkappa_{B_{p_2}(0,R_2)}(0; v_2)\}.$$

Proof. The function $p = \max\{p_1, p_2\}$ is a continuous semi-norm on $\mathscr{E}_1 \times \mathscr{E}_2$ for the product topology. Lemma 5.4 follows then from Lemma 5.3. ■

LEMMA 5.5. *For any $\varepsilon > 0$, there is a neighbourhood V of $(0, 0)$ in $B_p(0, R) \times \mathscr{E}$ such that*

$$\varkappa_{B_p(0,R)}(x; v) < \varepsilon$$

for all $(x, v) \in V$.

Proof. Consider the neighbourhood $V = B_p(0, R/2) \times B_p(0, R\varepsilon/4)$. For $x \in B_p(0, R/2)$, $v \in B_p(0, R\varepsilon/4)$ the function $f \in \mathrm{Hol}(\Delta, \mathscr{E})$ defined by $f(\zeta) = x + \frac{2}{\varepsilon}\zeta v$ maps Δ into $B_p(0, R)$. Since $f(0) = x$, $df(0)\frac{\varepsilon}{2} = v$, by Proposition 5.2,

$$\varkappa_{B_p(0,R)}(x; v) \leqslant \varkappa_{\Delta}(0; \varepsilon/2) = \varepsilon/2 < \varepsilon. \ ■$$

For $x \in D$, there is a continuous semi-norm p on $\mathscr{E}$ and some $R > 0$ such that $B_p(x, R) \subset D$. The translation $y \mapsto x+y$ maps $B_p(0, R)$ onto $B_p(x, R)$. A direct application of Proposition 5.2 and of (5.7) yields

$$\gamma_D(x; v) \leqslant \varkappa_D(x; v) \leqslant p(v)/R \quad \text{for all} \quad v \in \mathscr{E}. \tag{5.8}$$

Remark. Note that, if there is a continuous semi-norm p on $\mathscr{E}$ such that $B_p(x, R) \subset D$ for some $R > 0$, then $\gamma_D(x; v) = \varkappa_D(x; v) = 0$ whenever $p(v) = 0$. The conclusion for γ_D will follow also from Lemma 6.1.

Inequalities (5.8) and a trivial compactness argument yield

LEMMA 5.6. *For every compact subset K of D there is a continuous semi-norm p on $\mathscr{E}$ and a constant $k > 0$ such that*

$$\gamma_D(x; v) \leqslant \varkappa_D(x; v) \leqslant kp(v)$$

for all $x \in K$ and all $v \in \mathscr{E}$.

LEMMA 5.7. *For every $x \in D$, $\gamma_D(x; \cdot)$ is a continuous semi-norm on $\mathscr{E}$. If D is bounded, and if p is any continuous norm on $\mathscr{E}$ such that $B_p(x, r) \subset D$ for some $x \in D$ and some $r > 0$, then $\gamma_D(x; \cdot)$ is a norm equivalent to p.*

Proof. The first part of the lemma follows from (5.4), (5.5) and (5.8). Let now D be bounded. Then there is some $R > 0$ such that $D \subset B_p(x, R)$, and therefore

$$\gamma_D(x; v) \geqslant \gamma_{B_p(x,R)}(x; v) = p(v)/R \quad \text{for all } x \in D. \ ■$$

COROLLARY 5.8 [3]. *If D is a bounded domain in a complex Banach space $\mathscr{E}$, then, for every $x \in D$, $\gamma_D(x; \cdot)$ is a norm equivalent to the norm in $\mathscr{E}$.*

6. Let D be a domain in $\mathscr{E}$, and let p be a continuous semi-norm on $\mathscr{E}$ such that $\overline{B_p(x_0, r)} \subset D$ for some $x_0 \in D$ and some $r > 0$. Given any $f \in \mathrm{Hol}(D, \boldsymbol{C})$, suppose that f is bounded on $\overline{B_p(x_0, r)}$. Let $v \in \mathscr{E} \setminus \{0\}$, and suppose first that $p(v) = 0$. The function $\boldsymbol{C} \ni \zeta \mapsto f(x+\zeta v)$ is a bounded entire function for all $x \in B_p(x_0, r)$. Hence it is constant, i.e. $f(x+\zeta v) = f(x)$ for all $x \in B_p(x_0, r)$ and all $\zeta \in \boldsymbol{C}$. Let

$$f(x+y) = \sum_{q=0}^{+\infty} \frac{1}{q!}\, d^q f(x)(y)$$

be the Taylor expansion of f near x. Here $d^q f(x)$ is a continuous, complex-valued, q-homogeneous polynomial on $\mathscr{E}$ and, for $x \in B_p(x_0, r)$, there is a balanced neighbourhood V of 0 in $\mathscr{E}$ such that, for $y \in V$,

$$d^q f(x)(y) = \frac{q!}{2\pi}\int_0^{2\pi} e^{-iq\theta} f(x+e^{i\theta}y)\,d\theta.$$

Thus $d^q f(x)(v) = 0$ for all $q = 1, 2, \ldots$ and all $x \in B_p(x_0, r)$, hence ($x \mapsto d^q f(x)(v)$ being holomorphic) for all $x \in D$, i.e. $f(x+\zeta v) = f(x)$ for all $x \in D$. That proves

LEMMA 6.1. *Let p be a continuous semi-norm such that $B_p(x, r) \subset D$ for some $x \in D$ and some $r > 0$, and let $f \in \mathrm{Hol}(D, \boldsymbol{C})$ be bounded on $B_p(x, r)$. For every vector $v \in \mathscr{E}$, for which $p(v) = 0$, we have $d^q f(x)(v) = 0$ for all $x \in D$ and all $q = 1, 2, \ldots$ In other words, $f(x+\zeta v) = f(x)$ for all $x \in D$ and all $\zeta \in \boldsymbol{C}$. In particular, $\gamma_D(x; v) = 0$ at all $x \in D$.*

Suppose now that $p(v) > 0$. If $x \in B_p(x_0, r/2)$, $x + \frac{r}{2p(v)}\zeta v \in B_p(x_0, r)$ for all $\zeta \in \bar{\Delta}$. Thus, for $q = 1, 2, \ldots$, and for $x \in B_p(x_0, r/2)$,

$$d^q f(x)(v) = \frac{q!}{2\pi}\left(\frac{2p(v)}{r}\right)^q \int_0^{2\pi} e^{-iq\theta} f\left(x + \frac{r}{2p(v)}\, e^{i\theta} v\right) d\theta,$$

and therefore

$$|d^q f(x)(v)| \leqslant \left(\frac{2p(v)}{r}\right)^q q!\sup\{|f(y)|:\ y \in B_p(x_0, r)\} \tag{6.1}$$

for $q = 1, 2, \ldots$ and all $f \in \mathrm{Hol}(D, \boldsymbol{C})$.

Now, let $f \in \mathrm{Hol}(D, \Delta)$, and consider the complex valued holomorphic function $x \mapsto d^q f(x)(v)$. For $x_1, x_2 \in B(x_0, r/4)$ the function $t \mapsto d^q f(x_1 + t(x_2 - x_1))(v)$ is C^∞ on $[0, 1]$. Thus by the mean value theorem

$$|d^q f(x_2)(v) - d^q f(x_1)(v)| \leqslant \sup_x |d^1(d^q f(x)(v))(x_2 - x_1)|$$

for $x = x_1+t(x_2-x_1)$ with $0 < t < 1$. Since $p(x-x_0) < r/4$, if $p(x_2-x_1) > 0$, then

$$d^1\big(d^qf(x)(v)\big)(x_2-x_1) = q!\frac{p(x_2-x_1)}{2\pi}\int_0^{2\pi} e^{-i\theta}d^qf\left(x+\frac{r}{p(x_2-x_1)}e^{i\theta}(x_2-x_1)\right)d\theta;$$

if $p(x_2-x_1) = 0$, then $d^1\big(d^qf(y)(v)\big)(x_2-x_1) = 0$ for all $y \in D$. In any case, by (6.1),

$$\left|d^1\big(d^qf(x)(v)\big)(x_2-x_1)\right| \leqslant \frac{2p(x_2-x_1)}{r}\left(\frac{2p(v)}{r}\right)^q q!,$$

and therefore

$$|d^qf(x_2)(v)-d^qf(x_1)(v)| \leqslant q!\left(\frac{2}{r}\right)^{q+1}p(v)^q p(x_2-x_1) \tag{6.2}$$

for all $f \in \mathrm{Hol}(D, \Delta)$, all $x_1, x_2 \in B_p(x_0, r/4)$, $q = 1, 2, \ldots$

Note that r does not depend on f.

PROPOSITION 6.2. *Let D be a domain in $\mathscr{E}$. The function $\gamma_D\colon D\times\mathscr{E} \to \boldsymbol{R}_+$ is locally Lipschitz.*

Proof. 1. Let $v \in \mathscr{E}$, $x_0 \in D$, and let p be a continuous seminorm on $\mathscr{E}$ such that $B_p(x_0, r) \subset D$ for some $r > 0$. Let $x_1, x_2 \in B_p(x_0, r/4)$ and suppose that $\gamma_D(x_2; v) \geqslant \gamma_D(x_1; v)$. Since the Möbius transformations act transitively on Δ, we have, by (6.1), (6.2),

$$\begin{aligned}\gamma_D(x_2; v)-\gamma_D(x_1; v) &= \sup\{|df(x_2)(v)|\colon f \in \mathrm{Hol}(D, \Delta),\ f(x_2) = 0\}-\\ &\quad -\sup\left\{\frac{|df(x_1)(v)|}{1-|f(x_1)|^2}\colon f \in \mathrm{Hol}(D, \Delta),\ f(x_2) = 0\right\}\\ &\leqslant \sup\left\{|df(x_2)(v)|-\frac{|df(x_1)(v)|}{1-|f(x_1)|^2}\colon f \in \mathrm{Hol}(D, \Delta), f(x_2) = 0\right\}\\ &\leqslant \sup\{|df(x_2)(v)|-|df(x_1)(v)|\colon f \in \mathrm{Hol}(D, \Delta), f(x_2) = 0\}\\ &\leqslant \sup\{|df(x_2)(v)-df(x_1)(v)|\colon f \in \mathrm{Hol}(D, \Delta), f(x_2) = 0\}\\ &\leqslant (2/r)^2 p(x_2-x_1)p(v).\end{aligned}$$

Thus

$$|\gamma_D(x; v)-\gamma_D(x_0; v)| \leqslant (2/r)^2 p(x-x_0)p(v) \quad \text{for all } x \in B_p(x_0, r/4).$$

2. For $v_1, v_2 \in \mathscr{E}$ we have, by (5.5) and (6.1),

$$\begin{aligned}|\gamma_D(x_0; v_2)-\gamma(x_0; v_1)| &\leqslant \gamma_D(x_0; v_2-v_1)\\ &= \sup\{|df(x_0)(v_2-v_1)|\colon f \in \mathrm{Hol}(D, \Delta), f(x_0) = 0\}\\ &\leqslant \frac{2}{r}p(v_2-v_1).\end{aligned}$$

3. For $x_1 \in B_p(x_0, r/4)$, $v_1, v_2 \in \mathscr{E}$,

$$\begin{aligned}|\gamma_D(x_0; v_0)-\gamma_D(x_1; v_1)| &\leqslant |\gamma_D(x_0; v_0)-\gamma_D(x_0; v_1)|+|\gamma_D(x_0; v_1)-\gamma_D(x_1; v_1)|\\ &\leqslant \frac{2}{r}p(v_1-v_2)+\left(\frac{2}{r}\right)^2 p(x_1-x_0)p(v_1).\end{aligned}$$

That proves the proposition.

In view of (5.6) a direct computation yields

LEMMA 6.3. *The function $\log\gamma_\Delta\colon \Delta\times\boldsymbol{C} \to \boldsymbol{R}$ is plurisubharmonic.*

For u, y in $\mathscr{E}$, $V_{u,y}$ denotes, as before, the open set $V_{u,y} = \{\zeta \in \boldsymbol{C}\colon u+\zeta y \in D\} \subset \boldsymbol{C}$.

For any $f \in \mathrm{Hol}(D, \Delta)$ the complex-valued function $(\zeta, v) \mapsto df(u+\zeta y)(v)$ is holomorphic on the open set $V_{u,y}\times\mathscr{E} \subset \boldsymbol{C}\times\mathscr{E}$. Thus the function $(x, v) \mapsto df(x)(v)$ is holomorphic on $D\times\mathscr{E}$, and therefore $(x, v) \mapsto \log|df(x)(v)|$ is plurisubharmonic on $D\times\mathscr{E}$. By Lemma 6.3, the function $\zeta \mapsto \log\dfrac{1}{1-|f(u+\zeta y)|^2}$ is subharmonic on $V_{u,y}$. Hence $x \mapsto \log\dfrac{1}{1-|f(x)|^2}$ is plurisubharmonic on D. In conclusion we have proven

LEMMA 6.4. *For any $f \in \mathrm{Hol}(D, \Delta)$ the function $(x, v) \mapsto \log\dfrac{|df(x)(v)|}{1-|f(x)|^2}$ is plurisubharmonic on $D\times\mathscr{E}$.*

By definition, $\log\gamma_D\colon D\times\mathscr{E} \to \boldsymbol{R}$ is the upper envelope of a family of plurisubharmonic functions on $D\times\mathscr{E}$. Since γ_D is continuous (Proposition 6.2) the following theorem holds (cf. [7], p. 400).

THEOREM 6.5. *The function $\log\gamma_D\colon D\times\mathscr{E} \to \boldsymbol{R}$ is plurisubharmonic on $D\times\mathscr{E}$.*

Let l be a differentiable curve in D, and let $\sigma\colon [a, b] \to D$ be a differentiable parametrization of l. In view of Proposition 6.2 we can define the "length" of l with respect to γ_D by

$$L_\gamma(l) = \int_a^b \gamma_D\big(\sigma(t); \sigma'(t)\big)dt.$$

By (5.4) $L_\gamma(l)$ does not depend on the differentiable parametrization σ. The domain D, being connected and locally arcwise connected, is arcwise connected. Any two points x, y in D can be joined by a differentiable curve in D. For any x, y in D we define $\tilde{c}_D(x, y)$ as the infimum of $L_\gamma(l)$ over all differentiable curves joining x to y in D. It follows from the continuity of γ_D (Proposition 6.2) that this is the same as the infimum over all piece-wise differentiable curves joining x and y in D. Hence $\tilde{c}_D$ is a pseudo-distance.

For $\zeta_1, \zeta_2 \in \Delta$, $c_\Delta(\zeta_1, \zeta_2) = \omega(\zeta_1, \zeta_2) = \tilde{c}_\Delta(\zeta_1, \zeta_2)$.

Hence for any $f \in \mathrm{Hol}(D, \Delta)$ and for any differentiable curve l in D joining x to y we have, by Proposition 5.2,

$$\omega\big(f(x), f(y)\big) \leqslant \int_a^b \gamma_\Delta\big(f(\sigma(t)); f'(\sigma(t))\cdot\sigma'(t)\big)dt \leqslant \int_a^b \gamma_D\big(\sigma(t); \sigma'(t)\big)dt,$$

and therefore

$$\omega\big(f(x), f(y)\big) \leqslant \tilde{c}_D(x, y).$$

Thus

$$c_D(x, y) \leqslant \tilde{c}_D(x, y).$$

7. We shall now prove that the function $\varkappa_D$: $D \times \mathscr{E} \to \mathbf{R}_+$ is upper semi-continuous. This fact was established by H. L. Royden in [13] in the finite-dimensional case. Our proof will follow, *mutatis mutandis*, Royden's ideas.

LEMMA 7.1. *Let $R > 0$ and let $h \in \mathrm{Hol}(\Delta_R, D)$ $(\Delta_R = \{\zeta \in \mathbf{C}\colon |\zeta| < R\})$ be such that $h'(0) \neq 0$. There exists a continuous linear form $\lambda \neq 0$ on $\mathscr{E}$ and, for every $0 < r < R$, a neighbourhood V of 0 in $\mathscr{L} = \mathrm{Ker}\,\lambda$ and a function $g \in \mathrm{Hol}(\Delta_r \times V, D)$ such that $g_{|\Delta_r \times \{0\}} = h_{|\Delta_r}$; the differential $dg(0, 0)$ of g at $(0, 0)$ is a bi-continuous isomorphism of $\mathbf{C} \times \mathscr{L}$ onto $\mathscr{E}$, and g maps a neighbourhood of $(0, 0)$ bi-holomorphically onto a neighbourhood of $h(0)$.*

Proof. Assume $h(0) = 0$. Let λ be a continuous linear form on $\mathscr{E}$ such that $\lambda\big(h'(0)\big) = 1$, and let $f \in \mathrm{Hol}(\Delta_R \times \mathscr{L}, \mathscr{E})$ be defined by

$$f(\zeta, y) = h(\zeta) + y \qquad (\zeta \in \Delta_R,\ y \in \mathscr{L}).$$

Let $h(\zeta) = \sum_{\nu=1}^{+\infty} \zeta^\nu a_\nu$ $(a_\nu \in \mathscr{E}$ for $\nu = 1, 2, \ldots;\ a_1 \neq 0)$ be the power series expansion of h in Δ_R. The differential $df(0, 0)$ of f at $(0, 0)$ is defined by

$$df(0, 0)(\zeta, y) = \zeta a_1 + y \qquad (\zeta \in \mathbf{C},\ y \in \mathscr{L}).$$

Hence $df(0, 0)$ is a bi-continuous isomorphism of $\mathbf{C} \times \mathscr{L}$ onto $\mathscr{E}$, and

$$df(0, 0)^{-1}(x) = \big(\lambda(x), x - \lambda(x)a_1\big) \qquad (x \in \mathscr{E}).$$

For $x = f(\zeta, y)$

$$df(0, 0)^{-1}(x) = (\zeta, y) + \sum_{\nu=2}^{+\infty} \zeta^\nu df(0, 0)^{-1}(a_\nu),$$

i.e.

(7.1) $$\lambda(x) = \sigma(\zeta), \qquad x - \lambda(x)a_1 = y + h(\zeta) - \sigma(\zeta)a_1$$

where $\sigma \in \mathrm{Hol}(\Delta_R, \mathbf{C})$ is defined by

$$\sigma(\zeta) = \zeta + \sum_{\nu=2}^{+\infty} \zeta^\nu \lambda(a_\nu).$$

Since $\sigma'(0) = 1$, there is an open neighbourhood A of 0 in $\mathbf{C}$ which is mapped by σ bi-holomorphically onto an open neighbourhood A_1 of 0 in Δ_R. Let $\tau = (\sigma_{|A})^{-1}$, and let U be a neighbourhood of 0 in $\mathscr{E}$ such that $\lambda(U) \subset A_1$. Then, for $x \in U$, (7.1) yields

$$\zeta = \tau\big(\lambda(x)\big), \qquad y = x - \lambda(x)a_1 - h\big(\tau(\lambda(x))\big) + \sigma\big(\tau(\lambda(x))\big)a_1 = x - h\big(\tau(\lambda(x))\big).$$

That proves that there is a neighbourhood W of 0 in $\Delta_R \times \mathscr{L}$ whose image $f(W)$ is a neighbourhood of 0 in $\mathscr{E}$, and the restriction $f_{|W}$ is a bi-holomorphic map of W onto $f(W)$.

Since $f(\Delta_R \times \{0\}) = h(\Delta_R) \subset D$, then, for every $0 < r < R$, D contains the compact set $f(\bar{\Delta}_r \times \{0\})$. Hence there is an open neighbourhood V of 0 in $\mathscr{L}$ such that $\bar{\Delta}_r \times V \subset f^{-1}(D)$, i.e. $f(\bar{\Delta}_r \times V) \subset D$. Take $g = f_{|\Delta_r \times V}$. ∎

A neighbourhood V satisfying Lemma 7.1 can be described in terms of a semi-norm, i.e.

$$V = B_{\mathscr{L}, p} = \{y \in \mathscr{L}\colon p(y) < 1\},$$

where p is a suitable continuous semi-norm on $\mathscr{E}$.

LEMMA 7.2. *Given $\varepsilon > 0$, $x \in D$, $v \in \mathscr{E}$, there exist: $r > 0$, a continuous linear form λ on $\mathscr{E}$ $(\lambda \neq 0)$, a continuous semi-norm p on $\mathscr{E}$, and $g \in \mathrm{Hol}(\Delta_r \times B_{\mathscr{L}, p}, D)$ such that:*

$$g(0, 0) = x; \quad dg(0, 0)(1, 0) = v;$$

g maps an open neighbourhood of $(0, 0)$ bi-holomorphically onto a neighbourhood of x;

$$\varkappa_{\Delta_r \times B_{\mathscr{L}, p}}\big((0, 0); (1, 0)\big) < \varkappa_D(x; v) + \varepsilon.$$

Proof. Since, by (5.7), $\varkappa_{\Delta_R}(0; 1) = 1/R$, then

$$\varkappa_D(x; v) = \inf\{1/R\colon h \in \mathrm{Hol}(\Delta_R, D), h(0) = x, h'(0) = v\}.$$

Choose h such that

$$1/R < \varkappa_D(x; v) + \varepsilon,$$

and let r be such that $0 < r < R$ and

$$1/r < \varkappa_D(x; v) + \varepsilon.$$

Starting from h we define g as in Lemma 7.1. Then, by Lemma 5.4,

$$\varkappa_{\Delta_r \times B_{\mathscr{L}, p}}\big((0, 0); (1, 0)\big) = \varkappa_{\Delta_r}(0; 1) = 1/r < \varkappa_D(x; v) + \varepsilon. \ ∎$$

PROPOSITION 7.3. *The function $\varkappa_D$: $D \times \mathscr{E} \to \mathbf{R}_+$ is upper semi-continuous.*

Proof [13]. Same notations as in the proof of Lemma 7.2. Since, by (5.7) and Lemma 5.4, $\varkappa_{\Delta_r \times B_{\mathscr{L}, p}}$ is continuous at $(0, 0)$, for any $\varepsilon > 0$ there is an open neighbourhood W of $(0, 0, 0, 0)$ in $\Delta_r \times B_{\mathscr{L}, p} \times \mathbf{C} \times \mathscr{L}$ such that, for $(\zeta, y, \tau, w) \in W$,

$$\varkappa_{\Delta_r \times B_{\mathscr{L}, p}}\big((\zeta, y); (\tau, w)\big) < \varkappa_{\Delta_r \times B_{\mathscr{L}, p}}\big((0, 0); (1, 0)\big) + \varepsilon.$$

Since g is bi-holomorphic on a neighbourhood of $(0, 0)$ in $\Delta_r \times \mathscr{L}$ then — if W is sufficiently small — (g, dg) maps W onto a set containing a neighbourhood V of (x, v) in $D \times \mathscr{E}$. Let $(x', v') \in V$. Then $x' = g(\zeta, y)$, $v' = dg(\zeta, y)(\tau, w)$ for some $(\zeta, y, \tau, w) \in W$, and therefore

$$\varkappa_D(x'; v') = \varkappa_D\big(g(\zeta, y); dg(\zeta, y)(\tau, w)\big) \leqslant \varkappa_{\Delta_r \times B_{\mathscr{L},p}}\big((\zeta, y); (\tau, w)\big)$$
$$< \varkappa_{\Delta_r \times B_{\mathscr{L},p}}\big((0, 0); (1, 0)\big) + \varepsilon < \varkappa_D(x; v) + 2\varepsilon. \blacksquare$$

Any two points x, y in D can be joined by a differentiable curve l, expressed by a differentiable function $\sigma\colon [a, b] \to D$ ($\sigma(a) = x$, $\sigma(b) = y$). In view of Proposition 7.3 we may define the "length" $L_\varkappa(l)$ of l by the integral

$$L_\varkappa(l) = \int_a^b \varkappa_D\big(\sigma(t); \sigma'(t)\big)dt.$$

By (5.3) $L_\varkappa(l)$ is independent of the differentiable parametrization σ of l.

Let $\tilde{\mathfrak{k}}_D(x, y)$ be the infimum of $L_\varkappa(l)$ taken over all differentiable curves joining x and y in D. It follows from Lemma 5.5 that this is the same as the infimum over all piece-wise differentiable curves joining x and y in D. Hence $\tilde{\mathfrak{k}}_D(x, y)$ is a pseudo-distance.

THEOREM 7.4. *The Kobayashi pseudo-distance is the integrated form of* $\varkappa_D$, *i.e.*

$$\tilde{\mathfrak{k}}_D(x, y) = \mathfrak{k}_D(x, y).$$

This theorem has been proved by H. L. Royden for finite-dimensional complex manifolds; cf. Theorem 1 of [13]. Royden's proof rests essentially on the key Lemma 1 of [13] (which was proved in [13] for domains in C^n and in [14] for finite-dimensional manifolds). Our Lemma 7.1 extends Lemma 1 of [13] to domains in locally convex spaces. Replacing Lemma 1 by Lemma 7.1, Royden's proof of Theorem 1 can be adapted to establish Theorem 7.4. We give the proof here for the sake of completeness.

Proof. I. We prove first that $\tilde{\mathfrak{k}}_D(x, y) \leqslant \mathfrak{k}_D(x, y)$. Let $\varepsilon > 0$ and let $\zeta_1', \zeta_1'', \ldots, \zeta_\nu', \zeta_\nu'', f_1, \ldots, f_\nu$ be an analytic chain (notations as in n. 1) joining x and y in D, such that

$$\sum_{j=1}^{\nu} \omega(\zeta_j', \zeta_j'') < \mathfrak{k}_D(x, y) + \varepsilon.$$

Let l_j be the geodesic for the Poincaré metric, joining ζ_j', ζ_j'' in Δ, and let l be the piece-wise differentiable curve defined by $f_j(l_j)$ ($j = 1, \ldots, \nu$). Then, by Proposition 5.2,

$$\tilde{\mathfrak{k}}_D(x, y) \leqslant \int_l \varkappa_D = \sum_{j=1}^{\nu} \int_{f_j(l_j)} \varkappa_D \leqslant \sum_{j=1}^{\nu} \int_{l_j} \varkappa_\Delta = \sum_{j=1}^{\nu} \omega(\zeta_j', \zeta_j'') < \mathfrak{k}_D(x, y) + \varepsilon.$$

II. We prove now that $\tilde{\mathfrak{k}}_D(x, y) \geqslant \mathfrak{k}_D(x, y)$. For any $\varepsilon > 0$ there is a differentiable function $\varphi\colon [0, 1] \to D$ such that $\varphi(0) = x$, $\varphi(1) = 1$, and

$$\int_0^1 \varkappa_D\big(\varphi(t); \varphi'(t)\big)dt < \tilde{\mathfrak{k}}_D(x, y) + \varepsilon.$$

The function $t \mapsto \varkappa_D\big(\varphi(t); \varphi'(t)\big)$, being upper semi-continuous on $[0, 1]$, is the pointwise limit of a decreasing sequence of continuous functions. Hence, by the monotone convergence theorem, there is a positive continuous function σ on $[0, 1]$ such that $\sigma(t) > \varkappa_D\big(\varphi(t); \varphi'(t)\big)$ for all $t \in [0, 1]$, and

$$\int_0^1 \sigma(t)dt < \tilde{\mathfrak{k}}_D(x, y) + \varepsilon.$$

The function σ, being continuous, is Riemann integrable. Hence there is a $\delta > 0$ such that, for any choice of $0 = t_0 < t_1 < \ldots < t_n = 1$ with $t_{j+1} - t_j < \delta$ and any choice of $s_j \in [t_{j-1}, t_j)$, there is

$$\sum_{j=1}^{n} (t_j - t_{j-1})\sigma(s_j) < \tilde{\mathfrak{k}}_D(x, y) + \varepsilon. \tag{7.2}$$

Choose now $s \in [0, 1]$. By Lemma 7.2, there exist: $r > 0$, a continuous linear form λ on $\mathscr{E}$ ($\lambda \neq 0$), a continuous semi-norm p on $\mathscr{E}$ and a function $g \in \mathrm{Hol}(\Delta_r \times B_{\mathscr{L},p}, D)$ such that:

$$g(0) = \varphi(s), \quad dg(0, 0)(1, 0) = \varphi'(s);$$

g is a bi-holomorphic homeomorphism of a neighbourhood of $(0, 0) \in \Delta_r \times B_{\mathscr{L},p}$ onto a neighbourhood of $\varphi(s)$ in D;

$$\frac{1}{r} = \varkappa_{\Delta_r \times B_{\mathscr{L},p}}\big((0, 0); (1, 0)\big)$$
$$< \varkappa_D\big(\varphi(s); \varphi'(s)\big) + \sigma(s) - \varkappa_D\big(\varphi(s); \varphi'(s)\big) = \sigma(s).$$

Hence there is an open interval $I_s \supset s$ in $[0, 1]$ and a differentiable map $\mu\colon I_s \to \Delta_r \times B_{\mathscr{L},p}$ such that $\varphi = g \circ \mu$ on I_s, and $\mu(s) = (0, 0)$, $\mu'(s) = (1, 0) \in C \times \mathscr{L}$, i.e. there are differentiable maps $\mu_1\colon I_s \to \Delta_r$, $\mu_2\colon I_s \to B_{\mathscr{L},p}$, such that

$$\mu(t) = \big(\mu_1(t), \mu_2(t)\big)$$

and

$$\mu_1(t) = t - s + O(|t-s|^2), \quad p\big(\mu_2(t)\big) = O(|t-s|^2).$$

For $f \in \mathrm{Aut}(\Delta_r)$, the map $(\zeta, x) \mapsto \big(f(\zeta), x\big)$ defines a bi-holomorphic automorphism of $\Delta_r \times B_{\mathscr{L},p}$. Let $t', t'' \in I_s$. Choosing $f \in \mathrm{Aut}(\Delta_r)$ in such a way that $f\big(\mu_1(t')\big) = 0$, we have, by (1.3),

$$\mathfrak{k}_{\Delta_r \times B_{\mathscr{L},p}}\big(\mu(t'), \mu(t'')\big) = \mathfrak{k}_{\Delta_r \times B_{\mathscr{L},p}}\Big(\big(0, \mu_2(t')\big), \big(f(\mu_1(t'')), \mu_2(t'')\big)\Big)$$
$$\leqslant \mathfrak{k}_{\Delta_r \times B_{\mathscr{L},p}}\Big(\big(0, \mu_2(t')\big), \big(0, \mu_2(t'')\big)\Big) +$$
$$+ \mathfrak{k}_{\Delta_r \times B_{\mathscr{L},p}}\Big(\big(0, \mu_2(t'')\big), \big(f(\mu_1(t'')), \mu_2(t'')\big)\Big)$$

$$\leqslant \mathfrak{k}_{B_{\mathscr{L},p}}(\mu_2(t'),\mu_2(t''))+\mathfrak{k}_{\Delta_r}(0,f(\mu_1(t'')))$$
$$= \mathfrak{k}_{B_{\mathscr{L},p}}(\mu_2(t'),\mu_2(t''))+\mathfrak{k}_{\Delta_r}(\mu_1(t'),\mu_1(t'')),$$

where

$$\mathfrak{k}_{\Delta_r}(\mu_1(t'),\mu_1(t'')) = \omega\left(\frac{\mu_1(t')}{r},\frac{\mu_1(t'')}{r}\right),$$
$$\mathfrak{k}_{B_{\mathscr{L},p}}(\mu_2(t'),\mu_2(t''))\leqslant \mathfrak{k}_{B_{\mathscr{L},p}}(0,\mu_2(t'))+\mathfrak{k}_{B_{\mathscr{L},p}}(0,\mu_2(t''))$$
$$\leqslant \omega\big(0,p(\mu_2(t'))\big)+\omega\big(0,p(\mu_2(t''))\big).$$

Hence there is an open interval $J_s\subset I_s$, such that $s\in J_s$ and that for $t',t''\in J_s$ $(t'\neq t'')$

$$\mathfrak{k}_{\Delta_r\times B_{\mathscr{L},p}}(\mu(t'),\mu(t''))\leqslant(1+\varepsilon)\frac{|t'-t''|}{r}<(1+\varepsilon)|t'-t''|\sigma(s).$$

Choosing J_s with length $<\delta$, for $t',t''\in J_s$ $(t'\neq t'')$, we have then

$$\mathfrak{k}_D(\varphi(t'),\varphi(t''))=\mathfrak{k}_D\big(g(\mu(t')),g(\mu(t''))\big)\leqslant\mathfrak{k}_{\Delta_r\times B_{\mathscr{L},p}}(\mu(t'),\mu(t''))$$
$$<(1+\varepsilon)|t'-t''|\sigma(s).$$

Applying the Lebesgue covering lemma to the closed interval $[0,1]$ we see that there exists $\varrho>0$ such that, whenever $|t'-t''|<\varrho$, there is $s\in[0,1]$ with $t',t''\in J_s$. Choosing $\delta<\varrho$, $t_j-t_{j-1}<\varrho$ $(t_{j-1}\neq t_j)$ and s_j such that $t_{j-1},t_j\in J_{s_j}$, then, by (7.2),

$$\mathfrak{k}_D(x,y)=\mathfrak{k}_D(\varphi(0),\varphi(1))\leqslant\sum_{j=1}^{n}\mathfrak{k}_D(\varphi(t_{j-1}),\varphi(t_j))$$
$$<(1+\varepsilon)\sum_{j=1}^{n}(t_j-t_{j-1})\sigma(s_j)<(1+\varepsilon)(\tilde{\mathfrak{k}}_D(x,y)+\varepsilon).$$

Since $\varepsilon>0$ is arbitrary, that completes the proof of the theorem.

III. A fixed point theorem

8. In the following, $\mathscr{E}$ will be a locally bounded, locally convex, Hausdorff complex vector space, and D will be a bounded domain in $\mathscr{E}$. A set K is said to be *completely interior* to D — in symbols, $K\subset\subset D$ — if there exists an open neighbourhood U of 0 such that

(8.1) $$u+K\subset D\quad\text{for all } u\in U.$$

Equivalently, $K\subset\subset D$ if there is a continuous semi-norm p on $\mathscr{E}$ such that

(8.2) $$u+K\subset D\quad\text{whenever } p(u)<1.$$

The neighbourhoods U of 0 satisfying (8.1) constitute a fundamental system of neighbourhoods of 0. In other words, the family of semi-norms satisfying (8.2) defines a base of the uniform space $\mathscr{E}$.

LEMMA 8.1. *Let $f\in\mathrm{Hol}(D,\mathscr{E})$. If $f(D)\subset\subset D$, for every $x\in D$ $\{f^n(x)\}$ is a Cauchy sequence for the Kobayashi distance $\mathfrak{k}_D$.*

Proof. Let p be a continuous semi- norm as in (8.2) for $K=f(D)$. The domain D being bounded, there is a finite $M>0$ such that $p(x)<M$ for all $x\in D$. Let $t=1/2M$. For any fixed $x\in D$, let $g\in\mathrm{Hol}(D,\mathscr{E})$ be defined by

$$y\mapsto g(y)=(1+t)f(y)-tf(x)=f(y)+t\big(f(y)-f(x)\big).$$

Since, for any $y\in D$,

$$tp\big(f(y)-f(x)\big)\leqslant t\big(p(f(y))+p(f(x))\big)<2tM=1,$$

then $g(y)\in D$, i.e. $g(D)\subset D$. Thus, by Proposition 5.2,

$$\varkappa_D\big(g(x);dg(x)v\big)\leqslant\varkappa_D(x;v)\quad\text{for all } v\in\mathscr{E}.$$

Since $g(x)=f(x)$ and $dg(x)=(1+t)df(x)$, then we have

$$\varkappa_D\big(f(x);df(x)v\big)\leqslant\frac{1}{1+t}\varkappa_D(x;v),$$

for all $x\in D$ and all $v\in\mathscr{E}$. The Kobayashi distance $\mathfrak{k}_D$ being the integrated form of $\varkappa_D$ (Theorem 7.4), for x_1,x_2 in D

(8.3) $$\mathfrak{k}_D\big(f(x_1),f(x_2)\big)\leqslant\frac{1}{1+t}\mathfrak{k}_D(x_1,x_2).$$

Thus for any $x\in D$, and any $n=1,2,\ldots,$

$$\mathfrak{k}_D\big(f^n(x),f^{n+1}(x)\big)\leqslant\frac{1}{(1+t)^n}\mathfrak{k}_D\big(x,f(x)\big),$$

and that implies that $\{f^n(x)\}_{n=1,2,\ldots}$ is a Cauchy sequence for $\mathfrak{k}_D$. ∎

LEMMA 8.2. *Under the hypotheses of Lemma 8.1, for any $x\in D$ $\{f^n(x)\}$ is a Cauchy sequence in $\mathscr{E}$.*

Proof. We shall prove that for any continuous semi-norm q on $\mathscr{E}$, there is an index N such that, whenever $n,m\geqslant N$, $q\big(f^n(x)-f^m(x)\big)<1$.

Since D is bounded, there is some finite $R>0$ such that $D\subset B_q(0,R)$. Hence for any $z\in D$, $D\subset B_q(z,2R)$.

Let $N>0$ be such that $\mathfrak{k}_D(f^n(x),f^N(x))<1/4R$ for all $n\geqslant N$. Being $D\subset B_q\big(f^N(x),2R\big)$, by (1.3),

$$\mathfrak{k}_{B_q(f^N(x),2R)}\big(f^N(x),f^n(x)\big)\leqslant\mathfrak{k}_D\big(f^N(x),f^n(x)\big)<\frac{1}{4R},$$

i.e., by (2.1),

$$\omega\left(0,\frac{q\big(f^n(x)-f^N(x)\big)}{2R}\right)<\frac{1}{4R}\quad\text{for all } n\geqslant N.$$

Therefore, by (1.1),

$$q\big(f^n(x)-f^N(x)\big)<\tfrac12\quad\text{for all } n\geqslant N,$$

and, for $n,m\geqslant N$,

$$q\big(f^n(x)-f^m(x)\big)\leqslant q\big(f^n(x)-f^N(x)\big)+q\big(f^N(x)-f^m(x)\big)<\tfrac12+\tfrac12=1.\ ∎$$

If $\mathscr{E}$ is sequentially complete $\{f^n(x)\}$ converges to a point $x_0\in\overline{f(D)}\subset D$, which is a fixed point of f, and, by (8.3), is clearly the unique fixed point of f. Thus we may state the following theorem, which is due to C. J. Earle and R. S. Hamilton [3] for Banach spaces.

THEOREM 8.3. *If $\mathscr{E}$ is sequentially complete, any $f \in \mathrm{Hol}(D, \mathscr{E})$ such that $f(D) \subset\subset D$ has a unique fixed point.*

Remark. The proof of the above theorem can be carried out (as in [3]) using the Carathéodory differential metric γ_D and its integrated form $\tilde{c}_D$, instead of $\varkappa_D$ and $\mathfrak{k}_D$.

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THE HOLOMORPHIC FUNCTIONAL CALCULUS AS AN OPERATIONAL CALCULUS

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1. Introduction

1.1. I have been asked to speak on the ideology of the holomorphic functional calculus. And I have accepted, even though I do not believe that such a unique ideology exists. These talks, and these notes are an opportunity for presenting my own ideas on the subject.

These notes lay the stress on the h.f.c., they are related, but only partly, with [47], where non-Banach algebras are stressed.

While preparing this written text, I became increasingly aware that my ideology was that of an operational calculus. The fact that this involves holomorphic functions is a good surprise. The expression "holomorphic functional calculus" (h.f.c.) must be taken to mean "operational calculus involving holomorphic functions".

I must also mention the fact that Gelfand's papers ([11], [12], [13], [14]) were not available in Belgian libraries, or at the Institut Poincaré in Paris, even in 1953 when I completed research on my first paper [42]. This was a consequence of the disruption due to the war, and of lack of money in the post-war period.

I knew most of Gelfand's results. I had attended talks by mathematicians who had read Gelfand. I had read the Mathematical Reviews. But my knowledge was indirect and incomplete. This had an effect on my personal ideology at the outset, also later since I knew how much could be proved about Banach algebras without using Gelfand's results.

1.2. The following is the ideology of many mathematicians, and flows directly out of Gelfand's results.

A commutative Banach algebra is semi-simple modulo the radical. The radical is messy and of minor importance. A semi-simple Banach algebra is a function algebra.

Let X be a compact space $\mathscr{A} \subseteq C(X)$ a function algebra having X as structure space. Let $U \subseteq C^n$ be a set and f a continuous function on U. The function f operates on $\mathscr{A}$ if $f(a_1, \ldots, a_n) \in \mathscr{A}$ whenever $a_1, \ldots, a_n \in \mathscr{A}$ and

$$\{(a_1(x), \ldots, a_n(x)) |\ x \in X\} \subseteq U.$$