COROLLARY. *Let A_n be closed symmetric operators on the Hilbert space X with common domain $\mathcal{D}$. Let A be a closed operator with core $\mathcal{D}$ such that*

(i) $A_n x \to Ax$ *for all* $x \in \mathcal{D}$,

(ii) $\|(A_n - A)x\| \leqslant a\|A_n x\| + b\|x\|$ *for all* $x \in \mathcal{D}$.

Moreover, assume $\operatorname{def.} A_n = \operatorname{def.} A_{n+1}$. *Then* $\operatorname{def.} A \leqslant \operatorname{def.} A_n$.

These results have useful application to differential operators. For simplicity we apply these results only for Schrödinger operators on $\boldsymbol{R}^n$, $n \geqslant 2$.

THEOREM 4. (a) *Let* $T = -\Delta + q_1 + q_2 + q_3$ *be a Schrödinger operator on* $\mathcal{L}^2(\boldsymbol{R}^n)$, $n \geqslant 2$, *with* $\mathcal{D}_T = \mathcal{C}_0^\infty(\boldsymbol{R}_+^n)$, $\boldsymbol{R}_+^n \setminus \{0\}$ *and:*

(i) q_1 *is spherically symmetric and* $q_1 \in \mathcal{L}^2_{\mathrm{loc}}(\boldsymbol{R}_+^n)$,

(ii) $q_1(r) \geqslant -Kr^2$ *for large* r,

(iii) $\lim\limits_{r\to 0+} r^2 q_1(r) = c$ *exists,*

(iv) $\alpha(x) \geqslant q_2(x) \geqslant -K|x|^2$ *for some* $K > 0$ *and* α *continuous,*

(v) $q_3 \in \mathcal{L}^{n/2}_{\mathrm{loc}}(\boldsymbol{R}_+^n)$ *satisfies Stummel conditions.*

Then one has

$$\operatorname{def.} T = 0 \quad \text{if} \quad 4c > 3-(n-1)(n-3),$$

$$\operatorname{def.} T = 1 + \sum_{l=1}^{s} \frac{(2l+n-2)(n+l-3)!}{l!\,(n-2)!} = d(n,s) \ \text{if}$$

$$c(n,s) = 3-(n-1)(n-3)-4s(s+n-2) > 4c > c(n,s+1) \ \text{for} \ s = 0, 1, 2, \ldots$$

(b) *If instead of* (iii) q_1 *satisfies* $q_1(r) = \dfrac{f(r)}{r^2}$ *with f monotonic and* $\lim\limits_{r\to 0} f(r) = -\infty$, *then* $\operatorname{def.} T = \infty$.

Since Schrödinger operators with real potentials always have equal deficiency indices, we have written $\operatorname{def.} T = l$ instead of $\operatorname{def.} T = (l, l)$.

In the proof one treats q_2 by means of a series of cutoffs and uses Theorem 3. The potential is treated as an additive perturbation (Theorem 1) and the remaining operator $-\Delta + q_1$ is investigated by means of polar decomposition.

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THE SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR IN NONSEPARABLE HILBERT SPACES

(A shortened version of the lectures given at the Banach Center, November 1977)

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The Schrödinger operator $L = -\Delta + q(x)$, $x \in \boldsymbol{R}^n$ is usually considered in $L^2(\boldsymbol{R}^n)$. There are many reasons for this; one of the more important is that the mathematical models of quantum physics use $L^2(\boldsymbol{R}^n)$ as the basic space. Why, then, are we going in our lectures to consider the operator L in other, nonseparable spaces? Our idea is that if one investigates the spectral properties of L in a given Hilbert space, say, in $L^2(\boldsymbol{R}^n)$, then it can turn out useful to operate at the same time with spectral representations of L in other functional Hilbert spaces. We then have at our disposal several different points of view on the spectral properties of the operator L, and we may thus obtain better results in the basic space. From this point of view the nonseparable functional spaces $\mathfrak{H}(\boldsymbol{R}^n)$ are very useful because: 1. the essential spectra of L in $L^2(\boldsymbol{R}^n)$ and $\mathfrak{H}(\boldsymbol{R}^n)$ are in many cases equal, 2. the spectral resolutions of L in $L^2(\boldsymbol{R}^n)$ and $\mathfrak{H}(\boldsymbol{R}^n)$ are quite different, in particular, eigenfunctions of L which are not square-integrable belong to $\mathfrak{H}(\boldsymbol{R}^n)$.

This method was applied to the investigation of the spectral properties of L in the papers [4]–[9] and [3]. In the last few years in the papers [11]–[13] the nonseparable Hilbert spaces of almost periodic functions were used in the investigation of the spectral properties of the almost periodic elliptic differential and pseudo-differential operators.

I. The nonseparable functional Hilbert spaces

We will use the following norms:

$$\square f \square^2 = \limsup_{T\to\infty} \frac{1}{T^n} \int\limits_{|x|<T} |f|^2 dx, \tag{1}$$

$$\|f\|^2 = \lim_{T\to\infty} \frac{1}{T^n} \int\limits_{|x|<T} |f|^2 dx, \tag{2}$$

$$\|f\|_{L^2}^2 = \int_{R^n} |f|^2 dx, \tag{3}$$

where $f(x)$ is a complex-valued function, $x \in R^n$.

Let us put $M = \{f\colon \Box f \Box < +\infty\}$. Then, by the relation $f \sim g$ iff $\Box f-g \Box = 0$, the linear space

$$\mathfrak{M} = M/\sim$$

is a complete Banach space (see [10]). The elements of $\mathfrak{M}$ will be denoted by capital letters $F, U, \ldots$ If $f \in M$ represents $F \in \mathfrak{M}$, we write $F = (f)$.

We look for Hilbert spaces $\mathfrak{H} \subset \mathfrak{M}$. We take as the scalar product the quantity

$$(U, V) = \lim_{T\to\infty} \frac{1}{T^n} \int_{|x|<T} u\bar{v}\,dx, \quad U = (u),\ V = (v). \tag{4}$$

Therefore it is natural to consider the set

$$P = \{u\colon u \in M,\ \|u\| \text{ exists}\}.$$

But there arises the trouble that P is not a linear set (see [1]). Indeed, putting, in the case of R^1:

$$f(x) = 1, \quad g(x) = \begin{cases} e^{-i\log x}, & x \geqslant \alpha, \\ 0, & x < \alpha, \end{cases}$$

we have $f, g \in P$ but

$$\frac{1}{2T}\int_{-T}^{T} f\bar{g}\,dx = \frac{1}{2(i+1)} e^{-i\log T} - \frac{1}{2T}\left(\frac{\alpha}{i+1} e^{i\log\alpha}\right).$$

This means that the scalar product (f, g) does not exist, which is equivalent to the fact that $f+g \notin P$.

Two functions $f, g \in P$ will be called *comparable* if the limit (f, g) exists. In the same way we shall say that a function and a subset of P, or two subsets of P, are comparable.

But there exist linear subsets $E \subset P$, for example the set of trigonometric polynomials

$$T = \left\{u\colon u = \sum_{\mu=1}^{q} a_\mu e^{i(\lambda_\mu, x)},\ q < +\infty,\ \lambda_\mu \in R^n\right\} \subset P.$$

The following theorem holds (see [1]):

THEOREM 1. *If E is a linear subset of P, $f_n \in E$, $\|f_n - f_m\| \to 0$, then for $f \in M$ such that $\Box f-f_n \Box \to 0$ we have*

$$f \in P, \quad f \text{ is comparable with } E, \quad \|f - f_n\| \to 0.$$

Hence, for linear subsets $E \subset P$ the closure $\bar{E}$ is linear, too, and $\bar{E} \subset P$, i.e. each functional Hilbert space with the scalar product (4) is of the form

$$\mathfrak{H} = \bar{E}/\sim$$

where E an arbitrary linear subset of P. In what follows we shall use the notation

$$B^2(R^n) = \bar{T}/\sim\ = \overline{\text{A.P.}(R^n)}/\sim$$

where A.P.(R^n) denotes the set of almost periodic functions (for more information on A.P. functions see [2]).

The question arises: are there other nonseparable Hilbert spaces $\mathfrak{H}$: $B^2(R^n) \neq \mathfrak{H} \subset \mathfrak{M}$? There is a natural method of constructing linear subsets $E \subset P$. We take a set of complex valued functions u_λ, $\lambda \in \Lambda$, satisfying the following two conditions:

1. $u_\lambda \in P$,
2. $u_\lambda, u_{\lambda'}$ for each $\lambda, \lambda' \in \Lambda$ are comparable.

Then we put

$$E = \left\{u\colon u = \sum_{\mu=1}^{q} a_\mu u_{\lambda_\mu},\ q < +\infty,\ \lambda_\mu \in \Lambda,\ a_\mu \text{ complex}\right\}.$$

Let us introduce some definitions relevant to this method. A function u_λ will be called a *weak eigenfunction* and a complex number λ a *weak eigenvalue* of the operator $L = -\Delta + q$, $\sup_{x\in R^n} |q(x)| < +\infty$, iff:

(a) $u \in C^2(R^n) \cap P$, $\|u\| \neq 0$,
(b) $\sup_{R^n} (|u_\lambda|, |Du_\lambda|) < +\infty$,
(c) $\|(L-\lambda)u_\lambda\| = 0$.

The following lemmas hold (see [3]):

LEMMA 1. *If $u_\lambda, u_{\lambda'}$, are eigenfunctions, $\lambda \neq \lambda'$, $\operatorname{Im}\lambda = \operatorname{Im}\lambda' = 0$, then $(u_\lambda, u_{\lambda'}) = 0$, i.e. they are comparable.*

LEMMA 2. *If*

$$\varrho \in C^2(R^n) \cap P, \quad \|D^\alpha \varrho\| = 0, \quad |\alpha| \leqslant 2, \quad \|\varrho\| \neq 0, \tag{5}$$

then

$$\varrho(x) e^{i(\lambda_\mu, x)} = u_{|\lambda_\mu|^2}, \quad \mu \in R^n,$$

are weak eigenfunctions of the operator $L = -\Delta$.

Using these lemmas one can prove the following two theorems (see [3]):

THEOREM 2. *There exist nonseparable Hilbert spaces $\mathfrak{H}_\tau \subset \mathfrak{M}$, $0 < \tau < 1$, with the scalar product* (4), *which are spectral for $-\Delta + q$, $q \in$ A.P.(R^n), such that*

1. *$B^2(R^n)$ and $\mathfrak{H}_\tau$, $0 < \tau < 1$, are noncomparable,*
2. *$\mathfrak{H}_\tau$ and $\mathfrak{H}_{\tau'}$, $\tau \neq \tau'$, are noncomparable.*

THEOREM 3. *There exist nonseparable Hilbert spaces $\mathfrak{N}_\tau \subset \mathfrak{M}$, $0 < \tau < 1$, with the scalar product* (4), *which are spectral for $-\Delta + q$, $q \in$ A.P.(R^n), such that*

1. *for $0 < \tau < 1$, $B^2(R^n) \subset \mathfrak{N}_\tau$ and $\mathfrak{N}_\tau \ominus B^2(R^n) = Q_\tau$ is nonseparable,*
2. *Q_τ and $Q_{\tau'}$, $\tau \neq \tau'$, are noncomparable.*

The Hilbert space $\mathfrak{H}$ is said to be *spectral for the differential operator L* if L can be densely defined in $\mathfrak{H}$ in such a way that it becomes essentially self-adjoint.

II. The spectral cutting problem

Let $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$ and $\varrho \in C^2(\boldsymbol{R}^n)$. Let us consider the differential operator $L = -\Delta + q$ and the operator $L_{\mathrm{cut}} = -\Delta + \varrho q$ with the potential "cut by ϱ". By A and A_{cut} we shall denote the unique self-adjoint extensions in $L^2(\boldsymbol{R}^n)$ of the operators L and L_{cut} with domains $D_L = D_{L_{\mathrm{cut}}} = C_0^2(\boldsymbol{R}^n)$. Now the spectral cutting problem may be formulated as follows: for which cut ϱq do the spectra $\sigma(A)$, $\sigma(A_{\mathrm{cut}})$ satisfy the inclusion

$$\sigma(A) \subset \sigma(A_{\mathrm{cut}})\,?$$

Let ϱ satisfy the condition (5) of Lemma 2. For such ϱ we introduce the sets

$$\mathrm{A.P.}^k(\boldsymbol{R}^n) = \{u\colon u \in \mathrm{A.P.}(\boldsymbol{R}^n),\ D^\alpha u \in \mathrm{A.P.}(\boldsymbol{R}^n),\ |\alpha| \leqslant k\},$$
$$\mathrm{A.P.}_\varrho^k(\boldsymbol{R}^n) = \{u\colon u = \varrho v,\, v \in \mathrm{A.P.}^k(\boldsymbol{R}^n)\}.$$

We shall solve the spectral cutting problem operating with spectral representations of the operators L, L_{cut} in the following spaces:

$$L^2(\boldsymbol{R}^n), \qquad B^2(\boldsymbol{R}^n), \qquad \mathfrak{N} = \overline{\mathrm{A.P.}\varrho(\boldsymbol{R}^n)}/\sim.$$

The following theorem holds:

THEOREM 4. *The operator* $L = -\Delta + q$, $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$, $D_L = \mathrm{A.P.}^2(\boldsymbol{R}^n)$, *is in the space* $B^2(\boldsymbol{R}^n)$ *essentially self-adjoint.*

The proof was first given in the papers [5], [6]. Different generalizations of this theorem may be found in [9], [7], [8] and [11]–[13]. In the papers [9], [7], [8], the potential q with local square-integrable singularities is considered. The papers [11]–[13] contain a generalization of the theorem to the case of almost periodic elliptic operators.

The unique self-adjoint extension of L, whose existence is guaranteed by Theorem 4 will be denoted by $\mathfrak{A}$. For the space $\mathfrak{N}$ we have the following, similar, theorem:

THEOREM 5. *If ϱ satisfies the conditions* (5), $\|\varrho(1-\varrho)\| = 0$, $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$, *then the operators* $L = -\Delta + q$, $L_{\mathrm{cut}} = -\Delta + \varrho q$, $D_L = D_{L_{\mathrm{cut}}} = \mathrm{A.P.}_\varrho^2(\boldsymbol{R}^n)$ *are essentially self-adjoint.*

In what follows we shall denote by $\underset{1}{\mathfrak{A}}$ and $\underset{1}{\mathfrak{A}}{}_{\mathrm{cut}}$ the unique self-adjoint extensions of the operators L and L_{cut} in $\mathfrak{N}$.

The following theorem is true.

THE CUTTING THEOREM. *If ϱ satisfies the conditions* (5), $\|\varrho(1-\varrho)\| = 0$ *and there exist numbers* $\alpha \geqslant \beta > 0$ *such that for each* $u \in \mathrm{A.P.}(\boldsymbol{R}^n)$

$$\alpha\|u\| \geqslant \|u\| \geqslant \beta\|u\|$$

then

$$\sigma(A) \subset \sigma(A_{\mathrm{cut}}).$$

The proof is based on the following chain of equalities and one inclusion:

$$\sigma(A) = \sigma(\mathfrak{A}) = \sigma(\underset{1}{\mathfrak{A}}) = \sigma(\underset{1}{\mathfrak{A}}{}_{\mathrm{cut}}) \subset \sigma(A_{\mathrm{cut}}).$$

The first equality is the following known theorem:

THEOREM 6. *If $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$ then $\sigma(A) = \sigma(\mathfrak{A})$.*

The proof of this theorem was first given for $\boldsymbol{R}^1$, q continuous and periodic, in the papers [4]–[6]. For $\boldsymbol{R}^3$ and q periodic with local square-integrable singularities the proof may be found in [7], [8]. The paper [11] contains the proof of the theorem in the case of almost periodic elliptic operators.

III. The method of integral operators

In this section we shall be concerned with the form of the operator $(\mathfrak{A}-\lambda)^{-1}$, $\lambda \notin \sigma(\mathfrak{A})$. In order to avoid iterations we will consider here only the case of $\boldsymbol{R}^3$.

The basic fact for the spectral theory of elliptic operators in $L^2(\boldsymbol{R}^3)$ is this:

$$((A-\lambda)^{-1}f)(x) = (f, H(x,\cdot,\lambda))_{L^2} = \int_{\boldsymbol{R}^3} H(x,y,\lambda)f(y)\,dy \tag{6}$$

where $H(x,y,\lambda)$ is the appropriate fundamental solution of Carleman type.

In our nonseparable spaces the situation is different. It can be shown that there does not exist any function $G(x,y,\lambda)$ such that for each $F \in B^2(\boldsymbol{R}^n)$:

$$(\mathfrak{A}-\lambda)^{-1}F = \big((F, G(x,\cdot,\lambda))\big), \quad \big(G(x,\cdot,\lambda)\big) \in B^2(\boldsymbol{R}^3).$$

But it can be shown that for $q \in \mathrm{A.P.}(\boldsymbol{R}^3)$ the following equality is true:

$$(\mathfrak{A}-\lambda)^{-1}F = \Big(\int_{\boldsymbol{R}^3} H(x,y,\lambda)f(y)\,dy\Big)$$

where $F = (f) \in B^2(\boldsymbol{R}^3)$ and $H(x,y,\lambda)$ is the resolvent kernel appearing in (6). This fact follows from some nice properties of $H(x,y,\lambda)$. To formulate them, let us denote by $\underset{0}{\mathfrak{A}}$ the self-adjoint extension of the operator $-\Delta$ in $B^2(\boldsymbol{R}^3)$, and let us put

$$e(|x-y|, \lambda^2) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \quad \text{for} \quad \operatorname{Im}\lambda > 0.$$

The quantity $e(|x-y|,\lambda^2)$ vanishes very rapidly if $|x-y| \to \infty$; hence

$$(\underset{0}{\mathfrak{A}}-\lambda^2)^{-1}F = \Big(\int_{\boldsymbol{R}^3} e(|x-y|,\lambda^2)f(y)\,dy\Big), \quad F = (f) \in B^2(\boldsymbol{R}^n).$$

Consider the general case of the potential q not necessarily almost periodic satisfying the condition

$$\sup_{\tau \in \boldsymbol{R}^3} \int_\Omega q(x+\tau)^2\,dx < +\infty, \quad \Omega \text{ a bounded set.}$$

Then the resolvent kernel $H(x, y, \lambda^2)$, i.e. the solution of the following integral equation

$$H(x, y, \lambda^2) = e(|x-y|, \lambda^2) - \int_{R^3} e(|x-s|, \lambda^2) q(s) H(s, y, \lambda^2) ds$$

has the properties expressed in the following theorem.

THEOREM 7. *If*

$$\Lambda(\omega) = \Big\{z\colon z = \lambda^2,\ \operatorname{Im}\lambda > \omega \sup_{\tau\in R^3} \int_{|x|<1} |q(x+\tau)|^2 dx\Big\}$$

and $\omega > 0$ is sufficiently large, then for $\lambda^2 \subset \Lambda(\omega)$, $x, y \in R^3$, $0 < k < \operatorname{Im}\lambda - \omega$, the following estimates hold:

$$|H(x, y, \lambda^2) - e(|x-y|, \lambda^2)| \leqslant L(\omega, k) \frac{e^{-k|x-y|}}{|x-y|^{1/2}} \tag{7}$$

where the number $L(\omega, k)$ depends only on ω and k,

$$|H(x+\tau, y, \lambda^2) - H(x, y-\tau, \lambda^2)| \leqslant L(k, \omega, \Omega) \Big(\sup_{p\in R^3} \int_{\Omega} |q(x+\tau+p) - q(x+p)|^2 dx\Big)^{1/2} \frac{e^{-k|x-y+\tau|}}{|x-y+\tau|^{1/2}} \tag{8}$$

where $\tau \in R^3$ and the number $L(k, \omega, \Omega)$ depends only on k, ω, Ω.

The proof of this theorem may be found in [8] and [9].

From the estimate (7) it follows that the operator

$$R_{\lambda^2} F = \Big(\int_{R^3} H(x, y, \lambda^2) f(y) dy\Big), \quad F = (f) \in \mathfrak{M}$$

is well defined in $\mathfrak{M}$ and is bounded.

Moreover, from (8) we obtain that for $q \in \text{A.P.}(R^3)$ we have:

$$R_{\lambda^2}(\text{A.P.}(R^3)) \subset \text{A.P.}(R^3)$$

and for $F \in B^2(R^3)$

$$R_{\lambda^2} F = (\mathfrak{A} - \lambda^2)^{-1} F. \tag{9}$$

IV. Applications of the integral method

The first application to mention here is the proof of the fact that the operator $L = -\Delta + q$ for some q with singularities such that $(q) \in B^2(R^3)$ may be made self-adjoint in $B^2(R^3)$.

Suppose that for each bounded measurable set $\Omega \subset R^3$

$$\sup_{\tau\in R^3} \int_{\Omega} |q(x+\tau)|^2 dx < +\infty \quad \text{and} \quad q \in \operatorname{Lip}_\alpha(R^3 \setminus N)$$

where $0 < \alpha < 1$ and N denotes a set of isolated points, curves and surfaces. Moreover, let $D(R^3 \setminus N)$ be the set of complex valued functions defined by the following conditions:

1. $D(R^3 \setminus N) \subset C^2(R^3 \setminus N)$,
2. For each $\vartheta \in C_0^2(R^3)$ and $u, v \in D(R^3 \setminus N)$

$$L(u\vartheta) \in L^2(R^3), \qquad \int_{R^3} u\vartheta \overline{\Delta(v\vartheta)} dx = \int_{R^3} \Delta(u\vartheta) \overline{v\vartheta} dx.$$

Then the following theorem holds.

THEOREM 8. *If for each $\varepsilon > 0$ the set of points $\tau \in R^3$ such that*

$$\sup_{p\in R^3} \int_{\Omega} |q(x+p+\tau) - q(x+p)|^2 dx < \varepsilon$$

is relatively dense in R^3, then the operator $L = -\Delta + q$, $D_L = \{U\colon U = (u), u \in D\}$, where $D = \{u\colon u \in D(R^3 \setminus N) \cap \text{A.P.}(R^3), Lu \in \text{A.P.}(R^3)\}$, is essentially self-adjoint in $B^2(R^3)$.

Now we shall consider the problem of eigenfunctions. For $q \in \text{A.P.}(R^3)$ we would like to know if each eigenelement E, $(\mathfrak{A} - \lambda)E = 0$, may be represented by classical eigenfunctions. That means, we ask whether we have $E = \sum E_i$, $(\mathfrak{A} - \lambda)E_i = 0$ and $E_i = (e_i)$ such that $e_i \in C^2(R^3)$, $-\Delta e_i + q e_i = \lambda e_i$, i.e. $e_i(x)$ is a classical eigenfunction.

There holds the following theorem.

THEOREM 9. *If $q(x)$ is periodic and continuous and $(\mathfrak{A} - \lambda)E = 0$, then $E = \sum E_\mu$, $E_\mu = \big(e^{i(\lambda_\mu, x)} v(x)\big)$ where $e^{i(\lambda_\mu, x)} v(x)$ is a classical eigenfunction and $v(x)$ satisfies the same periodicity conditions as $q(x)$.*

An analogous theorem holds for periodic q with singularities. For the proof see [6], [7], [8]. For almost periodic potentials only the following theorem is known (see [3]):

THEOREM 10. *If $q \in \text{A.P.}(R^3)$, $(\mathfrak{A} - \lambda)E = 0$, $E = (e)$ where $e \in \text{A.P.}(R^3)$, then e is a classical eigenfunction.*

Finally, using the integral method we may obtain very strong perturbation properties of the spectrum $\sigma(A) = \sigma(\mathfrak{A})$ (see [6], [3], [7], [8]). One may obtain the estimate of the following type:

$$\delta\big(\sigma(\overset{(1)}{\mathfrak{A}}), \sigma(\overset{(2)}{\mathfrak{A}})\big) < L\|q^2 - q^1\| \tag{10}$$

where $\overset{(i)}{\mathfrak{A}}$ denotes the self-adjoint extension in $B^2(R^n)$ of $-\Delta + q^i$, and δ denotes the Hausdorff distance. For instance, in the case of the operator $-d^2/dx^2 + q$, putting

$$\delta_{(\alpha,\beta)}(A, B) = \max\big(\sup_{x\in A\cap(\alpha,\beta)} d(x, B), \sup_{x\in B\cap(\alpha,\beta)} d(x, A)\big)$$

where $d(x, A)$ denotes the distance of the point x to the set A, we have the following theorem:

THEOREM 11. *If* $q^i(x+a^i n) = q^i(x)$, $a^i < a$, $\int_0^a |q^i|^2 dx < N$ $(i = 1, 2)$, *then*

$$\delta_{(\alpha,\beta)}\big(\sigma(\overset{(1)}{\mathfrak{A}}), \sigma(\overset{(2)}{\mathfrak{A}})\big) < \sqrt{a}\,\frac{2N+1}{1-e^{-\alpha}} M(\alpha, \beta)\|q^1-q^2\|$$

where

$$M(\alpha, \beta) = \sup_{\alpha\leqslant\lambda\leqslant\beta} \big(|\lambda|^2+1\big)^{-1/2}.$$

The perturbation properties of the type (10) in the case of periodic q follow from some estimates of classical eigenfunctions and from the fact that the eigenelements of $\mathfrak{A}$ span the space $B^2(\boldsymbol{R}^3)$.

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THE SPECTRAL CUTTING PROBLEM

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The Schrödinger operator $L = -\Delta+q$ with an almost periodic potential q (in symbols: $q \in \mathrm{A.P.}(\boldsymbol{R}^n)$) is of interest in the quantum theory of disordered systems (alloys, liquids).

Let $\varrho(x)$ denote a function for which the set $\operatorname{supp}\varrho \cap \operatorname{supp}(\varrho-1)$ is not too large; we shall call the operator

$$L_{\text{cut}} = -\Delta+\varrho q$$

the *Schrödinger operator* with potential "cut by ϱ". By A and A_{cut} we shall denote the unique self-adjoint extensions in $L^2(\boldsymbol{R}^n)$ of the operators L and L_{cut} defined in $D_L = D_{L_{\text{cut}}} = C_0^2(\boldsymbol{R}^n)$. Now, the spectral cutting problem may be formulated as follows: for which cut ϱq do the spectra $\sigma(A)$, $\sigma(A_{\text{cut}})$ satisfy the inclusion

$$\sigma(A) \subset \sigma(A_{\text{cut}})? \tag{1}$$

This inclusion has the following physical interpretation. Suppose that an electron is moving in the medium defined by the potential q. Then the set of numbers $\sigma(A)$ represents the admissible levels of energy. In other words, $\sigma(A)$ represents the energetical characteristics of an electron moving in the medium considered. The potential ϱq defines the medium arising as the portion cut from the original medium defined by q. The inclusion (1) means that the electron in a piece cut from the medium may have the same energy levels as one moving in the whole medium.

We will use the following norms:

$$\Box f \Box^2 = \limsup_{T\to\infty} \frac{1}{T^n} \int_{|x|<T} |f|^2 dx, \tag{2}$$

$$\|f\|^2 = \lim_{T\to\infty} \frac{1}{T^n} \int_{|x|<T} |f|^2 dx, \tag{3}$$

$$\|f\|_{L^2}^2 = \int_{\boldsymbol{R}^n} |f|^2 dx, \tag{4}$$

where $f(x)$, $x \in \boldsymbol{R}^n$, is a complex-valued function.