we may state the following theorem, which is due to C. J. Earle and R. S. Hamilton [3] for Banach spaces.

THEOREM 8.3. *If $\mathscr{E}$ is sequentially complete, any $f \in \mathrm{Hol}(D, \mathscr{E})$ such that $f(D) \subset\subset D$ has a unique fixed point.*

Remark. The proof of the above theorem can be carried out (as in [3]) using the Carathéodory differential metric γ_D and its integrated form $\tilde{c}_D$, instead of $\varkappa_D$ and $\mathfrak{k}_D$.

Bibliography

[1] T. J. Barth, *The Kobayashi distance induces the standard topology*, Proc. Amer. Math. Soc. 35 (1972), 439–441.

[2] R. Braun, W. Kaup and H. Upmeier, *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, to appear.

[3] C. J. Earle and R. S. Hamilton, *A fixed point theorem for holomorphic mappings*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R. I., Vol. 16, 61–65.

[4] L. A. Harris, *Schwarz's lemma in normed linear spaces*, Proc. Nat. Acad. Sci. USA 62 (1969), 1014–1017.

[5] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Dekker, New York 1970.

[6] —, *Some remarks and questions concerning the intrinsic distance*, Tohoku Math. J. 25 (1973), 481–486.

[7] —, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. 82 (1976), 357–416.

[8] P. Noverraz, *Fonctions plurisousharmoniques et analytiques dans les espaces vectoriels topologiques*, Ann. Inst. Fourier (Grenoble) 19 (1969), 419–493.

[9] —, *Pseudo-convexité, convexité polynomiale et domaines d'holomorphie en dimension infinie*, North-Holland, Amsterdam 1973.

[10] H. J. Reiffen, *Die differentialgeometrischen Eigenschaften der invarianten Distanzfunktion von Carathéodory*, Schr. Math. Inst. Univ. Münster, No. 26 (1963).

[11] —, *Die Carathéodorysche Distanz und ihre zugehörige Differentialmetrik*, Math. Ann. 161 (1965), 315–324.

[12] W. Rinow, *Die innere Geometrie der metrischen Räume*, Springer-Verlag, Berlin 1961.

[13] H. L. Royden, *Remarks on the Kobayashi metric*, *Several complex variables*, II (Proc. Internat. Conf. University of Maryland, College Park, Md. 1970), Lecture Notes in Math., vol. 185, Springer-Verlag, Berlin 1971, 125–137.

[14] —, *The extension of regular holomorphic maps*, Proc. Amer. Math. Soc. 43 (1974), 306–310.

[15] E. Thorp and R. Whitley, *The strong maximum modulus theorem for analytic functions into a Banach space*, ibid. 18 (1967), 640–646.

[16] E. Vesentini, *Variations on a theme of Carathéodory*, Ann. Scuola Norm. Sup. Pisa, to appear.

[17] —, *Automorphisms of the unit ball*, *Several complex variables*, Lecture Notes in Math., Springer-Verlag, Berlin 1978.

[18] J. P. Vigué, *Le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe. Application aux domaines bornés symétriques*, Ann. Sci. Ec. Norm. Sup. (4) 9 (1976), 203–282.

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THE HOLOMORPHIC FUNCTIONAL CALCULUS AS AN OPERATIONAL CALCULUS

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1. Introduction

1.1. I have been asked to speak on the ideology of the holomorphic functional calculus. And I have accepted, even though I do not believe that such a unique ideology exists. These talks, and these notes are an opportunity for presenting my own ideas on the subject.

These notes lay the stress on the h.f.c., they are related, but only partly, with [47], where non-Banach algebras are stressed.

While preparing this written text, I became increasingly aware that my ideology was that of an operational calculus. The fact that this involves holomorphic functions is a good surprise. The expression "holomorphic functional calculus" (h.f.c.) must be taken to mean "operational calculus involving holomorphic functions".

I must also mention the fact that Gelfand's papers ([11], [12], [13], [14]) were not available in Belgian libraries, or at the Institut Poincaré in Paris, even in 1953 when I completed research on my first paper [42]. This was a consequence of the disruption due to the war, and of lack of money in the post-war period.

I knew most of Gelfand's results. I had attended talks by mathematicians who had read Gelfand. I had read the Mathematical Reviews. But my knowledge was indirect and incomplete. This had an effect on my personal ideology at the outset, also later since I knew how much could be proved about Banach algebras without using Gelfand's results.

1.2. The following is the ideology of many mathematicians, and flows directly out of Gelfand's results.

A commutative Banach algebra is semi-simple modulo the radical. The radical is messy and of minor importance. A semi-simple Banach algebra is a function algebra.

Let X be a compact space $\mathscr{A} \subseteq C(X)$ a function algebra having X as structure space. Let $U \subseteq C^n$ be a set and f a continuous function on U. The function f operates on $\mathscr{A}$ if $f(a_1, \ldots, a_n) \in \mathscr{A}$ whenever $a_1, \ldots, a_n \in \mathscr{A}$ and

$$\{(a_1(x), \ldots, a_n(x)) |\ x \in X\} \subseteq U.$$

The fact that a holomorphic function operates on $\mathscr{A}$ in the above sense is, in a way, the h.f.c.

With this ideology, the fact that the h.f.c. mapping $f \to f[a]$ is an algebra homomorphism is of secondary importance. Arens and Calderón [7] do not mention that property. I have never seen a translation of G. E. Šilov's [27] original paper on the subject and cannot say for sure whether he mentions that fact. Certainly, in 1960 [28] he gives a short summary of the theory of analytic functions in a normed ring and does not inform the reader that the h.f.c. mapping is a homomorphism.

1.3. My viewpoint is different. Assume that $\mathscr{A}$ is a topological algebra, which may be commutative, and may have a unit. Let $a_1, \ldots, a_n$ be elements of $\mathscr{A}$. We can add, multiply the a_i. We can take inverses when they exist. We can take limits of rational functions. Try to organize these operations.

If $a_1, \ldots, a_n$ are commuting elements of $\mathscr{A}$, if P is a polynomial in n indeterminates, we can define $P(a_1, \ldots, a_n)$.

We start out with some operations, which we assume can be perfomed. In a first phase, these would be taking inverses of polynomials, limits of rational functions. But later it appears that other operations are worth studying.

We would like to find a unital algebra $\mathcal{O}$, containing elements $z_1, \ldots, z_n$, on which the operations considered can be performed. And each time we have an algebra $\mathscr{A}$ with unit, and elements $a_1, \ldots, a_n$ on which these operations can be performed, we would like to find a unique homomorphism $\mathcal{O} \to \mathscr{A}$, mapping z_i on a_i and unit on unit. In other words, we would look for the solution of a universal problem.

This search for universal solutions is not always the most appropriate. The operations we investigate may be such that no universal solution exists. Or the universal solution may exist, but be unmanageable. A non-universal solution may give more information.

The viewpoint described above can be called the search for an operational calculus. It turns out that the operational calculus algebra $\mathcal{O}$ is often an algebra of holomorphic functions. We find a holomorphic functional calculus.

1.4. Once we stop looking for universal solutions, uniqueness of the operational calculus is less important than its naturality.

Let $\mathscr{B}$ be a new topological algebra with unit, and $\varphi\colon \mathscr{A} \to \mathscr{B}$ a morphism (a continuous homomorphism mapping unit on unit). Let $b_i = \varphi a_i$. Whatever operations can be performed on a_i can also be performed on b_i. A morphism $\mathcal{O} \to \mathscr{B}$ must therefore exist, which maps z_i on b_i.

The operational calculus we have constructed is natural if this morphism is the composition of the operational calculus morphism $\mathcal{O} \to \mathscr{A}$ and $\varphi\colon \mathscr{A} \to \mathscr{B}$.

1.5. In the following statement by B. Mitjagin, S. Rolewicz, and W. Żelazko [19], the functional calculus is taken with the meaning I prefer.

Let $\mathscr{A}$ be a commutative Fréchet algebra (locally convex). Assume that $\sum_0^\infty c_n a^n$ converges whenever $\sum_0^\infty c_n z^n$ is an entire function. Then $\mathscr{A}$ is locally multiplicatively convex.

One can say (Mitjagin, Rolewicz and Żelazko do) that $f(z) = \sum_0^\infty c_n z^n$ operates on a if $\sum_0^\infty c_n a^n$ converges. In a way, the "entire functions" are sequences of coefficients c_n such that $|c_n|^{1/n} \to 0$.

1.6. I differ ideologically from the majority on another point. For the majority, Banach algebra theory is the center around which the h.f.c. revolves.

My original motivation was trying to solve problems about operators. And not all operators have compact spectra. It is not too difficult also to construct an operator with an empty spectrum, i.e. such that $a - sI$ has an inverse for all complex numbers s.

Operators with non compact spectra, or with empty spectra fall outside the scope of Banach algebra theory.

1.7. I have not been as successful as I like with that part of the program. I have not been as successul as I thought I was. I have obtained an operational calculus involving elements of the center of a topological algebra, making hypotheses about what can be called a resolvent, or an asymptotic resolvent of $(a_1, \ldots, a_n)$.

It is not clear that the hypotheses on the resolvent are easy to check. But that is not the worst.

My original motivation was the theory of partial differential equations. All the operational calculi I have obtained are functional calculi, function algebras are commutative. My operational calculi can only apply to commuting operators.

I can study partial differential equations with constant coefficients. I obtain results which are essentially equivalent with the n-dimensional Heaviside calculus, but the Heaviside calculus existed before my results. At present at least, partial differential equations theory is not concerned, not mainly concerned with equations with constant coefficients.

I may be dismissing too fast the operators which can be built up from commuting operators, e.g. operators with separating variables. But this class of operators is not stable under perturbations.

1.8. Function algebras are commutative. The best application I know of my results are due to I. Cnop [8] and J. P. Ferrier [9], [10], and apply to algebras of holomorphic functions satisfying growth conditions.

1.9. This is the place to speak of non-commutative operational calculus. An operational calculus is an attempt to the solution of a universal problem. The attempt can only be useful if the algebra we find has properties which are not apparent when just looking at the operations postulated as applicable to the given elements.

In the commutative case, the h.f.c. has proved useful because the algebras of holomorphic functions on Stein manifolds are understood well enough. Of course, this understanding has come in the last thirty years.

It may be that the useful non commutative operational calculi will involve algebras whose structure is not yet understood but will be understood in the next thirty years.

J. L. Taylor presents a very abstract program ([33], [34], [35]). But it is not unlikely that one would find, at the end of the road, algebras which have the good properties.

E. Nelson adopts another approach to the problem (cf. the paper by E. Albrecht in the same proceedings [2]). He changes the multiplication defined on $\mathscr{A}$ and makes it commutative. This would be interesting, if the spectra were computable, and not too large.

It seems unfortunately that $\mathrm{sp}(e_{12}, e_{21})$ in M_2 is not easy to find, if M_2 is the algebra of 2×2 matrices,

$$e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

L. Hörmander and other mathematicians are busy carrying out computations with partial differential operators, pseudo-differential operators, and *tutti quanti.* They are finding properties of operators on function spaces, and are localizing, even microlocalizing these properties. I believe that their work is the beginning of an operational calculus involving multiplication operators and derivations along vector fields.

2. Some pre-existing results

2.1. A functional calculus exists for self-adjoint operators on Hilbert spaces and continuous functions on their spectra.

If a is such an operator, $\|a^2\| = \|a\|^2$. Therefore

$$\max\{|s| \mid s \in \mathrm{sp}\, a\} = \lim \|a^n\|^{1/n} = \|a\|.$$

Also, $a+iI$ has an inverse, the spectrum of a is real, if $z = x+iy$, $y \neq 0$,

$$a - zI = y\left[\frac{a-xI}{y} - iI\right]$$

has an inverse. The spectral mapping theorem is easy to prove for polynomials of a single variable, $P(\mathrm{sp}\, a) = \mathrm{sp}\, P(a)$. If $P \in \mathbf{R}[x]$, $P(a)$ is self-adjoint

$$\|P(a)\| = \max\{|s| \mid s \in \mathrm{sp}\, P(a)\} = \max\{|P(s)| \mid s \in \mathrm{sp}\, a\}.$$

The Stone–Weierstrass theorem and completion allow one to conclude.

2.2. The Gelfand holomorphic functional calculus applies to elements of a Banach algebra.

Let $\mathscr{A}$ be a Banach algebra with unit, and $a \in \mathscr{A}$. Let U be a neighbourhood of $\mathrm{sp}\, a$, let V be a neighbourhood of $\mathrm{sp}\, a$ with rectifiable boundary and relatively compact in U. Let f be holomorphic on U. Let

$$f[a] = \frac{1}{2\pi i} \int_{\partial V} f(s)(s-a)^{-1} ds.$$

It is clear that $f[a]$ does not depend on the choice of V, the class of homology of ∂V is well determined in $U \setminus \mathrm{sp}\, a$. If we put on the algebra $\mathcal{O}(U)$ of holomorphic functions on U the compact open topology, the mapping $f \to f[a]$ is continuous. A calculus of residues shows that $f[a] = P(a)Q(a)^{-1}$ if $f(z) = P(z)/Q(z)$ is a rational function. So, this continuous mapping is a homomorphism on a dense subalgebra, it is a homomorphism.

2.3. The following observation looks trivial. But if we combine it with the Oka–Cartan results and the Arens–Calderón trick, we get the full h.f.c. (see Section 3.7).

Let $f(z) = \sum_0^\infty c_n z^n$ be holomorphic on a disc of radius larger than $\|a\|$. Then $\sum_0^\infty c_n a^n$ converges. We can define

$$f[a] = \sum_0^\infty c_n a^n.$$

This is, of course, a good "substitution" of a into f.

2.4. Both the Gelfand h.f.c. and the baby h.f.c. of Section 2.3 have n-dimensional analogues.

Let f be holomorphic on a neighbourhood U of $\mathrm{sp}\, a_1 \times \ldots \times \mathrm{sp}\, a_n$. Let $V_1, \ldots, V_n$ be neighbourhoods of $\mathrm{sp}\, a_1, \ldots, \mathrm{sp}\, a_n$ respectively, with rectifiable boundaries, and such that $V_1 \times \ldots \times V_n$ is relatively compact in U. We can define

$$f[a_1, \ldots, a_n] = \int_{\partial V_1 \times \cdots \times \partial V_n} f(s_1, \ldots, s_n)(s_1 - a_1)^{-1} \ldots (s_n - a_n)^{-1} ds_1 \ldots ds_n.$$

Let f be holomorphic on a polydisc of polyradius $(\varrho_1, \ldots, \varrho_n)$ with each $\varrho_i > \|a_i\|$. Assume that

$$f(z_1, \ldots, z_n) = \sum c_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

We can define

$$f[a_1, \ldots, a_n] = \sum c_{k_1 \ldots k_n} a_1^{k_1} \ldots a_n^{k_n}.$$

2.5. The Heaviside calculus, and its n-dimensional form given by Leray in 1950 are among the premices of my work. The following is a variation on that theme:

Γ is a tube in C^n, i.e. $\Gamma = U+iR^n$ where U is open and convex in R^n. Let $\mathcal{O}_1(\Gamma)$ be the holomorphic functions which satisfy an estimate

$$|u(z)| \leqslant M(V)(1+|z|^2)^{k(V)/2}$$

on each closed set $V+iR^n$ at a positive distance from the boundary of $U+iR^n$.

The inverse Laplace transform $\mathcal{L}^{-1}u$ of $u \in \mathcal{O}_1(\Gamma)$ is the product of $e^{\xi x}$ and the inverse Fourier transform of the restriction of u to the vertical $\xi+iR^n$. This is a distribution, because u has polynomial growth on the vertical. The product does not depend on ξ. And Paley–Wiener show that the support of this distribution is contained in $-C$ where C is the polar cone to the asymptotic cone of Γ.

We call $u\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ this distribution. The mapping $u \to u(\partial/\partial x)$ is a continuous homomorphism of $\mathcal{O}_1(\Gamma)$ into a convolution algebra of distributions. This homomorphism maps the constant 1 onto the Dirac measure δ, and maps z_i onto $\partial\delta/\partial x_i$.

2.6. E. R. Lorch [17], [18] studies $\mathcal{A}$-holomorphic functions when $\mathcal{A}$ is a Banach algebra. These are defined on an open subset of $\mathcal{A}$, $\mathcal{A}$-valued, and are Fréchet differentiable with $\mathcal{A}$-linear differential. It is clear that the h.f.c. defines Lorch holomorphic functions.

Lorch's research is independent of Gelfand's. His results are published after Gelfand's. The material covered is not exactly the same.

But the main reason why it must be mentioned here is that his papers were available in Western European libraries.

3. The h.f.c. and Banach algebras

3.1. The functional calculus for functions near to products of spectra is not a final result.

Instead of considering $(a_1, \ldots, a_n)$ we could have considered n linearly independent linear combinations $(b_1, \ldots, b_n)$ of $(a_1, \ldots, a_n)$. We would like the operational calculus in $(b_1, \ldots, b_n)$ to be an immediate transform of the operational calculus in $(a_1, \ldots, a_n)$.

And this is not the case. If $b = Ta$ where T is an invertible linear transformation, we define $g[b]$ when g is holomorphic near to $\mathrm{sp}\, b_1 \times \ldots \times \mathrm{sp}\, b_n$. Letting $f = g \circ T$, this is equivalent to defining $f[a]$ when f is holomorphic near to $T^{-1}(\mathrm{sp}\, b_1 \times \ldots \times \mathrm{sp}\, b_n)$. This is not (in general) the direct product of n subsets of C.

3.2. The most reasonable thing here is to try to increase the operational calculus by considering next to $a_1, \ldots, a_n$ a finite number of polynomials $P_1(a), \ldots, P_N(a)$ in $a_1, \ldots, a_n$.

The h.f.c. for functions holomorphic near to direct products gives a homomorphism $f(z, y) \to f[a, P(a)]$,

$$\mathcal{O}\left(\prod \mathrm{sp}\, a_i \times \prod \mathrm{sp}\, P_j(a)\right) \to \mathcal{A}$$

which maps unit on unit, z_i on a_i, and y_j on $P_j(a)$ (if z_i stands for the mapping $(s, t) \to s_i$ and y_j for $(s, t) \to t_j$, $C^{n+N} \to C$). This homomorphism vanishes on the ideal α generated by the functions $y_j - P_j(z)$ and induces therefore a homomorphism

$$\mathcal{O}\left(\prod \mathrm{sp}\, a_i \times \prod \mathrm{sp}\, P_j(a)\right) / \alpha \to \mathcal{A}.$$

The operational calculus algebra is the quotient algebra considered above, or a direct limit of such algebras.

3.3. Fortunately, results of Oka and Cartan show that the operational calculus algebra is the algebra of holomorphic functions near to a compact subset of C^n.

PROPOSITION. *Let $U_1, \ldots, U_n$, $V_1, \ldots, V_N$ be open subsets of C. Let $P_1, \ldots, P_N$ be polynomials in n variables. Let*

$$\Delta = \{(z_1, \ldots, z_n) |\ z_i \in U_i,\ P_j(z) \in V_j\}.$$

The mapping $f(z, y) \to f(z, P(z))$ is a surjective homomorphism $\mathcal{O}(\prod U_i \times \prod V_j) \to \to \mathcal{O}(\Delta)$, whose kernel is generated by the functions $y_j - P_j(z)$.

This follows from the theory of functions of several complex variables.

DEFINITION. Let $a_1, \ldots, a_n$ be elements of $\mathcal{A}$. The *rationally convex joint spectrum* $\tilde{\mathrm{sp}}(a_1, \ldots, a_n)$ is the set of $s \in C^n$ such that $P(s) \in \mathrm{sp}\, P(a)$ for all polynomials in n variables.

PROPOSITION. *There is a unique continuous homomorphism $\mathcal{O}(\tilde{\mathrm{sp}}(a_1, \ldots, a_n)) \to \mathcal{A}$ which maps z_i on a_i and unit on unit.*

The algebra $\mathcal{O}(\tilde{\mathrm{sp}}\, a)$ can be considered as the operational calculus algebra, if $(a_1, \ldots, a_n) \in \mathcal{A}^n$ are elements such that $\mathrm{sp}\, P(a)$ is given for all P.

Uniqueness of the mapping is a straightforward application of Runge's theorem. Rational functions are dense in $\mathcal{O}(\tilde{\mathrm{sp}}(a_1, \ldots, a_n))$. A morphism mapping unit on unit and z_i on a_i maps $P(z)/Q(z)$ on $P(a)/Q(a)$.

To prove the existence of the mapping, we start with an open neighbourhood W of $\tilde{\mathrm{sp}}(a_1, \ldots, a_n)$. By compactness argument, we find a finite number of polynomials $P_1, \ldots, P_N$, and neighbourhoods $U_1, \ldots, U_n$, $V_1, \ldots, V_N$ resp. of $\mathrm{sp}\, a_i$ and of $\mathrm{sp}\, P_j(a)$ in such a way that $W \supseteq \Delta$

$$\Delta = \{(z_i, \ldots, z_n) |\ z_i \in U_i,\ P_j(z) \in V_j\}.$$

The Oka–Cartan results combined with the h.f.c. on direct products gives a homomorphism $\mathcal{O}(\Delta) \to \mathcal{A}$, $z_i \to a_i$, $1 \to 1$. We compose this with the restriction map $\mathcal{O}(W) \to \mathcal{O}(\Delta)$ and obtain a continuous homomorphism $\mathcal{O}(W) \to \mathcal{A}$.

We shall have the required continuous homomorphism $\mathcal{O}(\tilde{\mathrm{sp}}\, a) \to \mathcal{A}$ if we prove that the above composition does not depend on the choice of P, U, V. But let P', U', V' be new polynomials and open neighbourhoods, let $P'' = (P, P')$, let $U'' = U \cap U'$ and $V'' = (V, V')$. We are led to the consideration of

$$\Delta' = \{z \in C^n |\ z_i \in U'_i,\ P'_j(z) \in V'_j\},$$
$$\Delta'' = \{z \in C^n |\ z_i \in U_i \cap U'_i,\ P_j(z) \in V_j,\ P'_j(z) \in V'_j\}.$$

The proof of the fact that the compositions $\mathcal{O}(W) \to \mathcal{O}(\Delta) \to \mathcal{A}$ and $\mathcal{O}(W) \to \to \mathcal{O}(\Delta'') \to \mathcal{A}$ are equal is straightforward and left to the reader, with one hint. If $f \in \mathcal{O}(W)$, to find the image of f by the composition mapping, we must choose a function $F_1(z, y, y') \in \mathcal{O}(\prod U_i \times \prod V_j \times \prod V_j')$ such that $f(z) = F_1(z, P(z), P'(z))$ when $z \in \Delta''$, and we let $f[a] = F_1[a, P(a), P'(a)]$. Of course, a function $F(z, y)$ exists, in $\mathcal{O}(\prod U_i \times \prod V_j)$ such that $F(z, P(z)) = f(z)$ when $z \in \Delta' \supseteq \Delta''$. We can define F_1 in such a way that it is independent of y' and $F_1(z, y, y') = F(z, y)$. Then $F_1[a, P(a), P'(a)] = F[a, P(a)]$.

3.4. This is the place to speak of polynomial and of rational convexity.

Definition. Let U be a set. Let $\mathcal{F}$ be a set of complex-valued functions on U. The *$\mathcal{F}$-convex hull of X* is

$$\tilde{X} = \{z \in U \mid \forall f \in \mathcal{F}\colon\ |f(z)| \leqslant \sup_{x \in X} |f(x)|\}.$$

When speaking of rationally convex hulls, we must make the convention that $f(z_0) = \infty$ when $f = P/Q$ is a rational function, and in lowest terms we have $Q(z_0) = 0$. This is less obvious than it seems. Consider $f(z) = z_1/z_2$ on C^2, and let

$$X = \{(z, 0) \mid |z| \leqslant 1\}.$$

Then f is well determined at all points of X except the origin, and vanishes where it is determined on X. But $f(0, 0) = \infty$ anyway.

With this convention, the rationally convex hull of X is the set of $(z_1, \ldots, z_n) \in C^n$ such that $|f(z)| \leqslant \sup_{x \in X} |f(x)|$ for every rational function $f = P/Q$ whose denominator does not vanish on X.

Proposition. *The rationally convex hull of a compact subset X of C^n is the set*

$$\tilde{X} = \{(z_1, \ldots, z_n) \in C^n \mid \forall P\colon P(z) \in P(X)\}.$$

This is easy. If $P(z) \notin P(X)$, the rational function $1/(P - P(z))$ is bounded on X and takes the value ∞ at z, so z is not in the rationally convex hull of X. Conversely, assume that z is not in the rationally convex hull of X, let r be a rational function such that

$$|r(z)| > \max_{x \in X} |r(x)|.$$

Then $r(z) \notin r(X)$. Let $r = P/Q$ where P and Q are polynomials. Then $P(z)Q - Q(z)P$ vanishes at z but does not vanish on X.

3.5. $\tilde{\mathrm{sp}}(a_1, \ldots, a_n)$ is not a joint spectrum as defined by Z. Słodkowski and W. Żelazko ([29], [30], [62]). It does not have the spectral mapping property for polynomials, not even the projection property. It is a "rationally convex spectrum" if we accept the

Definition. A *rationally convex spectrum on $\mathcal{A}$*, a commutative Banach algebra with unit, is a mapping associating to every finite system $(a_1, \ldots, a_n)$ of elements of $\mathcal{A}$, a rationally convex set $\tilde{\sigma}(a_1, \ldots, a_n)$ in such a way that

I. $\tilde{\sigma}(a_1, \ldots, a_n) \subseteq \prod_1^n \mathrm{sp}\, a_i$,

II. If $n = 1$, if $a \in \mathcal{A}$, $\tilde{\sigma}(a) = \mathrm{sp}\, a$,

III. If $b_1, \ldots, b_N$ are elements of $\mathcal{A}$, if $P_1, \ldots, P_n$ are polynomials in n variables, if $a_i = P_i(a)$, then

$$\tilde{\sigma}(a_1, \ldots, a_n) = \operatorname{hull} P(\tilde{\sigma}(b_1, \ldots, b_N)),$$

where "hull" stands for the rationally convex hull.

Of course, not more than one rationally convex joint spectrum in the sense of this definition can be found, so $\tilde{\mathrm{sp}}$ is the rationally convex joint spectrum. A "projective limit" argument allows us to reduce the rationally convex spectrum of $(a_1, \ldots \ldots, a_n)$. We want to take $(s_1, \ldots, s_n)$ out of $\tilde{\mathrm{sp}}(a_1, \ldots, a_n)$ if elements $b_1, \ldots, b_N$ can be found, and polynomials in N variables $P_1, \ldots, P_n$, such that $a_i = P_i(b)$, but $s \notin P(\tilde{\mathrm{sp}}\, b)$. More precisely

Definition. The *joint spectrum*, $\mathrm{sp}(a_1, \ldots, a_n)$, *of* $(a_1, \ldots, a_n)$ is the set

$$\mathrm{sp}(a_1, \ldots, a_n) = \bigcap P(\tilde{\mathrm{sp}}(b_1, \ldots, b_N)),$$

where we let N range over N, b range over $\mathcal{A}^N$, P over n-tuples of polynomials in N variables, subject to the relations $a_i = P_i(b)$.

It is easy to see that the set of sets $P(\tilde{\mathrm{sp}}(b_1, \ldots, b_N))$ that we intersect has the finite intersection property. Let $(c_1, \ldots, c_M) \in \mathcal{A}$ and $Q_1, \ldots, Q_n$ be polynomials in M variables such that $a = Q(c)$. Consider $(b, c) \in \mathcal{A}^{N+M}$. Let $\bar{P}_i, \bar{Q}_i$ be polynomials in $N+M$ indeterminates, respectively independent of the M last, and of the N first indeterminates, and such that $\bar{P} = P$, $\bar{Q} = Q$. Then $\bar{P}_i(b, c) = \bar{Q}_i(b, c) = a_i$.

For all $(s, t) \in \tilde{\mathrm{sp}}(b, c)$, we have $\bar{P}_i(s, t) - \bar{Q}_i(s, t) \in \mathrm{sp}\, 0$ so $P(s) = Q(t)$. Of course, $\tilde{\mathrm{sp}}(b, c) \subseteq \tilde{\mathrm{sp}}\, b \times \tilde{\mathrm{sp}}\, c$, hence

$$\bar{P}(\tilde{\mathrm{sp}}(b, c)) = \bar{Q}(\tilde{\mathrm{sp}}(b, c)) \subseteq P(\tilde{\mathrm{sp}}\, b) \cap Q(\tilde{\mathrm{sp}}\, c).$$

This finite intersection property allows us, by compactness, to show that $\tilde{\mathrm{sp}}(a_1, \ldots \ldots, a_n)$ is the rationally convex hull of $\mathrm{sp}(a_1, \ldots, a_n)$, because it is the rationally convex hull of each $P(\tilde{\mathrm{sp}}(b_1, \ldots, b_N))$. It is also clear that the joint spectrum now has the good spectral mapping property, i.e. if $(a_1, \ldots, a_n) = (P_1, \ldots, P_n)(b_1, \ldots, b_N)$, then

$$\mathrm{sp}(a_1, \ldots, a_n) = P(\mathrm{sp}(b_1, \ldots, b_N)).$$

Proposition. *The joint spectrum of $(a_1, \ldots, a_n)$ is the intersection of the projections of the sets $\tilde{\mathrm{sp}}(a, b)$ where $N \in N$ and $b \in \mathcal{A}^N$.*

One inclusion is trivial. The projection $C^{n+N} \to C^n$ is a polynomial mapping, which maps (a, b) onto a. This shows that each projection contains the joint spectrum.

To obtain the inverse inclusion, we consider $b_1, \ldots, b_N$ and a polynomial mapping $P = (P_1, \ldots, P_n)$ such that $P(b) = a$. We also consider $(a, b) \in \mathcal{A}^{n+N}$, and observe that $P_i(t) = s_i$ whenever $(s, t) \in \tilde{\mathrm{sp}}(a, b)$. The projection of $\tilde{\mathrm{sp}}(a, b)$ is contained in $P(\tilde{\mathrm{sp}}\, b)$. This ends the proof.

PROPOSITION. *The joint spectrum of* $(a_1, \ldots, a_n)$ *is the set of* $(s_1, \ldots, s_n) \in C^n$ *such that* $(a_1-s_1, \ldots, a_n-s_n)$ *generates a proper ideal.*

If $(a_1-s_1, \ldots, a_n-s_n)$ do not generate a proper ideal we find $u_1, \ldots, u_n$ such that $\sum(a_i-s_i)u_i = 1$. For all $(z, y) \in \tilde{\mathrm{sp}}(a, u)$ we have $\sum(z_i-s_i)y_i = 1$. Hence $(s, y) \notin \tilde{\mathrm{sp}}(a, u)$, this for all choices of $y \in C^n$, and s is not in the projection of $\tilde{\mathrm{sp}}(a, u)$. This proves one inclusion.

The converse inclusion is easy to prove if we accept Gelfand's theory, i.e. if we accept the axiom of choice (or at least, the existence of maximal ideals). I believe that the following considerations rely much less on the axiom of choice.

Assume that $(s_1, \ldots, s_n) \notin \mathrm{sp}(a_1, \ldots, a_n)$. Choose $(b_1, \ldots, b_N)$ in such a way that s is not in the projection of $\tilde{\mathrm{sp}}(a, b)$. Then, for no choice of $(t_1, \ldots, t_N)$ is (s, t) in $\tilde{\mathrm{sp}}(a, b)$. Consider the functions $(z_1-s_1, \ldots, z_n-s_n, y_1-t_1, \ldots, y_N-t_N)$. They belong to $\mathcal{O}(\tilde{\mathrm{sp}}(a, b))$, they have no common zeroes, and generate therefore the improper ideal of $\mathcal{O}(\tilde{\mathrm{sp}}(a, b))$. The holomorphic functional calculus shows that $(a_1-s_1, \ldots, a_n-s_n, b_1-t_1, \ldots, b_N-t_N)$ generate the improper ideal of $\mathscr{A}$. We must still establish the following result.

PROPOSITION. $(a_1, \ldots, a_n)$ *generates the improper ideal of* $\mathscr{A}$ *if* $b \in \mathscr{A}$ *exists such that* $(a_1, \ldots, a_n, b-t)$ *generates the improper ideal of* $\mathscr{A}$ *for all* $t \in C$.

If the ideal generated by $(a_1, \ldots, a_n)$ is proper, its closure is proper too. Let α be this closure. Let $\bar{b}$ be the equivalence class of b in $\mathscr{A}/\alpha$. Our hypothesis ensure that $\bar{b}$ has empty spectrum in $\mathscr{A}/\alpha$. This is impossible.

3.6. Now comes the Arens–Calderón trick [7]. Let U be a neighbourhood of $\mathrm{sp}(a_1, \ldots, a_n)$. A compactness argument shows that $(b_1, \ldots, b_N)$ exists such that $\pi\colon (s, t) \to s$ maps $\tilde{\mathrm{sp}}(a, b)$ into U. Let $f \in \mathcal{O}(U)$, $f \circ \pi$ is holomorphic on a neighbourhood of $\tilde{\mathrm{sp}}(a, b)$. We can define $(f \circ \pi)\ [a, b]$.

If $c_1, \ldots, c_M$ are new elements such that projection maps $\tilde{\mathrm{sp}}(a, c)$ into U, we consider (a, b, c) and their rationally convex joint spectrum. Let $\pi_1\colon (z, y, x) \to z$ be projection $C^{n+N+M} \to C^n$. Then $f \circ \pi_1$ is the direct product of $f \circ \pi$ and the constant one. Hence $f \circ \pi_1[a, b, c] = f \circ \pi[a, b] \cdot 1[c] = f \circ \pi(a, b)$.

If $\pi_2\colon (z, x) \to z$ were projection $C^{n+M} \to C^n$, we would show similarly that

$$f \circ \pi_1[a, b, c] = f \circ \pi_2[a, c],$$

hence

$$f \circ \pi_1[a, b] = f \circ \pi_2[a, c].$$

This proves that the element of $\mathscr{A}$ associated to $f \in \mathcal{O}(U)$ does not depend on the choice of $b_1, \ldots, b_N$.

PROPOSITION. *For every* $(a_1, \ldots, a_n) \in \mathscr{A}^n$, *a continuous homomorphism* $\mathcal{O}(\mathrm{sp}(a_1, \ldots, a_n)) \to \mathscr{A}$ *exists, which maps* z_i *on* a_i *and unit on unit.*

3.6. This is the place to sketch a path to the h.f.c. which is slightly shorter than the one we have trodden. We start with the h.f.c. as defined in Section 2.3, i.e. if $f(z_1, \ldots, z_n)$ is holomorphic on the polydisc $|z_1| < r_1, \ldots, |z_n| < r_n$ and $r_i > \|a_i\|$ we let $f[a] = \sum c_k a^k$ if $f(z) = \sum c_k z^k$. Let now $P_1, \ldots, P_N$ be polynomials in n indeterminates, let $r_i > \|a_i\|$, $r'_j > \|P_j(a)\|$. The same operation allows us to define $F[a, P(a)]$ when $F(z, y)$ is holomorphic for $|z_i| < r_i$, $|y_j| < r'_j$.

The Oka–Cartan results allow us now to define $f[a]$ when f is holomorphic on the polyhedron $|z_i| < r_i$, $|P_j(z)| < r'_j$, hence $f[a]$ when f is holomorphic near to

$$\overline{\mathrm{sp}}(a_1, \ldots, a_n) = \{(s_1, \ldots, s_n) \in C^n |\ \forall\ P \text{ a polynomial: } |P(s)| \leqslant \|P(a)\|\}.$$

The set $\overline{\mathrm{sp}}(a_1, \ldots, a_n)$ is polynomially convex. We can call this set a "polynomially convex spectrum".

Not every compact subset of C is polynomially convex. It makes sense to speak of the polynomial hull of a subset of C. The polynomially convex spectrum has properties

I. $\overline{\mathrm{sp}}(a_1, \ldots, a_n) \subseteq \prod_1^n \mathrm{hull\,sp}\, a_i$,

II. $\overline{\mathrm{sp}}\, a = \mathrm{hull\,sp}\, a$ when $a \in \mathscr{A}$,

III. Let $P_1, \ldots, P_n$ be polynomials in N variables, let $b_1, \ldots, b_N$ be elements of $\mathscr{A}$ and $a_i = P_i(b)$. Then

$$\overline{\mathrm{sp}}(a_1, \ldots, a_n) = \mathrm{hull}\, P(\overline{\mathrm{sp}}(b_1, \ldots, b_N)).$$

Here of course, hull stands for the polynomial hull.

A compact set can of course be defined by

$$\mathrm{sp}(a_1, \ldots, a_n) = \bigcap P(\overline{\mathrm{sp}}(b_1, \ldots, b_N)),$$

where $N \in N$, $b \in \mathscr{A}^N$, P is a n-tuple of polynomials in n indeterminates, and $a = P(b)$. And the same proofs as in Section 3.5 shows that

$$\mathrm{sp}(a_1, \ldots, a_n) = \{(s_1, \ldots, s_n) \in C^n |\ \text{ideal } (a_1-s_1, \ldots, a_n-s_n) \text{ is proper}\}.$$

Hence we have the same joint spectrum as previously.

But the substitution of an element in a convergent power series is easier than in a Cauchy integral. It is remarkable that the Arens–Calderón procedure makes the two resulting operational calculi equivalent.

4. Classical applications

4.1. The Šilov idempotent theorem [27], the Arens–Royden theorem [5], Aren's theorem on the group of components of $\mathrm{GL}_n(\mathscr{A})$, [6], the application of this theorem to K-theory [36], are some of the classical applications of the holomorphic functional calculus.

These are applications of two ideological facts. The structure space of a Banach algebra $\mathscr{A}$ is a compact, polynomially convex subset of the dual $\mathscr{A}'$ of $\mathscr{A}$. It is a projective limit of Stein manifolds. Part (only part, unfortunately) of the morality of analytic function theory in several complex variables is that the algebra of analytic functions on a Stein manifold is not too different from the algebra of continuous functions on that manifold.

If X is the structure space of $\mathscr{A}$, the holomorphic functional calculus is a mapping $\mathcal{O}(X) \to \mathscr{A}$. The Gelfand homomorphism is a mapping $\mathscr{A} \to C(X)$. The composition of these two mappings is the restriction mapping $\mathcal{O}(X) \to C(X)$. The algebra $\mathscr{A}$ is "sandwiched" between two algebras which are not too different from each other.

In each specific application, these remarks must be made precise, but it is not too surprising that many properties of a Banach algebra depend only on the topological structure of the maximal ideal space.

4.2. Above, I have used the analytic functions near a compact subset of the dual $\mathscr{A}'$ of $\mathscr{A}$. Of course, these must be defined, but their definition is not too difficult. $\mathscr{A}'$ has a weak topology (i.e. $\sigma(\mathscr{A}', \mathscr{A})$). Weak holomorphic functions near compact subsets of $\mathscr{A}'$ do not depend effectively on an infinite number of variables.

By definition, a holomorphic function on an open subset of a topological vector space is a continuous function on that subset, whose restriction to each one-dimensional affine subspace is holomorphic. We look first at the germs of holomorphic functions at $x \in \mathscr{A}'$, where $\mathscr{A}'$ has the weak-star topology $\sigma(\mathscr{A}', \mathscr{A})$.

If U is a neighbourhood of x, and f is holomorphic on U, then f is bounded on a neighbourhood V of x. This neighbourhood contains

$$W_x = \{z \in \mathscr{A}' \mid |\langle z-x, a_1\rangle| \leqslant \varepsilon, \ldots, |\langle z-x, a_k\rangle| \leqslant \varepsilon\}$$

for some finite subset $\{a_1, \ldots, a_k\}$ of $\mathscr{A}$ and some $\varepsilon > 0$. Liouville's theorem shows that f is constant on the affine subspaces parallel to $\{a_1, \ldots, a_k\}^{\perp}$ contained in W_x, i.e. f depends only on $(a_1, \ldots, a_k)$.

At each $x \in \mathscr{A}'$, we have a ring $\mathcal{O}_x$ of germs of holomorphic functions near x. The union of these rings can be organized into a sheaf. We are interested in a compact subset $X \subseteq \mathscr{A}'$, and the ring $\mathcal{O}(X)$ of sections of the sheaf $\mathcal{O}$ over X.

A classical compactness argument shows that for each such section $f \in \mathcal{O}(X)$, we can find a finite number, $a_1, \ldots, a_k$, of elements of $\mathscr{A}$, a neighbourhood U of $\{(\hat{a}_1(x), \ldots, \hat{a}_k(x)) \mid x \in X\}$, and a function $f_1 \in \mathcal{O}(U)$ in such a way that

$$f = f_1(\hat{a}_1, \ldots, \hat{a}_k).$$

If $X \subseteq \mathscr{A}'$ is polynomially convex, a second compactness argument allows us to find $b_1, \ldots, b_l$ in such a way that the polynomial hull $\tilde{F}$ of

$$F = \{(\hat{a}_1(x), \ldots, \hat{a}_k(x), \hat{b}_1(x), \ldots, \hat{b}_l(x)) \mid x \in X\}$$

projects into U by the projection mapping $C^{k+l} \to C^k$. Let U_1 be the reciprocal image of U by this projection, U_1 is a neighbourhood of $\tilde{F}$, and f is the composition of some $f_2 \in \mathcal{O}(U_1)$ and the mapping $x \to (\hat{a}(x), \hat{b}(x))$, of neighbourhoods of X into U_1.

In other words, *the algebra $\mathcal{O}(X)$ of sections of $\mathcal{O}$ near X is the direct limit of the algebras of holomorphic functions on neighbourhoods of $\{(\hat{a}_1(x), \ldots, \hat{a}_k(x)) \mid x \in X\}$ when $\{a_1, \ldots, a_k\}$ ranges over finite subsets of $\mathscr{A}$, when X is $\sigma(\mathscr{A}', \mathscr{A})$-compact in $\mathscr{A}'$.*

If X is compact and polynomially convex in $\mathscr{A}'$, $\mathcal{O}(X)$ is the direct limit of the algebras of holomorphic functions on the neighbourhoods of the polynomial hulls of

$$\{(\hat{a}_1(x), \ldots, \hat{a}_k(x)) \mid x \in X\},$$

where $\{a_1, \ldots, a_k\}$ again ranges over the finite subsets of $\mathscr{A}$.

These facts allow the reader to understand better the Arens–Calderón trick.

4.3. Let us mention now these classical applications of the holomorphic functional calculus.

PROPOSITION (Arens–Calderón [7]). *Let $F(z_1, \ldots, z_n, y)$ be a holomorphic function for $|z_i| < 1$, $|y| < 1$. Let $a_1, \ldots, a_n$ belong to a commutative Banach algebra, and have a spectral radius less than one. Assure that a continuous function φ can be found on the maximal ideal space, and that*

$$F(\hat{a}_1(m), \ldots, \hat{a}_n(m), \varphi(m)) \neq 0,$$

$$\frac{\partial F}{\partial y}(\hat{a}_1(m), \ldots, \hat{a}_n(m), \varphi(m)) \neq 0$$

this for every maximal ideal m. An element b then exists such that $\hat{b} = \varphi$, $F(a, b) = 0$.

The implicit function theorem allows us to extend φ holomorphically to a neighbourhood of the maximal ideal space in $\mathscr{A}'$. The holomorphic functional calculus maps this holomorphic extension onto the solution b of our equations.

The Šilov idempotent theorem is a special case of the above.

PROPOSITION (Šilov [27]). *Assume that the maximal ideal space X is not connected, let $X = X_1 \cup X_2$ with X_1, X_2 disjoint and open. An idempotent e then exists such that $\hat{e} = 0$ on X_1, $\hat{e} = 1$ on X_2.*

In fact, Šilov proved the above result when X is finitely generated. The extension to infinitely generated algebras is due to Arens and Calderón.

PROPOSITION (Arens–Royden [5]). *The groups of components of the group of invertible elements of $\mathscr{A}$ is $H^1(X, \mathbb{Z})$, where X is the maximal ideal space of $\mathscr{A}$ and $H^1(X, \mathbb{Z})$ is the first Čech cohomology group of X with integer coefficients.*

It will be easier to mention first the following result of Arens.

PROPOSITION (Arens [6]). *The group of components of $\mathrm{GL}_n(\mathscr{A})$, the group of $n \times n$ invertible matrices with coefficients in $\mathscr{A}$, is isomorphic with the group of homotopy classes of mappings $X \to \mathrm{GL}_n$, where homotopy classes are multiplied by pointwise multiplication.*

The special case $n = 1$ of this result of Arens shows that the group of components of $\mathscr{A}^*$, the invertible elements of $\mathscr{A}$, is isomorphic with the group of homotopy classes of mappings $\mathscr{A} \to \mathbb{C} \setminus \{0\}$. This is the quotient $C(X, \mathbb{C} \setminus \{0\})/\exp C(X, \mathbb{C})$. Consider the exact sequence of sheaves

$$0 \to \underset{\sim}{Z} \to \underset{\sim}{C} \to \underset{\sim}{C \setminus \{0\}} \to 0$$

where the first mapping is multiplication by $2\pi i$ and the second exponentiation. We have a long exact sequence

$$\ldots \to H^0(X, \underset{\sim}{C}) \to H^0(X, \underset{\sim}{C\setminus\{0\}}) \to H^1(X, \underset{\sim}{Z}) \to H^1(X, C) \to \ldots$$

The H^0-spaces are spaces of sections. Sheaf theory shows that $H^1(X, \underset{\sim}{C}) = 0$.

In other words, $H^1(X, \underset{\sim}{Z})$ is the cokernel of the exponential map $H^0(X, \underset{\sim}{C}) \to$ $\to H^0(X, \underset{\sim}{C\setminus\{0\}})$.

To prove his result, Arens uses the following results of Grauert. Hence, X will be a compact analytically convex subset of a Stein manifold.

1. A continuous mapping $X \to \mathrm{GL}_n$ is homotopic to an analytic one.
2. If an analytic mapping $X \to \mathrm{GL}_n$ is homotopic to a constant, there is a homotopy φ: $X \times I \to \mathrm{GL}_n$ of the given analytic mapping with the constant mapping such that, for each $t \in I$, $\varphi(\cdot, t)$ is analytic on X.

It is by combining these two results and the Arens–Calderón trick that Arens proves his result on the group of components of $\mathrm{GL}_n(\mathscr{A})$.

K-theory allows one to define functors in a fairly abstract way. These functors give interesting invariants of the algebras. Aren's theorem has, as consequences, that the K-theory of a Banach algebra only depends on the maximal ideal space of the algebra. The best exposition that I know of these results is due to J. L. Taylor [36].

5. Naturality and uniqueness

5.1. The naturality of the holomorphic functional calculus is a very important property. The following "naturality" result is obvious.

PROPOSITION. *Let $\mathscr{A}$ and $\mathscr{B}$ be two commutative unital Banach algebras, and φ: $\mathscr{A} \to \mathscr{B}$ a continuous unital homomorphism. Let $a_i \in \mathscr{A}$, and $b_i = \varphi a_i$ for $i = 1, \ldots, n$. Then*

$$\mathrm{sp}(b_1, \ldots, b_n) \subseteq \mathrm{sp}(a_1, \ldots, a_n).$$

Let $f \in \mathcal{O}(\mathrm{sp}(a_1, \ldots, a_n))$. We can define $f[a]$, $f[b]$ and have the equality

$$f[b] = \varphi f[a].$$

We shall discuss here the consequences of naturality. Naturality is not unrelated to uniqueness. It is reasonable to discuss the uniqueness properties of the h.f.c. in this section also.

The h.f.c. has been constructed when $a_1, \ldots, a_n$ are elements of a commutative unital Banach algebra. We shall describe in Section 6 a class of good non Banach algebras to which the h.f.c. generalizes. The algebra of holomorphic functions near to a compact set is such an algebra. We shall feel free to use this fact, and the naturality of the h.f.c. constructed in this way.

5.2. DEFINITION. A compact subset $X \subseteq C^n$ is *analytic* if

$$\mathrm{sp}(z_1, \ldots, z_n; \mathcal{O}(X)) = X,$$

where $z_1, \ldots, z_n$ are the coordinate functions in C^n.

PROPOSITION. *Let $X \subseteq C^n$ be analytic. A bounded homomorphism $\mathcal{O}(X) \to \mathscr{A}$ mapping z_i on a_i and unit on unit exists iff $\mathrm{sp}(a_1, \ldots, a_n) \subseteq X$. This homomorphism is unique.*

Such a bounded homomorphism φ can only exist if $\mathrm{sp}(a_1, \ldots, a_n) \subseteq X$ since $\mathrm{sp}(\varphi z_1, \ldots, \varphi z_n) \subseteq \mathrm{sp}(z_1, \ldots, z_n)$. If $\mathrm{sp}(a_1, \ldots, a_n) \subseteq X$, the h.f.c. is a homomorphism with the required properties.

To prove uniqueness, we observe that the h.f.c. allows us to define $f[z] \in \mathcal{O}(X)$ if $\mathrm{sp}\, z \subseteq X$. It turns out that $f[z] = f$. Assume this is the case,

$$\varphi(f) = \varphi(f[z]) = f[\varphi(z)] = f[a]$$

if φ: $\mathcal{O}(X) \to \mathscr{A}$ is a bounded unital homomorphism mapping z_i on a_i.

To show that $f[z] = f$, we remember that an $f \in \mathcal{O}(X)$ is determined by its germ at each $x \in X$. Call f_x this germ. The mapping $f \to f_x$ is a bounded homomorphism $\mathcal{O}(X) \to \mathcal{O}(\{x\})$. It maps $f[z]$ onto $f[z_x] = f_x[z_x]$. A singleton is polynomially convex. Runge's theorem shows that the identity is the only endomorphism of $\mathcal{O}(\{x\})$ which leaves the $z_{i\cdot x}$ invariant ($z_{i\cdot x}$ is the germ of z_i at x). Therefore $f[z_x] = f_x$, and $f[z] = f$.

COROLLARY. *Let $X \subseteq C^n$ be analytic. All multiplicative linear functionals on $\mathcal{O}(X)$ can be represented by points of X.*

In Section 5.3, we shall see that this corollary is not as obvious as one can expect. The converse of this result is trivial, X is analytic if all multiplicative functionals on $\mathcal{O}(X)$ can be represented.

The result of W. R. Zame that we shall discuss in Section 5.7 implies that only one bounded homomorphism $\mathcal{O}(X) \to \mathscr{A}$ can exist if all multiplicative linear functionals can be represented.

It can be argued that the corollary is the only result in this section which does not follow from Zame's result.

5.3. DEFINITION. An element x of a unital topological algebra $\mathscr{A}$ is *a function of* $a_1, \ldots, a_n \in \mathscr{A}$ if two unital continuous homomorphisms, φ_1, φ_2: $\mathscr{A} \to \mathscr{B}$ are such that $\varphi_1 x = \varphi_2 x$ if $\varphi_1 a_i = \varphi_2 a_i$ for $i = 1, \ldots, n$.

I do not like the expression "is a function of", but I have not found any better expression, and the notion is worth considering.

The set of "functions of $(a_1, \ldots, a_n)$" is clearly a closed sub-algebra of $\mathscr{A}$ which contains $a_1, \ldots, a_n$, and the unit. This closed sub-algebra also contains every element which can be obtained from $(a_1, \ldots, a_n)$ by applying a natural operational calculus. If $\mathscr{A}$ is a commutative Banach algebra, if $f \in \mathcal{O}(\mathrm{sp}(a_1, \ldots, a_n))$, then $f[a_1, \ldots, a_n]$ is a function of $a_1, \ldots, a_n$. If X is analytic, all elements of $\mathcal{O}(X)$ are functions of $z_1, \ldots$ $\ldots, z_n$ within $\mathcal{O}(X)$.

PROPOSITION. *Let $\mathscr{A}$ be a unital Banach algebra. Let $a_1, \ldots, a_n$ be elements of $\mathscr{A}$ and x a function of $a_1, \ldots, a_n$. Let M be an $\mathscr{A}$-bimodule and D: $\mathscr{A} \to M$ a derivation. Then $Dx = 0$ if $Da_i = 0$.*

The statement "D is a derivation" is equivalent with the statement "the mapping $x \to x \oplus Dx$ is a homomorphism $\mathscr{A} \to \mathscr{A} \oplus M$" when we put on $\mathscr{A} \oplus M$ the multiplication

$$(a_1 \oplus m_1)(a_2 \oplus m_2) = a_1 a_2 \oplus (a_1 m_2 + m_1 a_2).$$

The result follows.

COROLLARY. *A function of $a_1, \ldots, a_n$ belongs to the bicommutant of $(a_1, \ldots, a_n)$.*

This follows from the consideration of the inner derivations D_b: $x \to bx - xb$, with b in the commutant of $a_1, \ldots, a_n$.

If we accept the idea that a good functional calculus must be natural, we see that a natural functional calculus maps into the bicommutant.

$\bar{z}$ is not a function of z in $C_1(D)$, if D is the unit disc, because $\partial/\partial\bar{z}$ is a derivation $C_1(D) \to C(D)$ which vanishes at z and not at $\bar{z}$. On the other hand Berkson, Dawson, and Elliott [63] have shown that all elements of $C(D)$ are "functions of z" in $C(D)$.

5.4. The uniqueness properties of the h.f.c. are related to the properties of the holomorphic hull of a compact set in C^n. But not all specialists of Banach algebra theory know what a holomorphic hull is.

Let U be an open set in C^n. It can happen that all functions holomorphic on U can be extended to an open set U_1 larger than U. If U is the half spherical shell

$$U = \{(z_1, \ldots, z_n) |\ 1 - \varepsilon < |z| < 1, \operatorname{Re} z_1 > 0\}$$

all holomorphic functions on U can be extended to the half-ball

$$U_1 = \{(z_1, \ldots, z_n) |\ |z| < 1, \operatorname{Re} z_1 > 0\}.$$

It can also happen that all functions on U can be extended in two essentially different ways to some point out of U. Let for instance U' be the union of U with a neighbourhood of a path j winding from the half-sphere $|z| = 1$, $\operatorname{Re} z_1 > 0$ back to the origin after having wound once around the complex hyperplane $z_1 = 2$. All functions on U, hence all functions on U' can be extended to U_1, hence to the origin. But the functions on U' take already a value at the origin. And the extension of a function to the interior of the half-shell may take a different value at the origin from the extension of this function to the neighbourhood of the path j. Think of the function $\log(z_1 - 2)$.

Fortunately, if all functions in U extend to z_1 along a path j from $z_0 \in U$ to z_1, all functions in U extend to some ε-neighbourhood of z_1 along j, where $\varepsilon > 0$ depends on j and z_1, but does not depend on the function. The common domain to which all elements of $\mathcal{O}(U)$ extend looks locally like an open subset of C^n.

DEFINITION. A domain $\tilde{U}$ spread over C^n is an *analytic manifold* with a projection mapping π: $\tilde{U} \to C^n$ which is a local isomorphism of analytic manifolds.

PROPOSITION. *To every open subset $U \subseteq C^n$, one can associate a domain $\tilde{U}$ spread over C^n, and containing U, in such a way that every holomorphic function on U extends uniquely to $\tilde{U}$. $\tilde{U}$ is called the envelope of holomorphy of U.*

Now, let V be an open subset of U. The envelope of holomorphy $\tilde{V}$ of V is spread over C^n. It is even spread over $\tilde{U}$, the projection π_V: $V \to C^n$ factors $\pi_V = \pi_U \circ \pi_{UV}$ where π_U is the projection $\tilde{U} \to C^n$ and π_{UV} a projection $\tilde{V} \to \tilde{U}$.

If V is relatively compact in U, π_{UV}: $\tilde{V} \to \tilde{U}$ has a relatively compact range in $\tilde{U}$.

The above results are not trivial. But they are part of the present-day classical theory of functions of several complex variables.

5.5. Let now X be a compact subset of C^n. For every open neighbourhood U of X, we have an envelope of holomorphy $\tilde{U}$. If V is relatively compact in U, we have a mapping π_{UV}: $\tilde{V} \to \tilde{U}$, with relatively compact range.

DEFINITION. The holomorphic hull $\tilde{X}$ of X is the *projective limit* of the holomorphic hulls of the neighbourhoods of X.

$\tilde{X}$ is a countable projective limit of manifolds, with structural mappings having relatively compact ranges. It is a compact metrizable space. There is a projection mapping $\tilde{X} \to C^n$. The counterimage of any $z \in C^n$ is a projective limit of finite sets. Such a projective limit is totally disconnected, but it can be uncountable. I do not have any example where an uncountable set of points of $\tilde{X}$ map onto the same point of C^n, but I do not doubt that such examples exist, and have been found.

PROPOSITION. *The holomorphic hull $\tilde{X}$ of X is the maximal ideal space of $\mathcal{O}(X)$.*

The holomorphic functions on neighbourhoods of X separate $\tilde{X}$, because the holomorphic functions on U (or their extensions to $\tilde{U}$) separate $\tilde{U}$. We must still show that a finite number $u_1, \ldots, u_k$ of elements of $\mathcal{O}(X)$ generate the improper ideal of $\mathcal{O}(X)$ if $u_1, \ldots, u_k$ have no common zero on $\tilde{X}$.

Let U be a neighbourhood of X to which $u_1, \ldots, u_k$ all extend. Let $\tilde{U}$ be the envelope of holomorphy of U. Let $\tilde{u}_1, \ldots, \tilde{u}_k$ be the extensions of $u_1, \ldots, u_k$ to $\tilde{U}$.

Call θ: $\tilde{X} \to \tilde{U}$ the *natural projection*, and let $\tilde{Y} = \theta\tilde{X}$. The extensions of u_i to $\tilde{X}$ are the compositions $\tilde{u}_i \circ \theta$. We assume that these extensions have no common zero on $\tilde{X}$. The functions $\tilde{u}_i$ have no common zero on $\tilde{Y}$. Let V be a neighbourhood of $\tilde{Y}$ in $\tilde{U}$, small enough that $\tilde{u}_1, \ldots, \tilde{u}_k$ have no common zero on V, and then W a neighbourhood of X whose holomorphic hull $\tilde{W}$ is mapped into V by the projection map π_{UW}: $\tilde{W} \to \tilde{U}$.

The extensions of $u_1, \ldots, u_k$ to $\tilde{W}$ have no common zero on $\tilde{W}$. But $\tilde{W}$ is a Stein manifold. The functions $u_1, \ldots, u_k$ generate the improper ideal of $\mathcal{O}(W)$, and *a fortiori* of $\mathcal{O}(\tilde{X})$.

5.6. Let $X \subseteq C^n$ be a compact set. Let $\tilde{X}$ be its holomorphic hull. Let π: $\tilde{X} \to C^n$ be the projection mapping. If π is injective, $\pi\tilde{X}$ is an analytic set in C^n. Since $\mathcal{O}(X) \simeq \mathcal{O}(\tilde{X}) \simeq \mathcal{O}(\pi\tilde{X})$, only one bounded homomorphism $\mathcal{O}(X) \to \mathscr{A}$ can exist which maps unit on unit, and z_i on a_i.

If π is not injective, if $x_1 \in \tilde{X}$, $x_2 \in \tilde{X}$ are such that $\pi x_1 = \pi x_2$, the mappings $u \to \tilde{u}(x_1)$, $u \to \tilde{u}(x_2)$ map $z_1, \ldots, z_n$ onto the same elements of C, but are different continuous homomorphisms $\mathcal{O}(X) \to C$.

5.7. The following uniqueness theorem for the h.f.c. is an easy consequence of Runge's theorem and the polynomial convexity of the set of multiplicative linear forms on $\mathscr{A}$.

PROPOSITION. *Assume that a bounded unital homomorphism $\mathcal{O}(\mathrm{sp}\,a_1, \ldots, a_n) \to \mathscr{A}$ is given for each finite system $a_1, \ldots, a_n$ of elements of $\mathscr{A}$. Call $\varphi_{a_1 \ldots a_n}$ this homomorphism. Assume that*

(1) $\varphi_{a_1 \ldots a_n}(z_i) = a_i$,

(2) *If $n' < n$, and $p\colon C^n \to C^{n'}$ is the projection $C^n \to C^{n'}$, then*

$$\varphi_{a_1 \ldots a_n}(f \circ p) = \varphi_{a_1 \ldots a_{n'}}(f)$$

for all $f \in \mathcal{O}(\mathrm{sp}\,a_1, \ldots, a_{n'})$.

Then $\varphi(f) = f[a]$.

For applications, condition (2) can be annoying, because it involves the simultaneous consideration of all mappings $\varphi_{a_1 \ldots a_n}$. The following uniqueness theorem, by W. R. Zame [57], involves the consideration of only one such homomorphism.

PROPOSITION. *Let $\varphi\colon \mathcal{O}(\mathrm{sp}\,a_1, \ldots, a_n) \to \mathscr{A}$ be a bounded unital homomorphism, satisfying*

(1) $\varphi(z_i) = a_i$,

(3) *$\varphi(f)\hat{}\,(m) = f(\hat{a}(m))$ when m is a maximal ideal of $\mathscr{A}$, if $\hat{u}(m)$ is the value at m of the Gelfand transform of $u \in \mathscr{A}$.*

Then $\varphi(f) = f[a]$.

Note that Zame states mistakenly the classical uniqueness result as "$\varphi(f) = f[a]$ if (1), (2) and (3) hold". It is clear that (1) and (2) implies (3). His work does not remove a condition, it replaces this condition by another one. This remark does not reduce the value of his result.

In the functional calculus setting (described in Section 1.2), Zame states that $\varphi(f) = f[a]$ if φ is a bounded unital homomorphism $\mathcal{O}(\mathrm{sp}\,a) \to \mathscr{A}$ which maps z_i on a_i and behaves as it should on the Gelfand transforms.

In the operational calculus setting (this is the ideology that I adopt in most of this paper), naturality is an important property of the operational calculus. Zame says that $\varphi(f) = f[a]$ if $\varphi\colon \mathcal{O}(\mathrm{sp}\,a) \to \mathscr{A}$ is a bounded unital homomorphism which maps z_i on a_i, and is such that

$$\chi(f[a]) = f(\chi(a))$$

whenever χ is a multiplicative linear form on $\mathscr{A}$. In other words he requires "naturality relative to the multiplicative linear forms". In any case, the result is ideologically a good result.

The following result is a corollary of Zame's theorem.

COROLLARY. *Let $\mathscr{A}$ be a Banach algebra. Let $a_1, \ldots, a_n$ be elements of $\mathscr{A}$, let S be their joint spectrum and $\tilde{S}$ its holomorphic hull. Let $\pi\colon \tilde{S} \to C^n$ be the projection mapping, assume that $\pi^{-1}s$ has a single element for all $s \in S$. There is then a unique bounded homomorphism $\mathcal{O}(S) \to \mathscr{A}$ which maps z_i on a_i.*

6. Idempotent bounded structures

6.1. This section contains a small generalization of Banach algebra theory. Most of the results about commutative Banach algebras generalize to algebras with an idempotent bounded structure.

The spectrum of an element is still compact and not empty. The introduction of algebras with idempotent bounded structures is not justified by the observation (Section 1.6) that operators with unbounded spectrum and operators with empty spectrum want to be studied.

It is justified by the fact that non-Banach algebras with an idempotent bounded structure exist, and that some of these are useful in functional analysis. Examples will be found in Section 6.5.

The fact that the algebra of holomorphic functions near to a compact subset $X \subseteq C^n$ is such an example must be mentioned. We have already applied the fact that the h.f.c. can be applied to $\mathcal{O}(X)$.

6.2. Topological vector space structures are poorly adapted to the theory that we shall discuss. Bounded structures are better adapted. It may be one can get even better results when studying algebras with convergence structures. I have never tried. I am convinced that the initial theory would be less trivial. I like algebras with bounded structures because the preliminary bornological results used in the spectral theory are so easy to prove.

DEFINITION. A *b-space*, or a *space with a complete convex bounded structure* is the union $E = \bigcup_\alpha E_\alpha$ of a directed system of Banach spaces, where the inclusion maps $E_\alpha \to E_\beta$ are bounded when $\alpha \leqslant \beta$. The bornology $\mathscr{B}$ of E is the set of $B \subseteq E$ such that $B \subseteq E_\alpha$, and is bounded in E_α for α large enough.

DEFINITION. A *b-algebra* $(\mathscr{A}, \mathscr{B})$ is a b-space equipped with a bounded bilinear multiplication.

We assume that $B_1 \cdot B_2 \in \mathscr{B}$ when $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$. In other words $\mathscr{A} = \bigcup_\alpha \mathscr{A}_\alpha$, where the $\mathscr{A}_\alpha$ are Banach spaces, and for every α, β there is a γ such that $\mathscr{A}_\alpha \cdot \mathscr{A}_\beta \subseteq \mathscr{A}_\gamma$, multiplication $\mathscr{A}_\alpha \times \mathscr{A}_\beta \to \mathscr{A}_\gamma$ being continuous.

The b-algebras are important in spectral theory. The reader is sent to paragraph 8 if he wants some spectral considerations which really involve the considerations of b-algebras. Except in Section 6.5, example (e), we shall not be dealing with general b-algebras in this section, but with the b-algebras having an idempotent bounded structure. We shall call these *mb-algebras* (*multiplicatively convex b-algebras*).

DEFINITION. A subset B of an algebra $\mathscr{A}$ is *idempotent* if $B^2 \subseteq B$. A complete convex bornology $\mathscr{B}$ on an algebra $\mathscr{A}$ is *idempotent* if every bounded set B is contained in MB_1 for some $M \in R_+$ and some bounded idempotent B_1. An algebra with an idempotent bornology is an mb-algebra.

If B_1 is idempotent if we let

$$B_2 = \Big\{\sum_1^\infty \lambda_n x_n \big| \ \forall n\colon\ x_n \in B_1, \forall n\colon\ \lambda_n \in \boldsymbol{C}, \sum |\lambda_n| \leqslant 1\Big\}$$

then B_2 is bounded, idempotent, and its Minkowski (gauge) functional is a Banach algebra norm on the vector space generated by B_2.

PROPOSITION. *An mb-algebra $\mathscr{A}$ is the union $\mathscr{A} = \bigcup \mathscr{A}_\alpha$ of a directed family of Banach algebras, a subset B being bounded in $\mathscr{A}$ if and only if B is contained and bounded in $\mathscr{A}_\alpha$ for some α.*

This class of algebras has been introduced by R. G. Allan, H. G. Dales, McClure [4], who called these pseudo-Banach algebras. H. Hogbe-Nlend [16] considered these and spoke of m-convex bornologies.

6.3. Let $a_1, \ldots, a_n$ be elements of the commutative mb-algebra with unit $\mathscr{A} = \bigcup_\alpha \mathscr{A}_\alpha$. Choose α large enough, that $a_i \in \mathscr{A}_\alpha$ for $i = 1, \ldots, n$. We can define

$$\mathrm{sp}\big((a_1, \ldots, a_n), \mathscr{A}_\alpha\big).$$

This is a compact non empty subset of $\boldsymbol{C}^n$. If $\alpha \leqslant \beta$,

$$\mathrm{sp}\big((a_1, \ldots, a_n; \mathscr{A}_\alpha\big) \supseteq \mathrm{sp}\big((a_1, \ldots, a_n); \mathscr{A}_\beta\big).$$

DEFINITION. The *spectrum of* $(a_1, \ldots, a_n)$ is the intersection

$$\mathrm{sp}\big((a_1, \ldots, a_n); \mathscr{A}\big) = \bigcap_\alpha \mathrm{sp}\big((a_1, \ldots, a_n); \mathscr{A}_\alpha\big).$$

This is a compact non empty subset of $\boldsymbol{C}^n$. And clearly

$$(s_1, \ldots, s_n) \notin \mathrm{sp}\big((a_1, \ldots, a_n); \mathscr{A}\big) \Leftrightarrow \exists (u_1, \ldots, u_n) \in \mathscr{A}^n; \quad \sum_1^n (a_i - s_i) u_i = 1.$$

If U is a neighbourhood of $\mathrm{sp}\big((a_1, \ldots, a_n); \mathscr{A}\big)$, we find α large enough, that

$$U \supseteq \mathrm{sp}\big((a_1, \ldots, a_n), \mathscr{A}_\alpha\big)$$

and an h.f.c. mapping $\mathscr{O}(U) \to \mathscr{A}_\alpha$. If $\beta > \alpha$, the h.f.c. mapping $\mathscr{O}(U) \to \mathscr{A}_\beta$ is the composition of the h.f.c. mapping $\mathscr{O}(U) \to \mathscr{A}_\alpha$ and inclusion $\mathscr{A}_\alpha \to \mathscr{A}_\beta$. This is true because inclusion is a morphism of unital Banach algebras, which maps the elements a_i on themselves, and we know that the h.f.c. is natural.

This shows that the morphism $\mathscr{O}(U) \to \mathscr{A}_\alpha$ does not depend on the choice of α, if α is large enough.

PROPOSITION. *For every system $(a_1, \ldots, a_n)$ of elements of $\mathscr{A}$, we have a bounded homomorphism $\mathscr{O}\big(\mathrm{sp}(a_1, \ldots, a_n)\big) \to \mathscr{A}$ which maps the coordinate function z_i on a_i, and the unit on the unit. If $n' < n$, the h.f.c. homomorphism*

$$\mathscr{O}\big(\mathrm{sp}(a_1, \ldots, a_{n'})\big) \to \mathscr{A}$$

is the composition of the h.f.c. homomorphism $\mathscr{O}\big(\mathrm{sp}(a_1, \ldots, a_n)\big) \to \mathscr{A}$ and the mapping $\to f\circ\pi\colon \mathscr{O}\big(\mathrm{sp}(a_1, \ldots, a_{n'})\big) \to \mathscr{O}\big(\mathrm{sp}(a_1, \ldots, a_n)\big)$ if $\pi\colon \boldsymbol{C}^n \to \boldsymbol{C}^{n'}$ maps $(z_1, \ldots, z_n)$ onto $(z_1, \ldots, z_{n'})$.

Of course, the compatibility of the h.f.c. with the projection mappings, as stated in the above proposition, ensures the uniqueness of the h.f.c., just as it does when $\mathscr{A}$ is a Banach algebra.

6.4. We have a broad choice of paths which allows us to show that the main results of commutative Banach algebra theory apply to mb-algebras.

The most obvious path is the classical one in Banach algebra theory. Let m be a maximal ideal of $\mathscr{A} = \bigcup_\alpha \mathscr{A}_\alpha$. For each α, let $m_\alpha = m \cap \mathscr{A}_\alpha$, and $\mathrm{cl}_\alpha m_\alpha$ the closure of m_α in the Banach algebra $\mathscr{A}_\alpha$. Then $\overline{m} = \bigcup_\alpha \mathrm{cl}_\alpha m_\alpha$ is a proper ideal and contains m, so $m = m$. The quotient $\mathscr{A}/m$ is in a natural way an mb-algebra, and it is a field. Of course $\mathscr{A}/m$ cannot be a field other than the complex field, because the spectrum of an element is never empty.

[The reader must be warned against a mistake. If $E = \bigcup_\alpha E_\alpha$ is a b-space, if $F \subseteq E$ is a vector subspace, we can let $F_\alpha = F \cap E_\alpha$, next $\mathrm{cl}_\alpha F_\alpha$ the closure of F_α in the Banach space E_α, finally $\mathrm{cl} F = \bigcup_\alpha \mathrm{cl}_\alpha F_\alpha$. The mapping $F \to \mathrm{cl} F$ is not idempotent in general, $\mathrm{cl} F \cap E_\alpha$ is the union of the sets $(\mathrm{cl}_\beta F_\beta) \cap E_\alpha$. Each of these is closed, but the union need not be closed. Above, m is equal to its closure because it is maximal, so it is closed and $\mathscr{A}/m$ is an mb-algebra.]

We can follow another path, sketched in Section 2.5 when $\mathscr{A}$ is Banach algebra. For each $(a_1, \ldots, a_n)$ we have a compact set $\mathrm{sp}(a_1, \ldots, a_n)$. When $n' < n$, $\mathrm{sp}(a_1, \ldots, a_n)$ is mapped onto $\mathrm{sp}(a_1, \ldots, a_{n'})$ by the projection $\boldsymbol{C}^n \to \boldsymbol{C}^{n'}$. The projective limit of the sets $\mathrm{sp}(a_1, \ldots, a_n)$ is a compact space, the "spectrum of $\mathscr{A}$". One can then identify the spectrum of $\mathscr{A}$ with the set of all maximal ideals, or of all multiplicative linear forms on $\mathscr{A}$.

We may develop the structure space theory and the holomorphic functional calculus in each $\mathscr{A}_\alpha$, and take the suitable inductive and projective limits. Or we may show directly that the structure space theory and the h.f.c. apply to $\mathscr{A}$. The end result is, of course, always the same.

6.5. The fact that interesting, non-Banach mb-algebras exist is of course very important.

(a) We justify the considerations of Section 5.2 by observing that $\mathscr{O}(K)$ has a natural mb-structure.

Let K be compact in $\boldsymbol{C}^n$, or in a locally convex space. A holomorphic function on a neighbourhood of K is continuous on a neighbourhood of K, and therefore bounded on a — possibly smaller — neighbourhood.

For each open set U, containing K, and each of whose components meets K, we consider the Banach algebra $\mathscr{O}_\infty(U)$ of bounded holomorphic functions on U. The restriction mapping $\mathscr{O}_\infty(U) \to \mathscr{O}(K)$ is injective. The set of open sets U that we consider is directed for inclusion, $\mathcal{O}(K)$ is the union of a directed family of Banach algebras.

(b) Let $\mathscr{A}$ be a function algebra on the compact space X. A function f is locally in $\mathscr{A}$ if

$$\forall x \in X\ \exists V,\ \text{neighbourhood of } x, \exists u \in \mathscr{A};\quad u|_V = f|_V.$$

It is known that function algebras exist, with functions that are locally in $\mathscr{A}$ but not globally in $\mathscr{A}$.

Let $X = \bigcup_1^k V_i$ be a finite open covering. We call $\mathscr{A}_{\{V_i\}}$ the set of $f \in C(X)$ such that $\forall\ i\ \exists\ u_i \in \mathscr{A}$ with $f|_{V_i} = u_i|_{V_i}$. This set of functions is normed by

$$n_{\{V_i\}}(f) = \max_i \min\{\|u\| \mid u|_{V_i} = f|_{V_i}\}.$$

Then $\mathscr{A}_{\{V_i\}}$ is a Banach algebra, and

$$\mathscr{A}_{\text{loc}} = \bigcup_{\{V_i\}} \mathscr{A}_{\{V_i\}}$$

is an mb-algebra.

(c) Let $\mathscr{A}$ be a commutative complete p-normed algebra with unit. For each finite subset $\{x_1, \ldots, x_k\}$ of $\mathscr{A}$ and each $\varepsilon > 0$, consider the closed balanced convex hull of

$$\left\{ \frac{x_1^{r_1} \ldots x_k^{r_k}}{(1+\varepsilon)^{\Sigma r_i} \|x_1\|^{r_1} \ldots \|x_k\|^{r_k}} \,\middle|\, r_1, \ldots, r_k \in N \right\}.$$

This is the unit ball of a Banach algebra $\mathscr{A}_\alpha$, itself contained in $\mathscr{A}$, and $\mathscr{A} = \bigcup \mathscr{A}_\alpha$.

To my knowledge this is the easiest proof of results of W. Żelazko [58], [59], [60], [61], B. Gramsch [15], D. Przeworska-Rolewicz and S. Rolewicz [21] about complete p-normed algebras.

On the other hand I do not see how the results of P. Turpin and myself [38], [39], [40], [46] can be proved along these lines.

(d) An operator f on a topological vector space is *bornifying* if $f(U)$ is bounded for some neighbourhood U of the origin. A set B of bornifying operators is equibornifying if a neighbourhood U of the origin exists such that

$$B(U) = \{b(u) \mid b \in B, u \in U\}$$

is bounded.

Let E be a sequentially complete locally convex space, or more generally one in which all closed convex, balanced, bounded subsets are completant. *With its equibornifying boundedness, the algebra of bornifying operators on E is a mb-algebra.*

The word completant has been used. A convex balanced subset B of a vector space E is completant if its Minkowski (gauge) functional is a Banach space norm. A closed, bounded, convex balanced subset of a topological vector space is completant if it is sequentially complete.

The existence of this bounded structure is important, implicit in P. Uss's theory of bornifying operators [41], though Uss does not speak of bounded structures.

Hogbe-Nlend [16] credits M. Akkar [1] with the description of this bounded structure.

(e) Let $\mathscr{A}$ be a commutative unital b-algebra (i.e. a b-space with a bounded bilinear multiplication). An element $a \in \mathscr{A}$ is regular ([42], [43]) when $(a-s)^{-1}$ is defined and bounded on some neighbourhood of infinity, i.e. for $|s| > M$. It is bounded [3] if a^n/M^n is bounded when M is a large real number.

The bounded and the regular elements are of course the same when $\mathscr{A}$ is a b-algebra. The only difference between these notions is one of applicability of the theory in the non-complete case.

Let $\mathscr{A}_r$ be the set of regular elements of $\mathscr{A}$. Then $\mathscr{A}_r$ is a subalgebra of $\mathscr{A}$. A set $B \subseteq \mathscr{A}_r$ will be called "Allan bounded" if $B \subseteq MB_1$ for some $M \in \boldsymbol{R}_+$ and some bounded idempotent B_1. Let

$$B_2 = \left\{ \sum_1^\infty \lambda_n x_n \,\middle|\, \sum_1^\infty |\lambda_n| \leqslant 1, \forall n\colon\ x_n \in B_1 \right\}.$$

Then B_2 is a Banach ball, is idempotent. With the Minkowski functional (gauge) of B_2 as norm, the vector space $\mathscr{A}_{B_2}$ absorbed by B_2 is a Banach algebra.

If B_2 and B_2' are two idempotent Banach balls, both containing the unit, then $B_2'' \supseteq B_2 \cup B_2'$ if

$$B_2'' = \left\{ \sum_1^\infty \lambda_n x_n y_n \,\middle|\, x_n \in B_2, y_n \in B_2', \sum_1^\infty |\lambda_n| \leqslant 1 \right\}$$

and B_2'' is a bounded idempotent Banach ball. We see that

$$\mathscr{A}_r = \bigcup \mathscr{A}_{B_2}$$

is the union of a directed family of Banach algebras. It is an mb-algebra.

6.6. This is the place to mention a class of algebras, which I have often called "continuous inverse algebras", and which are called Q-algebras by the Polish mathematicians. They are the algebras where the inverse is defined on an open set, and is continuous there. The Polish school is ready to call "continuous inverse" any algebra where the mapping $a \to a^{-1}$ is continuous on its domain. I am reluctant to give the qualification to an algebra if the set of invertible elements is not large enough.

However, these notes are being published in Poland, I shall adopt the Polish terminology.

All elements of a Q-algebra are regular. The structure space theory, the h.f.c. apply to complete locally convex Q-algebras. These algebras are the topological algebras to which one expects the statement "*the set $U_{\mathscr{A}}$ of $(a_1, \ldots, a_n)$ such that* $\mathrm{sp}(a_1, \ldots, a_n) \subseteq U$ *is open, the mapping $f \to f(a_1, \ldots, a_n)$, $U_{\mathscr{A}} \to \mathscr{A}$ is continuous*" to generalize... U is of course an open subset of $\boldsymbol{C}^n$ and $f \in \mathcal{O}(U)$. And as a matter of fact, this statement does generalize to Q-algebras, at least complete locally convex ones, and complete locally pseudo-convex ones [where locally pseudo-convex means that the topology is defined by a family of p-norms and p-semi-norms, where each p is between 0 and 1, $0 < p \leqslant 1$, but p depends eventually on the norm and the semi-norm].

I have never studied systematically the class of Q-algebras, and do not know whether such a systematic exposition exists.

Some facts can be mentioned. A separately continuous multiplication on a Fréchet topological vector space is jointly continuous. This algebra is a Q-algebra if the set of its invertible elements is open. By the way, this algebra is a continuous inverse algebra if and only if the set of invertible elements is a G_δ set. (Here, continuous inverse is given its Polish meaning.)

If $\mathscr{A}$ is a topological vector space with an associative algebra structure, if the set of invertible elements is a neighbourhood of the unit and $a^{-1} \to 1$ when $a \to 1$, Turpin shows that the Jordan multiplication of $\mathscr{A}$, $(a, b) \to \frac{1}{2}(ab+ba)$ is continuous, that the set of invertible elements is open, and that the mapping $a \to a^{-1}$ is continuous ([37]).

Further, if $\mathscr{A}$ is a locally convex, then $\mathscr{A}$ is locally m-convex (the topology of $\mathscr{A}$ can be defined by submultiplicative semi-norms). However, non locally m-convex Q-algebras exist whose topology can be defined by p-semi-norms, but cannot be defined by submultiplicative p-semi-norms ([37]).

Banach algebras are of course Q-algebras. So are separated countable direct limits of Banach algebras. This takes care of algebras like $\mathcal{O}(X)$ and $\mathcal{O}(X, \mathscr{A})$. With its usual topology, the algebra of functions of class C_∞ on a compact manifold is a Q-algebra. So is the quotient $\mathscr{E}(U)/\mathscr{E}_0(U, X)$ of the algebra of functions of class C_∞ on a manifold by the algebra of functions of class C_∞ on that manifold which vanish along with all their derivatives on a compact subset X of U.

For a locally convex algebra to be a Q-algebra, it is necessary and sufficient that

(i) it be locally m-convex,

(ii) the set of characters of $\mathscr{A}$ be compact,

(iii) the spectral semi-norm

$$q(a) = \max\{|s| \mid s \in \operatorname{sp} a\}$$

be continuous.

6.7. In examples (a), (b) of Section 6.5, the algebra $\mathscr{A}$ was given with, or almost with its mb-structure. The continuity properties of the h.f.c. follow directly from the consideration of this mb-structure. The continuity statements that one obtains are the statements that one expects.

The situation prevailing in example (e) is not unexpected. $\mathscr{A}_r$ is a smaller algebra than $\mathscr{A}$. The bounded structure of $\mathscr{A}_r$ is finer than the induced bornology of $\mathscr{A}$. The continuity properties of the h.f.c. relate to the Allan boundedness of $\mathscr{A}_r$ and not to its original bounded structure.

In example (c), the situation is less agreeable. It is a fact that the mapping $(a_1, \ldots, a_n) \to f[a_1, \ldots, a_n]$ is continuous. Unfortunately, I do not see that this follows obviously from the idempotent bounded structure that Allan, Dales, and McClure introduce on our complete p-normed algebra, nor is it clear that another, grosser bounded structure would allow us to obtain the continuity of $f[a_1, \ldots, a_n]$ as a corollary of the analogous continuity in the Banach algebra case.

6.8. Another remark can be made about example (e), Section 6.5. The setting is absolutely not unusual in spectral theory. $\mathscr{A}$ is an algebra, $\mathscr{A}_r$ is the set of its regular elements. We could think of general operators and bounded operators, or general and bounded functions. This was at the back of my head when I defined the regular elements. G. Allan [4] made the remark explicitly.

Nothing special has come out of this, to my knowledge. Maybe nobody had the correct idea. But I believe that there is a more serious obstacle on the way.

Think of the Heaviside functional calculus. The differential operator is not a regular element of the relevant algebra, e.g. the algebra of distributions with support in $\boldsymbol{R}_+$. It has an inverse, the primitive operator. This inverse is regular, its regular spectrum (its spectrum in $\mathscr{A}_r$) is the origin.

Considerations such as these allow us to define $f\left(\dfrac{d}{dt}\right)$ when f is holomorphic at infinity, and by extension when f has a pole at infinity. The Heaviside calculus makes it possible to define $f\left(\dfrac{d}{dt}\right)$ when f is holomorphic in a half-plane $\operatorname{Re} z > M$ with polynomial growth at infinity on its domain.

To recapture the Heaviside calculus with the spectral techniques described here, we must not only know the regular spectrum of d/dt. We must also know subsets of $\boldsymbol{C}$ on which the resolvent of d/dt is bounded.

7. Central elements of $\mathscr{A}$

7.1. The algebras considered up to now were commutative. This hypothesis will be weakened slightly. $\mathscr{A}$ will be a Banach algebra, $a_1, \ldots, a_n$ will be elements of the center. The operational calculus will still be essentially commutative. The results presented here will be the Banach algebra special case of results obtained in 1960 [44].

My motive for presenting these computations is the fact that they use an algebraic structure not unrelated to Nelson's full symmetric algebra (cf. the paper by E. Albrecht, [2], these proceedings). This will allow me to illustrate the fact that the full symmetric algebra is a very useful algebraic structure. But full results can only be obtained by introducing unsymmetric products in a limited number of critical places (see also Section 7.11).

7.2. The h.f.c. that will be introduced here is related to that of J. L. Taylor [31], [32]. Of course, Taylor's construction is module-related, while mine is algebra-related. But algebra-related functional calculi are special cases of module related ones. Taylor can chase through a diagram associated to a double complex and identify two classes of cohomology. He integrates one, I integrate the other.

The formula by which I define the h.f.c. appears more explicit than Taylor's, but Taylor's results reach further since his spectrum is smaller (when $\mathscr{A}$ operates on a module M).

7.3. I was led to the computations presented here by a coefficient. I had found a formula which defined the h.f.c. in the commutative case. This formula involved the differential form

$$\frac{(n+k)!}{k!} y^k \bar{\partial} u_1 \wedge \ldots \wedge \bar{\partial} u_n,$$

where $y, u_1, \ldots, u_n$ are $\mathscr{A}$-valued functions which will be described later. And $(n+k)!/k!$ is the number of different products which one can write, when $\mathscr{A}$ is associative but not commutative, with k factors equal to y, and n other factors different from each other.

Is this a coincidence, or would the kernel

$$\sum \pm \varphi_r$$

define an h.f.c. in the non-commutative case, if the φ_r are the $(n+k)!/k!$ different products of k factors equal to y and $\bar{\partial} u_1, \ldots, \bar{\partial} u_n$, the sign $\pm$ depending on the parity of the order of $\bar{\partial} u_1, \ldots, \bar{\partial} u_n$. It turns out that it does (when $a_1, \ldots, a_n$ are in the center of $\mathscr{A}$).

However, to make the proofs readable, I introduced an auxiliary algebra. The introduction of this algebra obscured the initial motivation of this research, i.e. the attempt to understand a coefficient. But it links up with the considerations of E. Nelson and E. Albrecht.

7.4. Let $\mathscr{A}$ be a unital Banach algebra. Let $a_1, \ldots, a_n$ be elements of the center of $\mathscr{A}$.

DEFINITION. The *spectrum of* $(a_1, \ldots, a_n)$ is the set of $(s_1, \ldots, s_n) \in C^n$ such that $1 \notin \sum_1^n (a_i - s_i)\mathscr{A}$.

Let $M_{a_i}: \mathscr{A} \to \mathscr{A}$ be the multiplication operators $x \to a_i x$. The spectrum of $(a_1, \ldots, a_n)$ is the J. L. Taylor spectrum of $(M_{a_1}, \ldots, M_{a_n})$. This is, or should be clear. If $(s_1, \ldots, s_n)$ is in the J. L. Taylor resolvent set of $(M_{a_1}, \ldots, M_{a_n})$, the mapping $(u_1, \ldots, u_n) \to \sum_1^n (a_i - s_i) u_i$, $\mathscr{A}^n \to \mathscr{A}$ is surjective. (This is part of the Taylor regularity condition.) And $\mathscr{A} = \sum_1^n (a_i - s_i)\mathscr{A}$. On the other hand, J. L. Taylor ([45], Lemma 1.1) proves that $(s_1, \ldots, s_n)$ is in his resolvent set of $(M_{a_1}, \ldots, M_{a_n})$ if operators $U_1, \ldots, U_n$ commuting with $(M_{a_1}, \ldots, M_{a_n})$ exist, such that $\sum (M_{a_i} - s_i I) U_i = I$, I the identity operator. And this relation is verified if we put $U_i = M_{u_i}$, where $\sum (a_i - s_i) u_i = 1$ in $\mathscr{A}$.

Each representation of $\mathscr{A}$, each $\mathscr{A}$-module gives $(a_1, \ldots, a_n)$ a Taylor spectrum.

The spectrum of $(a_1, \ldots, a_n)$ that we are considering is the union of the spectra of $(a_1, \ldots, a_n)$ for all possible $\mathscr{A}$-modules, and it is the spectrum of $(a_1, \ldots, a_n)$ for the left regular representation of $\mathscr{A}$.

The spectrum of $(a_1, \ldots, a_n)$ is, for all these reasons, compact, non empty, and the spectrum of $(a_1, \ldots, a_{n'})$ is the projection of the spectrum of $(a_1, \ldots, a_n)$. (Because the Taylor spectrum has these properties.)

7.5. PROPOSITION. *Let U be a neighbourhood of* $\operatorname{sp}(a_1, \ldots, a_n)$. *$\mathscr{A}$-valued functions $u_1, \ldots, u_n, y$ of class C_∞ exist, where y has compact support in U, such that*

$$\sum_1^n (a_i - s_i) u_i + y = 1$$

on C^n.

This is simple. For each s in the resolvent set, choose $u_{i,s} \in \mathscr{A}^n$ such that $\sum (a_i - s_i) u_{i,s} = 1$. If s_i' is near to s_i, define

$$u_{i,s}(s') = \Big(1 - \sum (s_i' - s_i) u_{i,s}\Big)^{-1} u_{i,s}$$

so that

$$\sum_1^n (a_i - s_i') u_{i,s}(s') = 1$$

when s' is near to s. The function $u_{i,s}(s')$ is holomorphic in s', therefore of class C_∞ on a neighbourhood of s. A partition of unity allows us to find functions of class C_∞, say $u_i'(s)$, on the complement of the spectrum, and solutions of the equations $\sum (a_i - s_i) u_i'(s) = 1$.

Take next a function y, of class C_∞ on C^n, with compact support in U, and equal to one on a neighbourhood of the spectrum. Let $u_i = u_i'(1-y)$ on the resolvent set, $u_i = 0$ on the spectrum; $(u_1, \ldots, u_n, y)$ has the announced properties.

7.6. This is time to introduce the tensor algebra with symmetries, and the symmetric tensor algebra.

Let $\mathscr{A}$ be a Banach algebra with unit, and let $\mathscr{A}_1$ be the center of $\mathscr{A}$. Let M_1, M_2 be $\mathscr{A}$-modules. $M_1 * M_2$ will denote the $\mathscr{A}_1$-projective tensor product of M_1 and M_2, i.e. the quotient of $M_1 \hat{\otimes} M_2$ by the closed subspace generated by the elements $am_1 \otimes m_2 - m_1 \otimes am_2$, $m_1 \in M_1$, $m_2 \in M_2$, $a \in \mathscr{A}_1$.

$\mathscr{A}^{*N}$ is the $*$-tensor product of N copies of $\mathscr{A}$. We associate to $p \in S_N$ a linear transformation of $\mathscr{A}^{*N}$, defined on generators by

$$p(x_1 * \ldots * x_N) = x_{p_1} * \ldots * x_{p_N}$$

(S_N is the symmetric group, p maps i on p_i). We also define

$$\operatorname{sym}_N = \frac{1}{N!} \sum_{p \in S_N} p.$$

If Λ is an exterior algebra, we define p and sym_N on $\Lambda(\mathscr{A}^{*N}) = \mathscr{A}^{*N} \otimes \Lambda$, letting p or sym_N respectively operate on $\mathscr{A}^{*N}$ and the identity on Λ. This convention takes good care of the signs when factors involving differential forms are concerned. If $p \in S_3$ is the permutation $(1, 2, 3) \to (3, 1, 2)$ and u, v, w are $\mathscr{A}$-valued functions

$$p(u * \bar{\partial} v * \bar{\partial} w) = -\bar{\partial} w * u * \bar{\partial} v.$$

We call mult, or when doubt can arise over the degrees, mult_N the mapping $\mathscr{A}^{*N} \to \mathscr{A}$ induced by multiplication

$$a_1 * \ldots * a_n \to a_1 \ldots a_n.$$

Partial multiplication operators are also important. If $N = r_1 + \ldots + r_k$, such a partial multiplication maps $\mathscr{A}^{*N}$ into $\mathscr{A}^{*k}$, mapping $a_1 * \ldots * a_N$ onto

$$a_1 \ldots a_{r_1} * a_{r_1+1} \ldots a_{r_1+r_2} * \ldots * a_{r_1+\ldots+r_{k-1}+1} \ldots a_n.$$

7.7. The symmetric tensor algebra is the algebra

$$\mathscr{A}^{\circ} = \bigoplus \mathrm{sym}\, \mathscr{A}^{*N}$$

of symmetric tensors, with the convention $\mathscr{A}^{\circ 0} = \mathscr{A}_1$ if $\mathscr{A}_1$ is the center of $\mathscr{A}$. We write

$$\mathscr{A}^{\circ N} = \mathrm{sym}\, \mathscr{A}^{*N}$$

and, when $u \in \mathscr{A}^{\circ N}$, $v \in \mathscr{A}^{\circ M}$

$$u \circ v = \mathrm{sym}\, u * v \in \mathscr{A}^{\circ M+N}.$$

With these conventions, $\mathscr{A}^{\circ}$ is a commutative graded algebra.

Consider now the equation

$$\sum (a_i - s_i) u_i(s) + y(s) = 1.$$

Unfortunately, the relation can only hold in $\mathscr{A}^{\circ}$ if we interpret the right-hand side as the "unit of degree 1", i.e. the element of $\mathscr{A}^{\circ 1} = \mathscr{A}$ corresponding to the unit of $\mathscr{A}$. The unit of $\mathscr{A}^{\circ}$ is "the unit of degree 0", i.e. the element of $\mathscr{A}^{\circ 0} = \mathscr{A}_1$ (the center) corresponding to the unit of $\mathscr{A}$.

We can work in the quotient of $\mathscr{A}^{\circ}$ by the difference of these two "units". This is a unital commutative b-algebra, in which the relation

$$\sum (a_i - s_i) u_i(s) + y(s) = 1$$

holds, where 1 is the unit of the algebra. The quotient is not graded any more. It is "filtered", it is the union of a sequence of Banach spaces.

7.8. I would probably follow this path, work in that quotient of $\mathscr{A}^{\circ}$ if the path went all the way to the result we want. But it does not. Ninety percent of the results can probably be handled by this remark when $\mathscr{A}$ is a non-commutative Banach algebra, once the analogous result is proved in the commutative case.

But the theory is incomplete if we cannot prove the remaining 10% of the theorems.

For this reason, it seems reasonable to work in $\mathscr{A}^{\circ}$ and in $\mathscr{A}^{*} = \bigoplus \mathscr{A}^{*N}$. The mappings $u \to 1 \circ u$, $u \to 1 * u$, $\mathscr{A}^{\circ N} \to \mathscr{A}^{\circ N+1}$ or $\mathscr{A}^{*N} \to \mathscr{A}^{*N+1}$ map elements onto each other that we want to identify but choose not to.

The following result is in the good 90%. If $\mathscr{A}$ is a commutative Banach algebra, if $a_1, \ldots, a_n$ are elements of $\mathscr{A}$, if U is a neighbourhood of $\mathrm{sp}(a_1, \ldots, a_n)$, if u_i, y are $\mathscr{A}$-valued functions of class C_∞ which satisfy the relation

$$\sum_{1}^{n} (a_i - s_i) u_i(s) + y(s) = 1,$$

y having compact support in U, then

$$\frac{(n+k)!}{k!} y^k \bar{\partial} u_1 \wedge \ldots \wedge \bar{\partial} u_n$$

is in a class of $\bar{\partial}$-cohomology with compact support in U, which depends only on $a_1, \ldots, a_n$, and not on k, nor on u or on y. The reader can find a proof of this result in Bourbaki [64].

If $a_1, \ldots, a_n$ belong to the center of $\mathscr{A}$, if we consider the algebra $\mathscr{A}^{\circ}$, a straightforward application of the above result shows that the class of $\bar{\partial}$-cohomology with compact support in U, of the form

$$\frac{(n+k)!}{k!} \mathrm{mult}\, y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

does not depend on u, y or on k, where $\sum (a_i - s_i) u_i + y = 1$.

7.9. I shall outline two proofs in the remaining 10%. The reader will see that the symmetric tensor algebra is useful, even in these two proofs, only not all computations can be carried out in this algebra.

Letting $\mathscr{A}$, $a_1, \ldots, a_n$, $u_1, \ldots, u_n$, y be as at the end of the preceding section (i.e. $\mathscr{A}$ associative, a_i central, etc.) we define

$$\omega = \frac{(n+k)!}{k!} \mathrm{mult}\, y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n.$$

PROPOSITION. *The $\mathscr{A}^{*2}$-valued form*

$$1 * \omega - \omega * 1$$

is equal to $\bar{\partial}\varphi$ for some $\mathscr{A}^{*2}$-valued form φ with compact support in U.

This result is significant. Let $a \in \mathscr{A}$ be such that $1 * a = a * 1$. Then a is in the center of $\mathscr{A}$. The best way to see this is to observe that $1 * x * a = a * x * 1$ for all x in $\mathscr{A}$: apply a suitable permutation to the relation $x * 1 * a = x * a * 1$. Multiplication then shows that $xa = ax$.

The proposition, in a way, means that ω is cohomologically central.

The proof derives from the classical proof of the fact that the cohomology of

$$\frac{(n+k)!}{k!} y^k \bar{\partial} u_1 \wedge \ldots \wedge \bar{\partial} u_n$$

does not depend on k. But, in its application to the tensor algebra, we take care not to symmetrize everything. Let

$$\tilde{\omega}_k = \sum (-1)^{i-1} u_i * y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n$$

where, as usual, $\bar{\partial}\hat{u}_i$ means that the factor is omitted in the product. Then

$$\bar{\partial}\tilde{\omega}_k = \sum (-1)^{i-1} \bar{\partial} u_i * y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n +$$
$$+ k \sum (-1)^{i-1} u_i * y^{\circ k-1} \circ \bar{\partial} y \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n .$$

We differentiate the equation

$$\sum (a_i - s_i) u_i + y = 1$$

and obtain

$$\sum (a_i - s_i) \bar{\partial} u_i + \bar{\partial} y = 0 .$$

We replace $\bar{\partial} y$ by $-\sum (a_j - s_j) \bar{\partial} u_j$ in the second term of $\bar{\partial}\tilde{\omega}_k$. We remember that $\bar{\partial} u_j \circ \bar{\partial} u_j = 0$, also that the expression is $\mathscr{A}_1$-multilinear if $\mathscr{A}_1$ is the center of $\mathscr{A}$, the factor $(a_i - s_i)$ can be transferred from any factor in this product to any other factor,

$$k \sum (-1)^{i-1} u_i * y^{\circ k-1} \circ \bar{\partial} y \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n$$
$$= -k \sum (-1)^{i-1} u_i * y^{\circ k-1} \circ (a_i - s_i) \bar{\partial} u_i \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n$$
$$= -k \sum (a_i - s_i) u_i * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$
$$= k y * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n - k (1 * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n)$$

and therefore

$$\bar{\partial}\tilde{\omega}_k = \sum (-1)^{i-1} \bar{\partial} u_i * y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n +$$
$$+ k (y * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n) - k (1 * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n).$$

If we develop $y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$, we see that

$$(n+k) y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n = k y * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n +$$
$$+ \sum (-1)^{i-1} \bar{\partial} u_i * y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial}\hat{u}_i \circ \ldots \circ \bar{\partial} u_n$$

hence

$$\bar{\partial}\tilde{\omega}_k = (n+k) y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n - k (1 * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n).$$

This proves that the cohomology of

$$\frac{(n+k)!}{k!} y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

and

$$\frac{(n+k-1)!}{(k-1)!} 1 * y^{\circ k-1} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

are equal. Similarly

$$\frac{(n+k+r)!}{(k+r)!} y^{\circ k+r} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

and

$$\frac{(n+k)!}{k!} 1^{*r} * y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

are in the same class of cohomology with compact support. This class of cohomology is symmetric under the operation of the symmetric group of order $k+r$. A judiciously chosen permutation and multiplication operator shows that

$$\left[\frac{(n+k)!}{k!} \operatorname{mult} y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n \right] * 1$$

and

$$1 * \frac{(n+k)!}{k!} \operatorname{mult} y^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$$

are in the same class of $\bar{\partial}$-cohomology with compact support.

7.10. The following uses also more structure than that of the full symmetric algebra.

Let $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ be elements of the center of $\mathscr{A}$. Let U and V be open neighbourhoods of $\operatorname{sp}(a_1, \ldots, a_n)$ and of $\operatorname{sp}(b_1, \ldots, b_m)$ respectively. Then $U \times V$ is a neighbourhood of $\operatorname{sp}(a_1, \ldots, a_n, b_1, \ldots, b_m)$.

Classes of cohomology with compact supports in U, V, and $U \times V$ are associated to a, b, and (a, b) respectively. We shall call these $\omega(a)$, $\omega(b)$, and $\omega(a, b)$. It is expected that

$$\omega(a, b) = \omega(a) \otimes \omega(b).$$

Let us show that this is the case.

Start with the relations

$$\sum (a_i - s_i) u_i(s) + y(s) = 1,$$
$$\sum (b_j - t_j) v_j(t) + x(t) = 1,$$

where y and x have compact support in U and V, respectively. We have

$$\sum (a_i - s_i) u_i(s) + \sum (b_j - t_j) y(s) v_j(t) + y(s) x(t) = 1 .$$

The function $y(s)x(t)$ has compact support in $U \times V$. Therefore, $\omega(a, b)$ is the class of cohomology containing

$$\frac{(n+m+k)!}{k!} (y \cdot x)^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n \circ \bar{\partial} y v_1 \circ \ldots \circ \bar{\partial} y v_m .$$

We know that $\bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n$ is of maximum degree in $d\bar{s}_1, \ldots, d\bar{s}_n$, and that $\bar{\partial} y$ does not depend on $d\bar{t}_1, \ldots, d\bar{t}_m$. It follows that this form is equal to

$$\frac{(n+m+k)!}{k!} (y \cdot x)^{\circ k} \circ \bar{\partial} u_1 \circ \ldots \circ \bar{\partial} u_n \circ y \bar{\partial} v_1 \circ \ldots \circ y \bar{\partial} v_m .$$

Consider now

$$1 * (y \cdot x)^{\circ k} \circ y\bar{\partial}v_1 \circ \ \ldots \ \circ y\bar{\partial}v_m.$$

This is equal to

$$\widetilde{\mathrm{mult}}(1 * y^{*m+k} * x^{\circ k} \circ \bar{\partial}v_1 \circ \ \ldots \ \circ \bar{\partial}v_m),$$

where $\widetilde{\mathrm{mult}}\colon \mathscr{A}^{*(2m+2k+1)} \to \mathscr{A}^{*(m+k+1)}$ is a partial multiplication operator mapping

$$\alpha * \beta_1 * \ldots * \beta_{m+k} * \gamma_1 * \ldots * \gamma_{m+k}$$

onto

$$\alpha * \beta_1\gamma_1 * \ldots * \beta_{m+k}\gamma_{m+k}.$$

We also know that

$$1 * y^{*m+k} * x^{\circ k} \circ \bar{\partial}v_1 \circ \ \ldots \ \circ \bar{\partial}v_m$$

and

$$\mathrm{mult}(x^{\circ k} \circ \bar{\partial}v_1 \circ \ \ldots \ \circ \bar{\partial}v_m) * y^{*m+k} * 1^{*m+k}$$

are in the same class of $\bar{\partial}$-cohomology with compact support in V. It follows that

$$1 * (y \cdot x)^{\circ k} \circ y\bar{\partial}v_1 \circ \ldots \circ y\bar{\partial}v_m$$

and

$$\mathrm{mult}(x^{\circ k} \circ \bar{\partial}v_1 \circ \ \ldots \ \circ \bar{\partial}v_m) * y^{*m+k}$$

only differ by the coboundary of a differential form with compact support in V.

From the time we stopped writing $\bar{\partial}u_1 \circ \ldots \circ \bar{\partial}u_n$ in the computations, we have treated s as a constant or a parameter. This is legitimate, we must now multiply again by $\bar{\partial}u_1 \wedge \ldots \wedge \bar{\partial}u_n$, and symmetrize on the last $m+n+k$ factors. Any differential ds_i which appears in the process is mapped on zero when we multiply by that form. We see in this way that

$$1 * (y \cdot x)^{\circ k} \circ \bar{\partial}u_1 \circ \ldots \circ \bar{\partial}u_n \circ \bar{\partial}yv_1 \circ \ldots \circ \bar{\partial}yv_m$$

and

$$(-1)^{mn}\,\mathrm{mult}(x^{\circ k} \circ \bar{\partial}v_1 \circ \ldots \circ \bar{\partial}v_m) * y^{\circ m+k} \circ \bar{\partial}u_1 \circ \ldots \circ \bar{\partial}u_n$$

only differ by the $\bar{\partial}$-coboundary of a differential form with compact support in $U \times V$. We multiply this relation by

$$\frac{(m+n+k)!}{k!} = \frac{(m+n+k)!}{(m+k)!}\,\frac{(m+k)!}{k!}$$

and apply a mult operator. The required result follows

PROPOSITION. *$\omega(a, b)$ is the Cartesian product of $\omega(a)$ and $\omega(b)$.*

7.11. It is the talks by E. Albrecht on the operand algebra of E. Nelson that decided me to include the above considerations in this ideological series of talks. The algebra $\mathscr{A}^{\circ}$ is not unlike Nelson's full symmetric algebra.

The proofs given in Sections 7.9 and 7.10 involve both symmetric products of tensors and usual tensor products. In a way, it should be expected that non symmetric products would be useful if we must somehow approximate the structure of a non commutative algebra.

I do not know what a non-commutative operational calculus should be like. It should be useful, there are cases where Nelson's operand algebra is. But my ideology wills that such a non-commutative object should not be entirely commutative. Otherwise, we cannot write any commutator!

8. Non regular elements

8.1. There was a last talk in the series. I tried to present results involving the h.f.c. for elements which are not regular. These results were proved in 1960 [44]. The memoir which contains these is unavailable in many libraries, is at present out of print, and to say the least, is difficult to read.

I agree with several mathematicians that the time has come when these results must be presented anew. But such a new presentation does not make sense unless the results we are discussing are combined with those of J. L. Taylor [31], [32]. Such a combination should not be too difficult, but it has not been achieved yet (to my knowledge).

In any case, the enterprise of combining Taylor's results and mine goes beyond this ideological series of talks.

8.2. In my 1960 results, I considered the quotient of a b-algebra by a two sided b-ideal. This was initally an unmotivated generalization.

The generalization was introduced to carry one proof to the end. (The generalization of the result that $\mathrm{sp}(a_1, \ldots, a_n)$ is not empty.) Subsequent proofs remained feasible, but became more difficult. However in applications, it turns out that the originally unmotivated generalization is useful.

I have mentioned J. L. Taylor. There are places where he is hampered by the fact that he uses non-abelian (non-exact) categories in homological algebra. The category of quotient complete bornological spaces is an exact category. I do not doubt that [33] can be rewritten in that category. The theory would presumably be more elegant, if only one is convinced that quotient complete bornological spaces are relevant to functional analysis.

Not much has been published about these formal quotients. The reader will find something about quotient Banach spaces in these proceedings [52], [53], [54]. Results about quotient bornological spaces have been announced elsewhere [48], [49], [50], [51].

8.3. It seems reasonable in any case to speak about single elements of a b-algebra. The n-dimensional generalizations may be difficult, but the one-dimensional results do indicate in what direction one might find generalizations.

b-algebras have been defined in Section 6.1. The reader may not feel what is a b-algebra. Let him think about complete locally convex algebras. But the applications of the theory are richer when the theory is developed in the category of b-algebras.

8.4. DEFINITION. Let $\mathscr{A}$ be a b-algebra, and $a \in \mathscr{A}$. A set $S \subseteq \mathscr{A}$ is *spectral* for a if $(a-s)^{-1}$ is defined and bounded on the complement of S.

Note, if S has an unbounded complement, that it would be equivalent to assume that $(a-s)^{-1}$ has polynomial growth at infinity. This follows from the limited geometric expansion identity

$$(a-s)^{-1} = -s^{-1}-s^{-2}a- \ldots -s^{-k}a^{k-1}+s^{-k}a^k(a-s)^{-1}$$

which shows that the resolvent is bounded on a set if it grows at most like a polynomial of degree k.

PROPOSITION. *The set of S, spectral for a, is a filter with open basis. The resolvent is a function of class C_∞ on $C^*\setminus S$ if $C^* = C\cup\{\infty\}$ is the complex sphere and S is an open spectral set.*

Let us first prove that the spectral filter is a filter with open basis, i.e. that the interior of a spectral set is spectral, and that the empty set is not spectral. Let $(a-s)^{-1}$ be defined and bounded on $T\subseteq C$. Consider the resolvent identity

$$(a-s)^{-1}-(a-t)^{-1} = (s-t)(a-s)^{-1}(a-t)^{-1}.$$

We see that the resolvent is uniformly continuous on T. It extends to the closure of T. The extension is a bounded function on $\overline{T}$, and is the resolvent on $\overline{T}$. This shows already that the spectral filter has an open basis.

We do not yet know that it is a true filter, that $\emptyset$ is not spectral. But if $\emptyset$ is spectral, the resolvent identity shows that the resolvent is a bounded entire function. And it is not a constant.

We use the definition of differentiable functions on closed sets given by H. Whitney. A function of class C_∞ is of class C_r for all r. A function f of class C_r on a closed set $T\subseteq R^n$ is a system of functions $(f_{p_1\ldots p_n})$, $|p| = p_1+ \ldots +p_n \leqslant r$, where we must consider $f_{p_1\ldots p_n}$ as "derivatives" of f, and

$$f_p(x+h) = \sum_{|q|<r-|p|} f_{p+q}(x)\frac{h^q}{q!}+o(|h|^{r-|q|})$$

with the usual multi-index notations, the relation holding when $x\in T$, $x+h\in T$, the o-estimate holding uniformly when $h\to 0$.

The resolvent is complex-differentiable of class C_r on $T\cap C$. This follows from the limited geometric expansion relation,

$$(a-s-h)^{-1} = (a-s)^{-1}+h(a-s)^{-2}+ \ldots +h^r(a-s)^{-r-1}+h^{r+1}(a-s)^{-r-1}(a-s-h)^{-1}$$

which already shows that $(a-s-h)^{-1}$ is asymptotically the sum of the (not necessarily convergent) series $\sum(a-s)^{-k-1}h^k$. We must also show that the derivatives of the resolvent are asymptotically sums of series. The derivatives of the resolvent are products of constant factors and powers of the resolvent

$$\frac{d^k}{ds^k}(a-s)^{-1} = k!\,(a-s)^{-k-1}.$$

If we raise the asymptotic expansion of $(a-s-h)^{-1}\sim\sum(a-s)^{-l-1}h^l$ to the power $k+1$ and multiply by $k!$ we obtain an asymptotic expansion for $k!(a-s)^{-k-1}$, i.e. for $[d^k/ds^k](a-s)^{-1}$, and an inspection shows that this is the required Taylor expansion of the derivative.

To prove that the resolvent is differentiable at infinity, we must consider the differentiability of $(a-s^{-1})^{-1}$ for s near to zero. Of course

$$(a-s^{-1})^{-1} = -s(1-sa)^{-1}$$

so that we are led to investigate $(1-sa)^{-1}$. We have the asymptotic expansion

$$(1-(s+h)a)^{-1}\sim \sum(1-sa)^{-r-1}h^ra^r.$$

The derivatives of $(1-sa)^{-1}$ with respect to s have the form

$$r!a^r(1-sa)^{-r-1}.$$

Raising the asymptotic expansion of $(1-(s+h)a)^{-1}$ to the power $r+1$ and multiplying by $r!a^r$, we again obtain the required result.

8.5. We can now apply Whitney's result, about the extension of differentiable functions off closed sets.

PROPOSITION. *Let S be a spectral set, and T its complement. A function u of class C_r exists on the sphere $C^* = C\cup\{\infty\}$ such that u is C-differentiable to the order r for every $t\in T$. The Taylor expansion of $(a-t-h)^{-1}$ around $t\in T$ is*

$$(a-t-h)^{-1}\sim\sum_0^r(a-t)^{-k-1}h^k.$$

The Taylor expansion of $(a-s^{-1})^{-1} = -s(1-sa)^{-1}$ around the origin is

$$-s(1-sa)^{-1}\sim -\sum_0^r s^{k+1}a^k.$$

We notice that $(a-t)u(t) = 1$ when $t\in T$, the difference $1-(a-t)u(t)$ is of class C_{r-1} and vanishes with its derivatives to the order $r-1$ on T.

8.6. J. Sebastiaõ e Silva [22], [23], [24], [25], [26] has constructed operational calculi involving spectral sets with smooth boundaries. The following could be a technique for constructing such calculi.

Start by assuming that S is a bounded simply connected spectral set with a smooth boundary. Let $\delta_S(s) = \inf_{t\notin S}|s-t|$. Let $u(s)$ be holomorphic and such that $\delta_S^k u$ is bounded for some k. Then some primitive of u is bounded, even continuous to the boundary. Let $f = g^{(r)}$. We have, when $s\in S$

$$f(s) = \frac{r!}{2\pi i}\int_{\partial S} g(\xi)(\xi-s)^{-r-1}d\xi.$$

If S is multiply connected, we find a rational function $\varphi(s)$ in such a way that $f-\varphi$ has the required primitives. Then $f = g^{(r)}+\varphi$.

If S is unbounded, we let $\delta_S(s) = \inf_{t\notin S}[|s-t|, (1+|s|^2)^{-1/2}]$, and call $\mathcal{O}(\delta_S)$ the set of f, holomorphic on S, such that $\delta_S^k f$ is bounded for some k. Let $f \in \mathcal{O}(\delta_S)$ and let t be in the exterior (the interior of the complement) of S. Then $(z-t)^{-r_1}f-\varphi$ has a primitive which is summable on the boundary of S,

$$f(s) = (s-t)^{r_1}g^{(r)}(s)+\varphi(s).$$

This allows us to define, to try to define

$$f[a] = \frac{r!}{2\pi i}(a-t)^{r_1}\int_{\partial S} g(\xi)(\xi-a)^{-r-1}d\xi+\varphi(a)$$

and hope that the operation defines a reasonable h.f.c.

It does. It must. The integral

$$\int_{\partial S} g(\xi)(\xi-z)^{-1}d\xi$$

converges in the b-algebra $\mathcal{O}(\delta_S)$. If a bounded morphism $\mathcal{O}(\delta_S)\to\mathcal{A}$ exists, which maps unit on unit and z on a, this bounded morphism maps

$$f(z) = \varphi(z)+\frac{r!}{2\pi i}(z-t)^{r_1}\int_{\partial S} g(\xi)(\xi-z)^{-r-1}d\xi$$

onto $f[a]$. And such a bounded morphism does exist.

8.7. It is possible to prove the existence of this bounded homomorphism without assuming that S has a smooth boundary. Its construction uses a formula, derived by Stoke's formula from the usual Cauchy integral formula. We showed in Section 8.5 that the resolvent possesses a differentiable extension (of class C_r), to the interior of the spectral set S. If $u(s)$ is the extension of the resolvent, and $y(s) = 1-(a-s)u(s)$, then $y(s)$ is of class C_{r-1} on $C\cup\{\infty\}$, and vanishes on the complement of X, along with its derivatives to the order $r-1$.

[The extension of u is of class C_r on S. The function $y(s)$ is of class C_r on C, and vanishes on the complement of S. A derivative is lost at infinity because s has a simple pole there, y is of class C_{r-1} on $(C\cup\{\infty\})\setminus S$.]

A function which vanishes at the boundary of S along with its derivatives to the order $r-1$ tends relatively fast to zero at the boundary. If f is holomorphic and such that $\delta_S^k f$ is bounded, $f\cdot y$ is of class C_{r-k-1} and vanishes at the boundary to the order $r-k-1$. Such a function is summable on S.

We would like to integrate $\int_{\partial S} f(s)u(s)\,ds$. We cannot, unless S has a nice boundary. Applying Stoke's formula, we obtain the expression

$$\frac{1}{2\pi i}\int_S f(s)\,\bar\partial u(s)\,ds$$

which is formally equal to the Cauchy integral.

Differentiating the relation

$$(a-s)u(s)+y(s) = 1$$

we obtain

$$(a-s)\,\bar\partial u(s)+\bar\partial y(s) = 0.$$

We multiply this by $u(s)$, and remember that $(a-s)u(s) = 1-y(s)$. We have

$$\bar\partial u(s) = y(s)\,\bar\partial u(s)-u(s)\,\bar\partial y(s).$$

The form $f\bar\partial u$ that we integrate is therefore equal to

$$f\bar\partial u = fy\bar\partial u-fu\bar\partial y.$$

This is an $\mathcal{A}$-valued form, differentiable to the order C_{r-k-2}, zero along with its derivatives to the order $r-k-2$ on the complement of S. And $f\bar\partial u\,ds$ is summable if $r > k+4$.

If u' is another extension of the resolvent, $u'-u$ vanishes to the order r on the complement of our spectral set,

$$f\bar\partial u'-f\bar\partial u = \bar\partial[f(u''-u)]$$

is the coboundary of a form which vanishes to the order $r-k-1$ on the complement of S. The integral of such a form is zero.

PROPOSITION. *A bounded linear mapping $\mathcal{O}(\delta)\to\mathcal{A}$ is associated in a natural way to $a\in\mathcal{A}$.*

This maps f onto the common value of

$$\frac{1}{2\pi i}\int f\bar\partial u\,ds = f[a]$$

for all u, of class C_r (r large enough depending on f), and extending the resolvent.

If $f = 1$, we notice that $f[a] = 1$, because $u(s)$ is asymptotically equal to $-s^{-1}-$ $-s^{-2}a-\ldots$ and

$$\int\bar\partial u\,ds \doteq \lim\int_{|s|=R}[s^{-1}+s^{-2}a+\ldots]ds = 2\pi i.$$

Assume now that $f(s) = (s-a)g(s)$ with $g\in\mathcal{O}(\delta_S;\mathcal{A})$. Remember that $(a-s)\,\bar\partial u = -\bar\partial y$.

$$2\pi i f[a] = \int f(s)\,\bar\partial u\,ds = -\int g(s)\,\bar\partial y\,ds = 0.$$

If $r(z) = P(z)/Q(z)$ is a rational function, $r(z)-r(a)$ belongs to the ideal generated by $z-a$ (if $r(a) = P(a)Q(a)^{-1}$). It follows that $r[a] = r(a)$.

PROPOSITION. *The mapping $f\to f[a]$ is a bounded homomorphism $\mathcal{O}(\delta_S)\to\mathcal{A}$. The rational functions of z are dense in $\mathcal{O}(\delta_S)$.*

The mapping is bounded and linear. Its restriction to the algebra of rational functions is a homomorphism. To prove that it is a homomorphism, it is sufficient to prove that the rational functions are dense in $\mathcal{O}(\delta_S)$.

Let $\mathcal{O}_1(\delta_S)$ be the closure in $\mathcal{O}(\delta_S)$ of the algebra of rational functions. Then $z \in \mathcal{O}_1(\delta_S)$, if z is the coordinate function, and S is spectral for z in $\mathcal{O}_1(\delta_S)$. We have therefore a mapping $f \to f[z]$, $\mathcal{O}(\delta_S) \to \mathcal{O}_1(\delta_S)$.

The "holomorphic functional calculus" that we have constructed may not be a homomorphism, but is natural. If we compose it with the inclusion map $\mathcal{O}_1(\delta_S) \to \to \mathcal{O}(\delta_S)$, we get the holomorphic functional calculus mapping $\mathcal{O}(\delta_S) \to \mathcal{O}(\delta_S)$. But this h.f.c. mapping must be the identity mapping. Just compose it with the evaluation map $u \to u(z_0)$ and observe that $u[z_0] = u(z_0)$ when z_0 is scalar. $u[z]$ is the holomorphic function which, for all $z_0 \in S$ takes the value $u(z_0)$ at z_0. I.e. $u[z] = u$.

8.8. This is a good place to stop.

In the second half of this talk, I discussed the n-dimensional generalization of the above results. This generalization has been sketched elsewhere ([47], Sections 8–12), and in a more leisurely way.

Another approach to the construction of an operational calculus can be mentioned. We have extended the resolvent to the interior of a spectral set, and constructed a function which was an "asymptotic resolvent". Instead, we could have considered the growth of the resolvent at the boundary of the spectrum.

I have obtained such results when $n = 1$, and when we assume that the resolvent has growth properties such as those of $\|(a-s)^{-1}\|$ when a is an element of a Banach algebra [45]. These results have been generalized for larger values of n by C. Wrobel [55], [56] and by Nguyen The Hoc [20].

Quite a bit of work can still be done on the operational calculus, even in commutative algebras.

References

[1] M. Akkar, *Sur la théorie spectrale des algèbres d'opérateurs*, Séminaire d'Analyse Fonctionnelle, Bordeaux 1972.

[2] E. Albrecht, *Several variable spectral theory in the noncommutative case*, these proceedings, 9–30.

[3] G. R. Allan, *A spectral theory for Banach algebras*, Proc. London Math. Soc. (3) 15 (1965), 399–421.

[4] G. R. Allan, H. G. Dales, and J. P. McClure, *Pseudo-Banach algebras*, Studia Math. 40 (1971), 55–69.

[5] R. Arens, *The group of invertible elements of a commutative Banach algebra*, Studia Math. Ser. Spec. Z. I. (1963), 21–23.

[6] —, *To what extent does the group of invertible elements determine the algebra?* in: *Function Algebras*, Scott, Foresman and Co., 1966, 165–168.

[7] R. Arens and A. P. Calderón, *Analytic functions of several Banach algebra elements*, Ann. of Math. (2) 62 (1955), 204–216.

[8] I. Cnop, *Spectral study of holomorphic functions with bounded growth*, Ann. Inst. Fourier (Grenoble) 22 (1972), 239–309.

[9] J. P. Ferrier, *Approximation des fonctions holomorphes de plusieurs variables avec croissance*, ibid. 22 (1972), 67–87.

[10] —, *Spectral theory and complex analysis*, North Holland, Notas de Matematica, v. 4, 1973.

[11] I. M. Gelfand, *Normierte Ringe*, Math. Sbornik, N.S. 9 (51) (1941), 3–24.

[12] —, *Ideale und primäre Ideale in normierten Ringen*, ibid. 9 (51) (1941), 41–48.

[13] —, *Zur Theorie der Charaktere der Abelschen topologischen Gruppen*, ibid. 9 (51) (1941), 49–50.

[14] —, *Über absolut konvergente trigonometrische Reihen und Integralen*, ibid. 9 (51) (1941), 51–66.

[15] B. Gramsch, *Funktionalkalkül mehrerer Veränderlichen in lokalbeschränkten Algebren*, Math. Ann. 174 (1967), 311–344.

[16] H. Hogbe-Nlend, *Les fondements de la théorie spectrale des algèbres bornologiques*, Bol. Soc. Brasil Mat. 3 (1972), 19–56.

[17] E. R. Lorch, *The spectrum of linear transformations*, Trans. Amer. Math. Soc. 52 (1942), 238–248.

[18] —, *The theory of analytic functions in normed abelian vector rings*, ibid. 54 (1943), 414–425.

[19] B. Mitjiagin, S. Rolewicz, and W. Żelazko, *Entire functions in B_0-algebras*, Studia Math. 21 (1962), 291–306.

[20] Nguyen The Hoc, *Croissance des coefficients spectraux et calcul fonctionnel*, Ann. Inst. Fourier (Grenoble) 27 (1977).

[21] D. Przeworska-Rolewicz and S. Rolewicz, *On integrals of functions with values in a complete linear metric space*, Studia Math. 26 (1966), 121–131.

[22] J. Sebastiaõ e Silva, *Le calcul opérationnel au point de vue des distributions*, Portugaliae Math. 14 (1956), 105–132.

[23] —, *Sur le calcul symbolique des opérateurs différentiels à coefficients variables*, I et II, Rendiconti Acad. Naz. Lincei 8.28 (1959), 42–47 and 118–122.

[24] —, *Le calcul opérationnel pour des opérateurs à spectre non borné*, Acad. Naz. dei Lincei. Mem. Sc. Fis. mat., e nat. 8–6. fasc. 1 (1960).

[25] —, *Sur le calcul symbolique à une ou plusieurs variables pour une algèbre localement convexe*, Rendiconti Acad. Naz. Lincei 8.30 (1961), 167–172.

[26] —, *Sur le calcul symbolique d'opérateurs permutables à spectre vide ou non borné*, Ann. di Mat. P.e App. 4. 58 (1962), 219–276.

[27] G. E. Šilov, *On the decomposition of a normed ring in a direct sum of ideals* (Russian), Mat. Sbornik 32 (74) (1953), 353–364.

[28] —, *Analytic functions in a normed ring*, Uspehi Math. Nauk. N. S. 15 (1960) nº 3 (93), 181–183. Amer. Math. Soc. Translations Series 2. v. 28, 207–209.

[29] Z. Słodkowski, *An infinite family of joint spectra*, Inst. of Math. Pol. Acad. Science, Preprint 85.

[30] Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, Studia Math. 50 (1974), 127–148.

[31] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. 6 (1970), 172–191.

[32] —, *The analytic functional calculus for several commuting operators*, Acta Math. 125 (1970), 1–38.

[33] —, *Homology and cohomology for topological algebras*, Advances in Math. 9 (1972), 137–182.

[34] —, *A general framework for a multi-operator operational calculus*, ibid. 9 (1972), 183–252.

[35] —, *Functions of several non commuting variables*, Bull. Amer. Math. Soc. 79 (1973), 1–34.

[36] —, *Banach algebras and topology*; in: *Algebras in Analysis*, Academic Press, 1975, 118–186.

[37] P. Turpin, *Une remarque sur les algèbres à inverse continu*, C. R. Acad. Sci. Paris 270 (1970), 1686–1689.

[38] P. Turpin and L. Waelbroeck, *Sur l'approximation des fonctions différentiables à valeurs dans les espaces vectoriels topologiques*, ibid. 267 (1968), 94–97.

[39] —, —, *Intégration et fonctions holomorphes à valeurs dans les espaces localement pseudo-convexes*, ibid. 267 (1968), 160–162.

[40] P. Turpin et L. Waelbroeck, *Algèbres localement pseudo-convexes à inverse continu*, ibid. 267 (1968), 194–195.
[41] P. Uss, *Sur les opérateurs bornés dans les espaces localement convexes*, Studia Math. 37 (1970–71), 139–158.
[42] L. Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, J. Math. Pures et Appl. (9) 33 (1954), 147–186.
[43] —, *Algèbres commutatives: éléments réguliers*, Bull. Soc. Math. Belgique 9 (1957), 42–49.
[44] —, *Etude spectrale des algèbres complètes*, Acad. Roy. Belgique, Cl. Sci. Mém. Coll. in 8° (2) 31 (1960), n° 7.
[45] —, *Calcul symbolique lié à la croissance de la résolvante*, Rend. del Sem. Mat. e Fis. di Milano 34 (1964), 51–72.
[46] —, *Fonctions différentiables et petite bornologie*, C. R. Acad. Sci. Paris 267 (1968), 220–222.
[47] —, *The holomorphic functional calculus and non-Banach algebras*; in: *Algebras in Analysis*, Academic Press, 1975, 187–251.
[48] —, *Les quotients d'espaces bornologiques complets*, C. R. Acad. Sci. Paris 285 (1977), 899–901.
[49] —, *Les quotients d'espaces bornologiques complets; théorie multilinéaire*, ibid. 285 (1977), 949–951.
[50] —, *Les q-algèbres*, ibid. 285 (1977), 1053–1055.
[51] —, *Les quotients banachiques*, ibid. 286 (1978), 37–39.
[52] —, *Quotient Banach spaces*, these proceedings, 553–562.
[53] —, *Quotient Banach spaces. Multilinear theory*, these proceedings, 563–571.
[54] —, *The Taylor spectrum and quotient Banach spaces*, these proceedings, 573–578.
[55] C. Wrobel, *Extensions du calcul fonctionnel holomorphe et applications à l'approximation*, C. R. Acad. Sci. Paris 275 (1972), 175–177.
[56] —, *Extensions du calcul fonctionnel holomorphe*, Revue de Math. n° 1. Institut E. Cartan, Nancy, 1974–75.
[57] W. R. Zame, *Existence, uniqueness, and continuity of funtional calculus homomorphisms* (to appear).
[58] W. Żelazko, *On the locally bounded and m-convex topological algebras*, Studia Math. 19 (1960), 333–356.
[59] —, *On the analytic functions in p-normed algebras*, ibid. 21 (1962), 345–350.
[60] —, *On the radicals of p-normed algebras*, ibid. 21 (1962), p. 203–206.
[61] —, *On the decomposition of a commutative p-normed algebra into a direct sum of ideals*, Coll. Math. 10 (1963), 57–60.
[62] —, *An axiomatic approach to joint spectra. I*, Institute of Mathematics, Polish Academy of Science, Preprint 100.
[63] Berkson, Dowson, Elliott, Bull. London Math. Soc. 4 (1972), 13–16.
[64] N. Bourbaki, *Théories spectrales*, Paris, Hermann et C^ie, 1967.

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QUOTIENT BANACH SPACES

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We define here the category qB of quotient Banach spaces and study its linear properties. This is an exact category whose definition can be motivated by functional analytic considerations. It is to be expected that qB will be useful where functional analysis and homological algebra interact. The multilinear propertis of qB, and the J. L. Taylor spectrum of commuting operators in qB will be considered elsewhere.

The category q of quotients of complete bornological spaces can also be useful when functional analysis and homological algebra interact. It contains qB. More applications can be attained with q than with qB. But qB is a subcategory of E.V. (the category of vector spaces), q is not. Its properties can be explained in a much simpler language than the properties of q. It is the simpler language that will be used here.

I met the quotients of complete bornological spaces for the first time in 1960 [5], without defining their category, without showing that my constructions were natural in q. I withheld systematic research and publication when I realized that there would be more trivialities than applications in the incipient stages of the theory.

G. Noël gave a definition of q, and obtained some results on tensor products ([1], [2], [3], [4]). I do not like much his definition. It is too categorical, and not functional analytic enough. But it is not worse than the definition I had at the time and did not publish.

This paper, and the next two will stress the functional analytic aspects of the category qB. After all, it is the functional analysts who are expected to use this category. And the fact that qB is (isomorphic with) a subcategory of E.V. may make the pill easier to swallow for functional analysts who do not have much intuition of what homological algebra is about.

The results contained in this paper and the properties of quotients of complete bornological spaces have already been announced [6], [7], [8], [9].

1

Let E be a Banach space. A Banach subspace F of E is a vector subspace on which a Banach space norm exists which defines on F a stronger topology than the topology induced by E. Applying the closed graph theorem, we observe that two such