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QUOTIENT BANACH SPACES

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We define here the category qB of quotient Banach spaces and study its linear properties. This is an exact category whose definition can be motivated by functional analytic considerations. It is to be expected that qB will be useful where functional analysis and homological algebra interact. The multilinear propertis of qB, and the J. L. Taylor spectrum of commuting operators in qB will be considered elsewhere.

The category q of quotients of complete bornological spaces can also be useful when functional analysis and homological algebra interact. It contains qB. More applications can be attained with q than with qB. But qB is a subcategory of E.V. (the category of vector spaces), q is not. Its properties can be explained in a much simpler language than the properties of q. It is the simpler language that will be used here.

I met the quotients of complete bornological spaces for the first time in 1960 [5], without defining their category, without showing that my constructions were natural in q. I withheld systematic research and publication when I realized that there would be more trivialities than applications in the incipient stages of the theory.

G. Noël gave a definition of q, and obtained some results on tensor products ([1], [2], [3], [4]). I do not like much his definition. It is too categorical, and not functional analytic enough. But it is not worse than the definition I had at the time and did not publish.

This paper, and the next two will stress the functional analytic aspects of the category qB. After all, it is the functional analysts who are expected to use this category. And the fact that qB is (isomorphic with) a subcategory of E.V. may make the pill easier to swallow for functional analysts who do not have much intuition of what homological algebra is about.

The results contained in this paper and the properties of quotients of complete bornological spaces have already been announced [6], [7], [8], [9].

1

Let E be a Banach space. A Banach subspace F of E is a vector subspace on which a Banach space norm exists which defines on F a stronger topology than the topology induced by E. Applying the closed graph theorem, we observe that two such

Banach space norms on F are equivalent, i.e. that the Banach space structure of F is determined by that of E and by the set F.

DEFINITION 1. A *Banach quotient* E/F is the quotient of Banach space E by a Banach subspace F. Let E/F and E'/F' be two Banach quotients. A strict morphism u: $E/F \to E'/F'$ is a linear mapping u: $x+F \to u_1x+F'$ where u_1: $E \to E'$ is a bounded linear mapping such that $u_1 F \subseteq F'$.

We shall not analyse further the Banach quotient "structure" of E/F, but shall consider that it is given by its realization as the quotient of a Banach space and a Banach subspace.

If E/F and E'/F' are two Banach quotients, if u_1: $E \to E'$ is a bounded linear mapping such that $u_1 F \subseteq F'$, we shall say that u_1 *induces* u where

$$u(x+F) = u_1x+F'.$$

Note that u_1: $F \to F'$ is a continuous linear mapping when we place on each of F and F' its own Banach space topology. If $u = 0$, $u_1 E \subseteq F'$ and again u_1: $E \to F'$ is continuous.

The composition of strict morphisms is obviously a strict morphism. The class of Banach quotients and strict morphisms is a subcategory of E.V., where E.V. is the category of vector spaces and linear maps.

DEFINITION 2. This subcategory will be called $\tilde{q}\boldsymbol{B}$.

$\tilde{q}\boldsymbol{B}$ is a good category to work with, because it is usually easy to show that a notion is "natural" in $\tilde{q}\boldsymbol{B}$. Unfortunately it does not only have good properties.

If E is a Banach space, and F a closed subspace, $E_1 = E/F$ is a Banach space. It would be very nice if the Banach quotient E/F were isomorphic with $E_1/0$. Unfortunately, this is not the case in $\tilde{q}\boldsymbol{B}$ unless F is complemented.

2

We are led to enlarging $\tilde{q}\boldsymbol{B}$.

DEFINITION 3. Let E/F and E'/F' be two Banach quotients. A *pseudo-isomorphism* s: $E/F \to E'/F'$ is a strict morphism induced by a surjective bounded linear s_1: $E \to E'$ such that $s_1^{-1}F' = F$. The category $q\boldsymbol{B}$ is the subcategory of E.V. generated by $\tilde{q}\boldsymbol{B}$ and the inverses of the pseudo-isomorphisms.

$q\boldsymbol{B}$ has as objects the Banach quotients and as morphisms all compositions of strict morphisms and inverses of pseudo-isomorphisms. This makes sense because pseudo-isomorphisms are bijective linear maps.

A *morphism of Banach quotients* will simply be a morphism in the category $q\boldsymbol{B}$.

3

DEFINITION 4. A Banach space is *free* if it is isomorphic to $l_1(X)$ for some X. A Banach quotient E/F is *standard* if E is free.

PROPOSITION 1. *It is possible to associate to every Banach quotient E/F a standard quotient E_1/F_1 and a pseudo-isomorphism s: $E_1/F_1 \to E/F$. In particular, every Banach quotient is isomorphic to a standard one.*

Every Banach space is of course isomorphic (as a Banach space) to the quotient of a free Banach space by a closed subspace. Let E_1 be free and s_1: $E_1 \to E$ be a continuous linear surjection. Let $F_1 = s_1^{-1}F$. Then E_1/F_1 is standard, and s_1: $E_1 \to E$ induces a pseudo-isomorphism s: $E_1/F_1 \to E/F$.

PROPOSITION 2. *Every morphism from a standard Banach quotient is strict.*

By induction, it is sufficient to show that $s^{-1} \circ u$ is strict when u: $E_1/F_1 \to E/F$ is a strict morphism, when E_1 is free, and when s: $E'/F' \to E/F$ is a pseudo-isomorphism. Let u_1: $E_1 \to E$ induce u and s_1: $E' \to E$ induce s and be surjective.

It is known that a bounded linear v_1: $E_1 \to E'$ exists, such that $u_1 = s_1 \circ v_1$. Since $u_1 F_1 \subseteq F$, $v_1 F_1 \subseteq s_1^{-1}F = F'$, i.e. v_1 induces a strict morphism v: $E_1/F_1 \to E/F$. Clearly $s \circ v = u$, hence $v = s^{-1} \circ u$ is strict. [The existence of v_1 follows from the fact that E_1 is free and s_1 is surjective.]

PROPOSITION 3. *Every morphism u of $q\boldsymbol{B}$ can be expressed $u = u_1 \circ s^{-1}$ where s is a pseudo-isomorphism from a standard quotient and u_1 is a strict morphism from the same quotient.*

Just combine Propositions 1 and 2.

NOTE. $q\boldsymbol{B}$ is "equivalent" to the category of standard quotients and strict morphisms of such standard quotients. Restriction of $q\boldsymbol{B}$ to this subcategory is inessential. Some mathematicians may consider that this restriction leads to a more elegant presentation of the theory. But the functional analyst who wants to consider a specific E/F, e.g. $L_2(I)/L_\infty(I)$ may object to replacing it by a pseudo-isomorphic l_1/F_1.

4

PROPOSITION 4. *A functor from the strict category of Banach quotients extends to the full category when it maps pseudo-isomorphisms on isomorphisms. The extension is unique.*

PROPOSITION 5. *Let $\boldsymbol{K}$ be a category, and F, G: $q\boldsymbol{B} \to \boldsymbol{K}$ be two functors which extended to $q\boldsymbol{B}$. Let F_1, G_1 be their extensions. Let H: $F \to G$ be a functor homomorphism. Then H is a homomorphism of F_1 to G_1.*

These propositions mean that a construction is natural in $q\boldsymbol{B}$ if it behaves as it should under the action of a strict morphism and if the situation does not change essentially when we replace a problem by a pseudo-isomorphic one (when we replace the given Banach quotients by new, pseudo-isomorphic Banach quotients). In practice, the verification of each condition is within the realm of functional analysis.

The uniqueness of the extension of a functor is immediate. So is Proposition 5. Remember that each morphism u factors $u = s^{-1} \circ u_1$ strict, s being a pseudo-isomorphism. If F_1 is an extension of F to $q\boldsymbol{B}$,

$$F_1(u) = F_1(s^{-1}) \circ F_1(u_1) = F_1(s)^{-1} \circ F_1(u_1) = F(s)^{-1} \circ F(u_1).$$

Let F_1 and G_1 be extensions of F and G, and H: $F \to G$ be a functor homomorphism. Let u be a morphism $U/V \to U_1/V_1$ and s a pseudo-isomorphism $U'/V' \to U/V$. We must show that

$$H(U_1/V_1)F_1(u) = G_1(u)H(U/V)$$

and assume that

$$H(U/V)F(s) = G(s)H(U'/V'), \quad H(U_1/V_1)F(u) = G(u)H(U'/V').$$

Proposition 5 follows.

We must still prove that the functor F: $\tilde{q}B \to K$ has an extension if it maps a pseudo-isomorphism onto an isomorphism. If it has an extension F_1, if u: $U/V \to$ $\to U_1/V_1$ is a general morphism which factors $u = s^{-1} \circ u_1$ we know that $F_1(u)$ $= F(s)^{-1}F(u_1)$.

We must therefore prove that $F(s)^{-1} \circ F(u_1)$ does not depend on the factorization $u = s^{-1} \circ u_1$. This being done, we must show that the mapping

$$u \to F_1(u) = F(s)^{-1}F(u_1)$$

is a functor.

Lemma 1. *Let s': $U'/V' \to U/V$ and s'': $U''/V'' \to U/V$ be pseudo-isomorphisms. A Banach quotient X/Y and pseudo-isomorphisms t': $X/Y \to U'/V'$, t'': $X/Y \to U''/V''$ can be found in such a way that $s' \circ t' = s'' \circ t''$.*

We consider surjective maps s_1': $U' \to U$, s_1'': $U'' \to U$ which induce s' and s'', and define

$$X = \{(x', x'') \in U' \times U'' |\ s_1'x' - s_1''x'' \in V\}$$

with the norm

$$\|(x', x'')\|_X = \|x'\|_{U'} + \|x''\|_{U''} + \|s_1'x' - s_1''x''\|_V.$$

Then X is a Banach space, $Y = V' \times V''$ is a Banach subspace of X.

Letting t_1': $X \to U'$ and t_1'': $X \to U''$ be defined by

$$t_1'(x', x'') = x', \quad t_1''(x', x'') = x''$$

we see that t_1', t_1'' induce pseudo-isomorphisms t': $X/Y \to U'/V'$, t'': $X/Y \to U''/V''$ and $s' \circ t' = s'' \circ t''$. Lemma 1 is proved.

We can now show that the relation $F_1(u) = F(u_1)F(s)^{-1}$ when $u_1 = u \circ s$, defines a mapping. Let s_1: $U_1/V_1 \to U/V$ and s_2: $U_2/V_2 \to U/V$ be pseudo-isomorphisms, with U_1/V_1 and U_2/V_2 standard. Find X/Y and pseudo-isomorphisms t_1: $X/Y \to U_1/V_1$, t_2: $X/Y \to U_2/V_2$ such that $s_1 \circ t_1 = s_2 \circ t_2$. We must show that $F(u_1)F(s_1)^{-1} = F(u_2)F(s_2)^{-1}$ if $u_i = u \circ s_i$, but

$$u_1 \circ t_1 = u \circ s_1 \circ t_1 = u \circ s_2 \circ t_2 = u_2 \circ t_2$$

hence $F(u_1)F(t_1) = F(u_2)F(t_2)$. Multiply this on the right by the inverse of $F(s_1)F(t_1) = F(s_2)F(t_2)$. We obtain the required equality.

To prove that F_1 is a functor, we consider u_i: $U_i/V_i \to U_{i+1}/V_{i+1}$ $(i = 1, 2)$ and want to show that $F_1(u_1)F(u_2) = F_1(u_1 \circ u_2)$. Let U_i'/V_i' be standard Banach quotients, and s_i: $U_i'/V_i' \to U_i/V_i$ be pseudo-isomorphisms. Let $u_i' = u_i \circ s_i$, and $v' = u_2 \circ u_1 \circ s_1$. The morphisms u_i', v' are strict and

$$F_1(u_i) = F(u_i')F(s_i)^{-1}, \quad F_1(u_1 \circ u_2) = F(v')F(s_1)^{-1}.$$

We must show that

$$F(v') = F(u_2')F(s_2)^{-1}F(u_1').$$

This is the case, $s_2^{-1} \circ u_1'$ is strict because U_1'/V_1' is standard. The relation $s_2 \circ (s_2^{-1} \circ$ $\circ u_1') = u_1'$ shows that $F(s_2^{-1} \circ u_1') = F(s_2)^{-1}F(u_1')$ hence

$$F(u_2')F(s_2)^{-1}F(u_1') = F(u_2')F(s_2^{-1} \circ u_1') = F\big(u_2' \circ (s_2^{-1} \circ u_1')\big) = F(v)$$

which is the desired relation.

5

Proposition 6. *qB is an exact category. A sequence of morphisms of qB is exact when it is exact in the category* E.V. *of vector spaces.*

An exact category is one in which every mapping has a kernel, a cokernel, and in which the natural mapping of the cokernel of the kernel mapping in the kernel of the cokernel mapping is an isomorphism.

Definition 5. Let E/F be a Banach quotient. Let E' be a Banach subspace of E which contains F. Then E'/F is a *subobject* or *subquotient* of E/F, and E/E' is a *quotient object.* The map $E'/F \to E/F$ induced by inclusion $E' \to E$ is the *inclusion map,* or the *canonical injection.* The mapping $E/F \to E/E'$ induced by the identity $E \to E$ is the *quotient map* or the *canonical surjection.*

It is standard, in a vector category, to say that a morphism i is *monic* if $u = 0$ whenever $i \circ u = 0$, and that s is *epic* if $v = 0$ when $v \circ s = 0$, where we assume that $i \circ u$ and $v \circ s$ exist. Canonical injections are clearly monic, canonical surjections are epic (they are monic, or epic in E.V.).

Lemma 2. *A bijective morphism of qB is an isomorphism.*

Every Banach quotient is isomorphic to a standard one. Every morphism from a standard quotient is strict. It will be sufficient to show that a strict bijective morphism is an isomorphism. Let u: $E/F \to E'/F'$ be a bijective strict morphism, induced by u_1: $E \to E'$.

Bijectivity of u implies that $u_1E + F' = E'$ and $F = u_1^{-1}F'$. Consider E_1 $= E \oplus F'$, $F_1 = F \oplus F'$. The linear mapping $E_1/F_1 \to E'/F'$ induced by $x \oplus y \to$ $\to u_1x + y$ is a pseudo-isomorphism. The mapping $E/F \to E_1/F_1$ induced by $x \to$ $\to x \oplus 0$ is an isomorphism of the strict category. Its inverse is induced by the projection mapping $x \oplus y \to x$.

u is the composition of the two isomorphisms above, it is an isomorphism.

Lemma 3. *Every monomorphism u: $E_1/F_1 \to E/F$ of qB factors in a unique way $u = i \circ u_1$ where u_1: $E_1/F_1 \to E'/F$ is an isomorphism, where E'/F is a subquotient of E/F and where i: $E'/F \to E/F$ is the inclusion map.*

Replacing E_1/F_1 eventually by an isomorphic standard quotient, we may assume that u is strict. Assume that u_1 induces u.

u is monic, $F_1 = u_1^{-1}F$. Otherwise $u_1^{-1}F/F_1$ would be a non-null subquotient of E_1/F_1. The composition of the canonical injection $u_1^{-1}F/F_1 \to E_1/F_1$ with u_1 would be zero, but the canonical injection would not be zero.

Let next $E' = u_1E_1+F$. This is a Banach subspace of E. The mapping $u_1\colon E_1 \to \to E'$ induces a bijective morphism $u'\colon E_1/F_1 \to E'/F$. And u is the composition of the isomorphism u' and the canonical injection $E'/F \to E/F$.

To show that E' is determined by u, we assume that $u = i_2 \circ u_2$ where u_2 is an isomorphism and i_2 a canonical injection $E_2/F \to E/F$. Then E_2/F is the range of the linear mapping u, and this implies that $E_2 = E'$.

LEMMA 4. *Every epimorphism $u\colon E/F \to E_1/F_1$ factors in a unique way $u = u' \circ s$ where u' is an isomorphism $E/E' \to E_1/F_1$, where E/E' is a quotient object of E/F, and where $s\colon E/F \to E/E'$ is the canonical surjection.*

Assume first that u is strict (this will be the case when E/F is standard). Let $u_1\colon E \to E_1$ induce u. We have the equality $u_1E+F_1 = E_1$, otherwise the quotient map $E_1/F_1 \to E_1/(u_1E+F_1)$ would not be zero, but its composition with u would be, and u would not be epic.

Let $E' = u_1^{-1}F_1$. Then $u_1\colon E \to E_1$ induces a bijective morphism $E/E' \to \to E_1/F_1$ and the composition of this bijective morphism and the quotient mapping $E/F \to E/E'$ is equal to u.

E' is determined by u. Just as in the proof of Lemma 3, we observe that the quotient E/E' is determined by the linear mapping u, and E/E' determines E'.

If the epimorphism u is not strict, we consider a pseudo-isomorphism $\sigma\colon U/V \to \to E/F$, with U/V standard, then $u \circ \sigma$ is strict. Let $v = u \circ \sigma$. We find a quotient object U/U', and an isomorphism $U/U' \to E_1/F_1$ in such a way that v is the composition of the quotient map and the isomorphism.

The pseudo-isomorphism σ is induced by a bounded surjection $\sigma_1\colon U \to E$. Let $E' = \sigma_1 U'$, then σ_1 induces a pseudo-isomorphism $U/U' \to E/E'$ and u is the composition of the quotient surjection $E/F \to E/E'$, the inverse of the pseudo-isomorphism $U/U' \to E/E'$, and the isomorphism $U/U' \to E_1/F_1$.

LEMMA 5. *Every morphism of qB has a kernel and a cokernel.*

A *kernel* of $u\colon E_1/F_1 \to E_2/F_2$ is a monomorphism $i\colon U/V \to E_1/F_1$, such that $u \circ i = 0$, and such that v factors $v = i \circ v_1$ if $u \circ v = 0$. Applying Lemma 3, we see that we may take U/V to be a subobject of E_1/F_1, i.e. $U = E_1'$, $V = F_1$ with $F_1 \subseteq E_1' \subseteq E$, and i to be the canonical injection. It is standard practise to say that E_1'/F is the kernel of u.

If u is strict, induced by $u_1\colon E_1 \to E_2$, its kernel is E_1'/F_1 where $E_1' = u_1^{-1}F_2$. If u is not strict, we find a pseudo-isomorphism $\sigma\colon U/V \to E_1/F_1$, induced by a surjection $\sigma_1\colon U \to E_1$, in such a way that $u \circ \sigma = v$ is strict. The morphism v has a kernel U_1/V. Letting $E_1 = \sigma_1 U_1$, we see that E_1/F is a kernel of E/F.

A *cokernel* of $u\colon E_1/F_1 \to E_2/F_2$ is an epimorphism $s\colon E_2/F_2 \to U/V$ such that $s \circ u = 0$ and such that v factors $v = v_1 \circ s$ when $v \circ u = 0$. Applying Lemma 4, we see that we can take s to be a canonical surjection $s\colon E_2/F_2 \to E_2/E_2'$. And again, it is standard practise to say that E_2/E_2' is the cokernel of u.

We may assume that u is strict, replacing eventually E_1/F_1 by a pseudo-isomorphic standard object. Let u_1 induce u. We see that $E_2/(u_1E_1+F_2)$ is a cokernel.

The proof of Proposition 6 is now straightforward.

6

The following two definitions are convenient but do not follow from general categorical principles. The category B can be embedded in qB, mapping every Banach space E onto the quotient $E/0$. The "exact sequences" of B are the sequences of mappings of B which are exact in qB. The definitions of exact and right-exact functors from B are made to order, in order that Proposition 7 hold.

DEFINITION 6. A sequence (u, v), $u\colon E \to E'$, $v\colon E' \to E''$ of morphisms in the category B of Banach spaces is *exact* if it is exact in the category E.V. of vector spaces.

DEFINITION 7. A functor Φ from B into an exact category K is *right-exact* if it maps a short exact sequence $0 \to E' \to E \to E'' \to 0$ onto a right-exact sequence $\Phi(E') \to \Phi(E) \to \Phi(E'') \to 0$. It is exact if it is right-exact and maps a monomorphism on a monomorphism.

PROPOSITION 7. *Every right-exact functor $\Phi\colon B \to K$ extends to a right-exact functor $\Phi_1\colon qB \to K$. Two different extensions of Φ are isomorphic. If Φ and Ψ are right-exact $B \to K$ and if Φ_1, Ψ_1 are right-exact extensions of Φ and Ψ, every functor homomorphism $H\colon \Phi \to \Psi$ has a unique extension $H_1\colon \Phi_1 \to \Psi_1$.*

The right-exact extension of an exact functor $B \to K$ is exact $qB \to K$.

The proof is easy diagram chasing. We identify systematically a Banach space E and the quotient $E/0$.

E/F is the cokernel of inclusion $i\colon F \to E$. We define $\Phi_1(E/F) = \operatorname{coker}\Phi(i)$. If $u\colon E/F \to E'/F'$ is a strict morphism, induced by $u_1\colon E \to E'$, if $v_1\colon F \to F'$ is the restriction of u_1, $\Phi_1(u)$ will be the morphism $\Phi_1(E/F) \to \Phi_1(E'/F')$ making the following diagram commutative:

$$\begin{array}{ccccccc}
\Phi(F) & \to & \Phi(E) & \to & \Phi_1(E/F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Phi(F') & \to & \Phi(E') & \to & \Phi_1(E'/F') & \to & 0
\end{array}$$

(where $\Phi(u_1)$, $\Phi(v_1)$ respectively map $\Phi(E) \to \Phi(E')$, $\Phi(F) \to \Phi(F')$). It is straightforward that $\Phi_1(u)$ does not depend on the choice of u_1 inducing u, also that $\Phi_1(u \circ v) = \Phi_1(u)\Phi_1(v)$. In other words, Φ_1 is a functor $\tilde{q}B \to K$.

To show that Φ_1 extends to qB, we must prove that it maps a pseudo-isomorphism onto an isomorphism. Let $s\colon E/F \to E'/F'$ be such a pseudo-isomorphism, let s_1 induce s and be surjective, let t_1 be the restriction of s_1 to F, considered as a mapping from F to F'. Then t_1 is surjective, and has the same kernel K as s_1.

We have, in qB, the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & F_1 & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & E_1 & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & E_1/F_1 & \to & E/F & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Apply the functor Φ to the two upper rows of this diagram, Φ_1 to the third row, map $\Phi(E_1) \to \Phi_1(E_1/F_1)$, $\Phi(E) \to \Phi_1(E/F)$ by the cokernel mappings. We obtain a commutative diagram in which the two upper rows and the three columns are right-exact.

$$\begin{array}{ccccccc}
\Phi(K) & \to & \Phi(F_1) & \to & \Phi(F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Phi(K) & \to & \Phi(E_1) & \to & \Phi(E) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Phi_1(E_1/F_1) & \to & \Phi_1(E/F) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

It is well known that the exactness of the third row follows, i.e. $\Phi_1(E_1/F_1) \to \Phi_1(E/F)$ is an isomorphism.

The next item on the agenda is a proof that Φ_1 is right-exact, and that Φ_1 is exact when Φ is exact. Every monomorphism of qB is equivalent to canonical injection, every epimorphism is equivalent to a canonical surjection (Lemmas 3 and 4). It will be sufficient to show that Φ_1 maps a short exact sequence such as

$$0 \to E'/F \to E/F \to E/E' \to 0$$

onto a right-exact sequence, even onto an exact sequence when Φ is exact.

We consider the commutative diagram with exact rows and columns in qB

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F & \to & F & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E' & \to & E & \to & E/E' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E'/F & \to & E/F & \to & E/E' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and deduce from this a commutative diagram right in K whose upper two rows and whose three columns are right-exact in K (when Φ is right-exact).

$$\begin{array}{ccccccc}
\Phi(F) & \to & \Phi(F) & \to & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Phi(E') & \to & \Phi(E) & \to & \Phi(E/E') & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Phi_1(E'/F') & \to & \Phi_1(E/F) & \to & \Phi(E/E') & \to & 0. \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The mappings of this diagram are, respectively the images by Φ or Φ_1 of the corresponding mappings in B or qB, or are suitable cokernel mappings. The upper two rows of the diagram, and all three columns are right-exact. The third row is therefore right-exact.

If the functor Φ is exact, the morphisms $\Phi(F) \to \Phi(E')$, $\Phi(F) \to \Phi(E)$, $\Phi(E') \to \Phi(E)$, and of course $\Phi(F) \to \Phi(F)$ are monic. The mapping $\Phi_1(E'/F) \to \Phi_1(E/F)$ is then monic, Φ_1 is exact.

We must still show that the extension Φ_1 is unique modulo isomorphism, and that every functor homomorphism $H\colon \Phi \to \Psi$ has a unique extension $H_1\colon \Phi_1 \to \Psi_1$. It will of course be sufficient to show that every functor homomorphism has a unique extension. If Φ_1 and Φ_1' are two extensions of the same functor Φ, the identity $\Phi \to \Phi$ will extend to functor homomorphisms $\Phi_1 \to \Phi_1'$, $\Phi_1' \to \Phi_1$. The compositions of these homomorphisms are functor homomorphisms $\Phi_1 \to \Phi_1$, $\Phi_1' \to \Phi_1'$ extending the identity $\Phi \to \Phi$, these compositions are the identity homomorphisms Φ_1 and Φ_1' are isomorphic functors.

Let Φ_1 and Ψ_1 be right-exact functors $qB \to K$ extending Φ and Ψ, and let $H\colon \Phi \to \Psi$ be a functor homomorphism. There cannot be more than one extension $H_1\colon \Phi_1 \to \Psi_1$ of H. Consider a qB space E/F. We have the right-exact sequences $\Phi(F) \to \Phi(E) \to \Phi_1(E/F) \to 0$, $\Psi(F) \to \Psi(E) \to \Psi_1(E/F) \to 0$; only one mapping $\Phi_1(E/F) \to \Psi_1(E/F)$ makes the following diagram commutative.

$$\begin{array}{ccccccc}
\Phi(F) & \to & \Phi(E) & \to & \Phi_1(E/F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Psi(F) & \to & \Psi(E) & \to & \Psi_1(E/F) & \to & 0.
\end{array}$$

(The mappings $\Phi(F) \to \Psi(F)$, $\Phi(E) \to \Psi(E)$ in this diagram are $H(F)$ and $H(E)$ respectively.) This proves that the extension is unique if it exists.

$H_1(E/F)$ can be defined in this way. The left-hand square of the above commutative diagram can be constructed when only Φ and H are given. $H_1(E/F)$ is the morphism of the cokernel of $\Phi(F) \to \Phi(E)$ into the cokernel of $\Psi(F) \to \Psi(E)$ which makes the diagram commutative.

To prove that H_1 defined in this way is a functor homomorphism, it is sufficient (Proposition 5) to prove that it is such a homomorphism over $\tilde{q}B$, i.e. to prove that

$$\Phi_1(u) \circ H_1(E/F) = H_1(E'/F') \circ \Psi_1(u)$$

when $u\colon E/F \to E'/F'$ is a strict morphism. We assume that $u_1\colon E \to E'$ induces u and consider the restriction $v_1\colon F \to F'$ of u_1.

We have the two commutative squares

$$\begin{array}{ccc}
\Phi(F) & \to & \Phi(F') \\
\downarrow & & \downarrow \\
\Psi(F) & \to & \Psi(F')
\end{array}
\qquad
\begin{array}{ccc}
\Phi(E) & \to & \Phi(E') \\
\downarrow & & \downarrow \\
\Psi(E) & \to & \Psi(E')
\end{array}$$

and mappings $\Phi(F) \to \Phi(E)$, $\Phi(F') \to \Phi(E')$, $\Psi(F) \to \Psi(E)$ and $\Psi(F') \to \Psi(E')$ building up this diagram to a commutative cube. The square

$$\begin{array}{ccc} \Phi_1(E/F) & \to & \Phi_1(E'/F') \\ \downarrow & & \downarrow \\ \Psi_1(E/F) & \to & \Psi_1(E'/F') \end{array}$$

is the cokernel of this mapping of commutative squares. It is known that this cokernel is again a commutative square. Proposition 7 is proved.

Added in proof. Since March 1978, on the same subject, the author completed: *Les espaces de Banach plats sont ultraplats*, Bulletin de la Société Mathématique de Belgique; *Fonctions à valeurs dans les quotients banachiques*, Bulletin de l'Académie Belge, Classe des Sciences; *Holomorphic functional calculus*, Studia Math. vol. 75; and *Quasi-Banach algebras, ideals, and holomorphic functional calculus*, ibid., vol. 75, all four at present in publication.

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QUOTIENT BANACH SPACES; MULTILINEAR THEORY

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The multilinear structure of the category qB is defined by putting a qB-structure on the vector space $qB(E/F, E'/F')$. Multilinear mappings $E_1/F_1 \times \ldots \times E_k/F_k \to E'/F'$ are defined by induction.

Strict multilinear mappings are specially interesting. These are induced by bounded multilinear mappings $u_1\colon E_1 \times \ldots \times E_k \to E'$ such that $u(x_1, \ldots, x_k) \in F'$ as soon as one of the x_i belongs to the corresponding F_i. All qB-multilinear maps $E_1/F_1 \times \ldots \times E_k/F_k \to E'/F'$ are strict if the E_i/F_i are standard qB-spaces.

The tensor product which can be defined in qB is a right-exact functor as it should be. It is unfortunately not an extension of the tensor product which is defined in the category of Banach spaces. If $\bar{F}$ is the closure of F, $E/\bar{F}$ is the "Banachization" of E/F. The projective tensor product of two Banach spaces is the Banachization of their qB-tensor product.

We are interested in qB-algebras. These are qB-spaces $\mathscr{A}$ with a bilinear multiplication belonging to $q_2(\mathscr{A}, \mathscr{A}; \mathscr{A})$. The qB-algebra is strict if its multiplication is a strict bilinear mapping. It is commutative, or associative if its multiplication is commutative, or associative. The structure of a qB-subalgebra can be put on the center of a qB-algebra.

Every qB-algebra is isomorphic with a strict qB-algebra. A strict qB-algebra is the quotient of a Banach algebra by a two-sided Banach ideal. An associative qB-algebra is isomorphic with the quotient A/α of an associative Banach algebra by a Banach ideal. The isomorphic A/α can even be chosen in such a way that $Z(A/\alpha) = \big(Z(A)+\alpha\big)/\alpha$ where $Z(A/\alpha)$ and $Z(A)$ are the centers of A/α and of A. Every commutative and associative qB-algebra is isomorphic with the quotient of a commutative and associative Banach algebra by a Banach ideal.

This paper is a sequel of [2].

1

Let E/F and E'/F' be qB-spaces. Call $\tilde{q}B^1(E/F, E'/F')$ the space of bounded linear mappings $E \to E'$ which map F into F', and $\tilde{q}B^0(E/F, E'/F')$ the space of bounded