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THE TAYLOR SPECTRUM AND QUOTIENT BANACH SPACES

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We define the J. L. Taylor spectrum of commuting endomorphisms of a quotient Banach space. We prove that this is compact, not empty, and that it has the projection property.

Our proofs are often very similar to J. L. Taylor's. The proof that the spectrum is compact is slightly more complicated than Taylor's, it relies on a perturbation argument in the category of quotient Banach spaces. We must complete a perturbation theorem proved by Taylor in the category of Banach spaces.

The proof that the spectrum is not empty and that it has the projection property is easier than Taylor's. Taylor sketches the proof that we give ([2], p. 186) but he cannot use it because he does not know that the spaces he considers are quotient Banach spaces, and that the spectrum of an operator on a quotient Banach space is not empty.

The reader will find a definition of the category $\boldsymbol{qB}$ of quotient Banach spaces in [3].

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Let E/F be a quotient Banach space, and $a_1, \ldots, a_n$ be commuting endomorphisms of E/F. Let Λ_n be the exterior algebra in n indeterminates $e_1, \ldots, e_n$. Let Λ_n^q be the homogeneous elements of degree n in Λ_n. We define and identify

$$\Lambda_n(E/F) = \Lambda_n \otimes E/F = (\Lambda_n \otimes E)/(\Lambda_n \otimes F),$$
$$\Lambda_n^q(E/F) = \Lambda_n^q \otimes E/F = (\Lambda_n^q \otimes E)/(\Lambda_n^q \otimes F)$$

and put on these spaces the obvious $\boldsymbol{qB}$-structure. An element of $\Lambda_n^q(E/F)$ can be written

$$\omega = \sum_{i_1 < \ldots < i_q} \omega_{i_1 \ldots i_q} e_{i_1} \wedge \ldots \wedge e_{i_q},$$

where the $\omega_{i_1 \ldots i_q}$ belong to E/F. If b is an endomorphism of E/F, we let $b\omega$ be the element

$$b\omega = \sum b\omega_{i_1 \ldots i_q} e_{i_1} \wedge \ldots \wedge e_{i_q}.$$

We can associate to the indeterminate e_k the mapping $\Lambda_n^q(E/F) \to \Lambda_n^{q-1}(E/F)$ defined by

$$e_k \vee \sum \omega_{i_1\ldots i_q} e_{i_1} \wedge \ldots \wedge e_{i_q} = \sum \omega_{i_1\ldots i_q} (e_k \vee (e_{i_1} \wedge \ldots \wedge e_{i_q})),$$

where

$$e_k \vee (e_{i_1} \wedge \ldots \wedge e_{i_q}) = 0 \quad \text{if} \quad k \neq i_1, \ldots, i_q,$$

$$e_k \vee (e_{i_1} \wedge \ldots \wedge e_{i_q}) = (-1)^{r-1} e_{i_1} \wedge \ldots \wedge e_{i_{r-1}} \wedge e_{i_{r+1}} \wedge \ldots \wedge e_{i_q} \quad \text{if} \quad k = i_r.$$

Then δ_a^q: $\Lambda_n^q(E/F) \to \Lambda_n^{q-1}(E/F)$ is defined by

$$\delta_a^q \omega = \sum a_i (e_i \vee \omega) = \sum e_i \wedge (a_i \cdot \omega).$$

The commutativity of the system $(a_1, \ldots, a_n)$ is equivalent with the relation $\delta_a^{q-1} \circ \delta_a^q = 0$ for all q.

We also define δ_a: $\Lambda_n(E/F) \to \Lambda_n(E/F)$ by

$$\delta_a(\omega_0 \oplus \ldots \oplus \omega_n) = \delta_a \omega_1 \oplus \ldots \oplus \delta_a \omega_n \oplus 0.$$

DEFINITION 1. The commuting system $(a_1, \ldots, a_n)$ is *regular* if

$$(0, \delta_a^n, \delta_a^{n-1}, \ldots, \delta_a^1, 0)$$

is an exact sequence of morphisms. The spectrum of $(a_1, \ldots, a_n)$ is the set of $(s_1, \ldots, s_n) \in C^n$ such that $(a_1 - s_1 I, \ldots, a_n - s_n I)$ is not regular (is singular).

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Let now Λ_{n-1} be the exterior algebra in $(e_1, \ldots, e_{n-1})$. Then Λ_{n-1} is a subalgebra of Λ_n. Besides, the mapping ε: $\omega \to e_n \vee \omega$ is a linear surjective mapping $\Lambda_n \to \Lambda_{n-1}$ having Λ_{n-1} as kernel. We tensor by E/F and obtain the exact sequence of quotient Banach spaces

$$0 \to \Lambda_{n-1}(E/F) \xrightarrow{i} \Lambda_n(E/F) \xrightarrow{\varepsilon} \Lambda_{n-1}(E/F) \to 0,$$

i is the inclusion map, ε the map $\omega \to e_n \vee \omega$.

Let $a' = (a_1, \ldots, a_{n-1})$ and define $\delta_{a'}$: $\Lambda_{n-1}(E/F) \to \Lambda_{n-1}(E/F)$ in a manner similar to δ_a: $\Lambda_n(E/F) \to \Lambda_n(E/F)$. It is clear that

$$i \circ \delta_{a'} = \delta_a \circ i, \qquad \delta_{a'} \circ \varepsilon = \varepsilon \circ \delta_a,$$

the above short exact sequence is an exact sequence of complexes. It induces an exact triangle of homology in the category $\boldsymbol{qB}$:

$$\begin{tikzcd}[column sep=tiny] H(\Lambda_{n-1}(E/F), \delta_{a'}) \arrow[rr] & & H(\Lambda_n(E/F), \delta_a) \arrow[dl] \\ & H(\Lambda_{n-1}(E/F), \delta_{a'}), \arrow[ul] & \end{tikzcd}$$

where $H(\Gamma, \delta)$ represents the homology of the complex (Γ, δ) when Γ is a quotient Banach space, and δ: $\Gamma \to \Gamma$ an endomorphism such that $\delta^2 = 0$, i.e. $H(\Gamma, \delta)$ is the cokernel of the obvious mapping $\operatorname{Im}\delta \to \operatorname{Ker}\delta$.

Taylor remarks that the degrees of the mappings in the above triangle are somewhat unexpected, because the short exact sequences introduced in nursery books on homological algebra are associated to mappings of degree zero. Here, in our original short exact sequence, i has degree zero, ε has degree -1, the associated mapping $H(\Lambda_{n-1}) \to H(\Lambda_n)$ will have degree zero as usual, however the mapping $H(\Lambda_n) \to H(\Lambda_{n-1})$ will have degree -1 and the connecting homomorphism $H(\Lambda_{n-1}) \to H(\Lambda_{n-1})$ will have degree zero. When we unwind our triangle to a long exact sequence, we have a sequence

$$\ldots \to H^q(\Lambda_{n-1}) \to H^q(\Lambda_n) \to H^{q-1}(\Lambda_{n-1}) \to H^{q-1}(\Lambda_{n-1}) \to \ldots$$

Taylor's computation of the connecting homomorphism goes through without any essential change here. It is the endomorphism of $H(\Lambda_{n-1}(E/F), \delta_{a'})$ induced by the operation of a_n on $\Lambda_{n-1}(E/F)$. Note that a_n does induce an endomorphism of $H(\Lambda_{n-1}(E/F, \delta_{a'})$ because a_n commutes with $(a_1, \ldots, a_{n-1})$ and therefore with $\delta_{a'}$.

PROPOSITION 1. *The commuting system $(a_1, \ldots, a_n)$ is regular if and only if the operator a_n induces an invertible transformation on $H(\Lambda_{n-1}(E/F), \delta_{a'})$.*

A quick look at the exact triangle of homology will convince the reader that the vanishing of $H(\Lambda_n(E/F), \delta_a)$ is equivalent with the bijectivity of the connecting homomorphism.

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The next step will be the proof that the spectrum of a single operator is not empty. This result is interesting in its own right. It has two proofs. One gives more information when we wish to construct an operational calculus. We shall give the shorter proof here.

LEMMA 1. *The spectrum of a single endomorphism of a non-zero quotient Banach space is not empty.*

Let a be an operator on a quotient Banach space E/F. We may assume without loss of generality that E/F is standard, i.e. that $E = l_1(X)$ for some X. All endomorphisms of E/F are then strict. The algebra of endomorphisms of E/F will then be the quotient A/α where A is the algebra of bounded linear transformations of E which leave F invariant and α is the two-sided ideal of linear mappings of E into F.

A is a Banach algebra with the norm

$$\|u_1\|_A = \sup \{\|u_1 x\|_E, \|u_1 y\|_F \mid \|x\|_E \leqslant 1, \|y\|_F \leqslant 1\},$$

α is a true ideal of A (if $F \neq E$) and is therefore not dense. A_1 will be the quotient $A/\bar{\alpha}$. Then A_1 is a quotient of A/α.

Let $a_1 \in A_1$ be the quotient image of $a \in A/\alpha$. It is trivial that $\operatorname{sp} a_1 \subseteq \operatorname{sp} a$, easy to see but we shall not need the result that $\operatorname{sp} a_1 = \operatorname{sp} a$. Since $\operatorname{sp} a_1 \neq \emptyset$, it follows that $\operatorname{sp} a \neq \emptyset$.

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PROPOSITION 2. *The mapping* $(s_1, \ldots, s_n) \to (s_1, \ldots, s_{n-1})$ *maps* $\mathrm{sp}(a_1, \ldots, a_n)$ *onto* $\mathrm{sp}(a_1, \ldots, a_{n-1})$.

Consider first $(s_1, \ldots, s_{n-1}) \notin \mathrm{sp}(a_1, \ldots, a_{n-1})$ and $s_n \in \mathbf{C}$. We want to show that $(s_1, \ldots, s_n) \notin \mathrm{sp}(a_1, \ldots, a_n)$, i.e. that $a-sI$ is regular. Proposition 1 shows that this is the case if $a_n - s_n I$ induces an invertible transformation on $H(\Lambda_{n-1}(E/F), \delta_{a'-s'I})$. But we assume that $a'-s'I$ is regular, that $H(\Lambda_{n-1}(E/F), \delta_{a'-s'I}) = 0$, and all transformations of the null space are invertible.

Assume next that $(s_1, \ldots, s_{n-1})$ is not in the projection of $\mathrm{sp}(a_1, \ldots, a_n)$, i.e. that $(a_1 - s_1 I, \ldots, a_{n-1} - s_{n-1} I, a_n - tI)$ is regular for all choices of t. Proposition 1 again shows that $a_n - tI$ induces an invertible transformation of $H(\Lambda_{n-1}(E/F), \delta_{a'-s'I})$, this for all choices of t. The operator induced by a_n on this quotient Banach space has an empty spectrum. Lemma 1 shows that this is not possible unless the space $H(\Lambda_{n-1}(E/F), \delta_{a'-s'I})$ is null, unless $a'-s'I$ is regular.

COROLLARY. *The spectrum of* $(a_1, \ldots, a_n)$ *is not empty.*

Methodologically, Lemma 1 is more important than this corollary. But it is a special case, and may not be called a proposition.

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We must still prove that the spectrum is compact.

It is easy to show that it is bounded. After all $\mathrm{sp}(a_1, \ldots, a_n) \subseteq \prod_1^n \mathrm{sp}\, a_i$. It is therefore sufficient to show that $\mathrm{sp}\, a$ is bounded when a is an operator on a Banach quotient. We may assume that E/F is standard. All endomorphisms of E/F are then strict. Assume that u induces a, let

$$\|u\| = \sup\{\|ux\|_E, \|uy\|_F \mid \|x\|_E \leqslant 1, \|y\|_F \leqslant 1\}.$$

Then $u_1 - sI$ is invertible in the Banach algebra of linear transformations of E which leave F invariant, when $|s| > \|u\|$. *A fortiori*, $a-sI$ is invertible in $qB(E/F, E/F)$.

The proof of the fact that $\mathrm{sp}(a_1, \ldots, a_n)$ is closed uses the following result.

PROPOSITION 3. *Let* E, F, G *be Banach spaces. Let* $f\colon E \to F$ *and* $g\colon F \to G$ *be bounded linear mappings such that* (f, g) *is exact, and* g *has closed range. Assume that* $\varepsilon > 0$ *is small, that* $\|f - f'\| < \varepsilon$, *that* $\|g - g'\| < \varepsilon$. *Then* (f', g') *is exact.*

J. L. Taylor ([2], Lemma 2.1) proves this result, and uses it to prove that his spectrum is closed when $a_1, \ldots, a_n$ are operators on a Banach space. B. E. Johnson [1] proves this result also (Lemma 6.1), proves furthermore that g has closed range, and applies this to show that a "perturbed algebra" has properties similar to those of the given algebra.

To apply this result to Banach quotients, we need a result which expresses the exactness of mappings of quotients in terms of properties of the mappings inducing these mappings.

Let E_i/F_i $(i = 1, 2, 3, 4)$ be quotient vector spaces (or abelian groups), let $f_i\colon E_i/F_i \to E_{i+1}/F_{i+1}$ be linear mappings, induced by $u_i\colon E_i \to E_{i+1}$ (this for $i = 1, 2, 3$). Call $v_i\colon F_i \to F_{i+1}$ the restriction of u_i to F_i, and $t_i\colon F_i \to E_i$ the inclusion mapping. Assume that $f_{i+1} \circ f_i = 0$ when $i = 1, 2$. This means that $u_{i+1} \circ u_i(E_i) \subseteq F_{i+2}$, that $w_i\colon E_i \to F_{i+2}$ exists such that $u_{i+1} \circ u_i = t_{i+2} \circ w_i$. We have thus the commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & F_4 \arrow[d] \\
E_1 \arrow[r] \arrow[urr] & E_2 \arrow[r] \arrow[urr] & E_3 \arrow[r] & E_4
\end{tikzcd}$$

We now define $\varphi_i\colon E_i \oplus F_{i+1} \to E_{i+1} \oplus F_{i+2}$ by

$$\varphi_i(e_i \oplus f_{i+1}) = (u_i e_i - t_{i+1} f_{i+1}) \oplus (w_i e_i - v_{i+1} f_{i+1}).$$

We also define $\psi\colon E_3 \oplus F_4 \to E_4$ by

$$\psi(e_3 \oplus f_4) = u_3 e_3 - t_4 f_4.$$

The result that we shall need is

PROPOSITION 4. $(\varphi_1, \varphi_2, \psi)$ *is exact if* (f_1, f_2, f_3) *is exact. Also* (f_1, f_2) *is exact if* $(\varphi_1, \varphi_2, \psi)$ *is exact.*

The proof will be left to the reader.

PROPOSITION 5. *Let* E_i/F_i $(i = 1, \ldots, 4)$ *be Banach quotients and* $f_i\colon E_i/F_i \to E_{i+1}/F_{i+1}$ *be strict morphisms such that* (f_1, f_2, f_3) *is exact. Define* u_i, v_i $(i = 1, 2, 3)$ *and* w_1, w_2 *as above. Let now* $f'_i\colon E_i/F_i \to E_{i+1}/F_{i+1}$ *be new strict morphisms such that* $f'_{i+1} \circ f'_i = 0$. *Define* u'_i, v'_i, *and* w'_i *in a similar way to* u_i, v_i, w_i. *Assume that the choice can be made in such a way that*

$$\|u'_i - u_i\|_{\mathscr{L}(E_i, E_{i+1})} \leqslant \varepsilon, \quad \|v'_i - v_i\|_{\mathscr{L}(F_i, F_{i+1})} \leqslant \varepsilon, \quad \|w'_i - w_i\|_{\mathscr{L}(E_i, F_{i+2})} \leqslant \varepsilon$$

with ε *small enough. Then* (f'_1, f'_2) *is exact.*

Just apply Propositions 3 and 4.

PROPOSITION 6. *The J. L. Taylor spectrum of commuting endomorphisms of* E/F *is compact.*

We have already observed that it is bounded. We must show that it is closed. Assume that $s = (s_1, \ldots, s_n)$ is in the resolvent set of $(a_1, \ldots, a_n)$. Let s' be near to s. We assume that

$$0, \delta^n_{a-s}, \delta^{n-1}_{a-s}, \ldots, \delta^1_{a-s}, 0$$

is exact. The sequence

$$0, \delta^n_{a-s'}, \delta^{n-1}_{a-s'}, \ldots, \delta^1_{a-s'}, 0$$

is such that $\delta^{i-1}_{a-s'} \circ \delta^i_{a-s'} = 0$, is also induced by u'_i, v'_i, w'_i which are norm-wise near to the u_i, v_i, w_i defining the sequence δ^i_{a-s}. Proposition 5 shows that the new sequence is exact.

By the way, the same argument also shows that the spectrum is bounded. The complex $(\Lambda_n(E/F), \delta_{a-s})$ is exact if and only if $(\Lambda_n(E/F), \delta_{\alpha(s)})$ is exact where

$$\alpha(s) = \frac{a-s}{(1+|s|^2)^{1/2}}.$$

When s is large, $\alpha(s)$ is a perturbation in the sense of Proposition 5 of $s_0 = s/|s|$, and $\delta_{\alpha(s)}$ is a perturbation of $\delta_{s/|s|}$. We know that $s/|s|$ is a non-zero system of scalars, is therefore regular. The complex $(\Lambda_n(E/F), \delta_{a-s})$ is therefore exact.

PROPOSITION 7. *Assume that $(a_1, \ldots, a_n)$ is a regular system of strict commuting endomorphisms of E/F, a_i induced by a_i^1: $E \to E$. Let $b_1, \ldots, b_n$ be new commuting endomorphisms of E/F, induced by b_i^1, $E \to E$. Assume that*

$$\|a_i^1 - b_i^1\|_{\mathscr{L}(E,E)} < \varepsilon, \quad \|a_i^1 - b_i^1\|_{\mathscr{L}(F,F)} < \varepsilon, \quad \|a_i^1 a_j^1 - a_j^1 a_i^1 + b_j^1 b_i^1 - b_i^1 b_j^1\|_{\mathscr{L}(E,F)} < \varepsilon,$$

with ε small. Then $(b_1, \ldots, b_n)$ is regular.

For the proof to go through, at least, we must assume not only that $(b_1, \ldots, b_n)$ is near to $(a_1, \ldots, a_n)$ in the Banach algebra of linear transformations of E which leave F invariant. We must also assume that the commutation $[a_i, a_j]$ are near to $[b_i, b_j]$ in the two-sided ideal of linear mappings of E into F.

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PROPERTIES OF THE SPECTRAL RADIUS IN BANACH ALGEBRAS

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CONTENTS

Introduction

In this article we treat a number of topics, as indicated in the table of contents, on the interrelation between the algebraic properties of the spectral radius and those of the algebra. Originally these investigations were motivated by the well-known contrast between commutative and non-commutative Banach algebras. For example, spectra of elements behave much better in commutative algebras than in non-commutative algebras. Therefore we tried to explain to what extent spectral properties (nice in some reasonable sense) of elements in the algebra can effect commutativity, the essence of the Gelfand theory. It emerged that the key to this mystery is contained just in the notion of the spectral radius. It is indeed interesting to find if the properties of the spectral radius can give information about the properties of the whole spectrum and, moreover, about the structure of the whole algebra. Of course, we obtained these results first for Banach algebras over the complex field because some of the crucial steps were based on complex analytic tools like the Cauchy integral formula and the Beurling–Gelfand formula for the spectral radius. The work culminated in the dissertation [25].

Here we provide another approach, also simple, which is more algebraic and avoids the preceding analytic techniques. It consists in a more ingenious applica-

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