are equivalent. But for $v = \varrho u$, $u \in \mathrm{A.P.}^2(\boldsymbol{R}^n)$, we have

$$\|(\underset{1}{\mathfrak{A}}-\lambda)v\| = \|\varrho(\mathfrak{A}-\lambda)u\| \quad \text{and} \quad \|v\| = \|\varrho u\|.$$

Hence the equivalence of the inequalities is a simple consequence of (6).

Let us now introduce a function $\varphi_T \in C_0^2(|x| < T)$ such that $0 \leqslant \varphi_T \leqslant 1$, $\varphi_T(x) = 1$ for $|x| < T-1$ and for $u \in \mathrm{A.P.}_\varrho^2(\boldsymbol{R}^n)$ the following equalities take place:

$$\|u\|^2 = \lim_{T\to\infty} \frac{1}{T^n}\|\varphi_T u\|_{L^2}^2, \qquad (\underset{1}{\mathfrak{A}_{\mathrm{cut}}}u, u) = \lim_{T\to\infty} \frac{1}{T^n}\left(A_{\mathrm{cut}}(\varphi_T u), \varphi_T u\right)_{L^2},$$

$$\|\underset{1}{\mathfrak{A}_{\mathrm{cut}}}u\|^2 = \lim_{T\to\infty} \frac{1}{T^n}\|A_{\mathrm{cut}}(\varphi_T u)\|_{L^2}^2.$$

Hence and from the inequality

$$\|(A_{\mathrm{cut}}-\lambda)v\|_{L^2} \geqslant c\|v\|_{L^2}, \quad v \in C_0^2(\boldsymbol{R}^n), \quad c = \mathrm{const},$$

it follows that

$$\|(\underset{1}{\mathfrak{A}_{\mathrm{cut}}}-\lambda)u\| \geqslant c\|u\|, \quad u \in \mathrm{A.P.}_\varrho^2(\boldsymbol{R}^n).$$

This gives the last inclusion of (9) and ends the proof of the cutting theorem. For more information on spectral analysis in nonseparable spaces see [3]–[6], [8]–[11].

References

[1] J. Bass, *Espace de Besicovitch*, Bull. Soc. Math. France 91 (1963), 39–61.

[2] M. Burnat, *Die Spektraldarstellung einiger Differentialoperatoren mit periodischen Koeffizienten im Raume der fastperiodischen Funktionen*, Studia Math. 25 (1964), 33–64.

[3] —, *Perturbation properties of the Schrödinger operator spectrum with unbounded periodic potential*, Bull. Acad. Polon. Sci., Sér. math., astronom. phys. 20 (6) (1972), 457–460.

[4] —, *The Schrödinger operator with periodic potential in three dimensions*, preprint, Warsaw University, Institute of Mathematics, 1973.

[5] —, *On certain spectral problems in nonseparable Hilbert spaces*, Bull. Acad. Polon. Sci., Sér. math., astronom., phys., 26 (11) (1978), 895–900.

[6] M. Burnat and A. Palczewski, *On the spectral properties of the operator $-\Delta u + q(x^1, x^2, x^3)u$ with almost periodic $q(x^1, x^2, x^3)$*, ibid. 21 (10) (1973).

[7] J. Marcinkiewicz, *Une remarque sur les espaces de M. Besicovitch*, C. R. Acad. Sci. Paris 208 (1939), 157–159.

[8] М. А. Шубин, *Теоремы о совпадении спектров псевдо дифференциального почти-периодического оператора в пространствах $L^2(\boldsymbol{R}^n)$ и $B^2(\boldsymbol{R}^n)$*, Сиб. Мат. Сборник 17 (1976), 200–215.

[9] —, *Почти-периодические функции и дифференциальные операторы с частными производными*, Успехи Мат. Наук XXXIII, 2 (200) (1978), 3–47.

[10] J. Herczyński, *A note on Shubins theorem*, Bull. Acad. Polon. Sci., Sér. math., astronom., phys., 29 (1981), 65–71.

[11] — *On the spectrum of the Schrödinger operator*, ibid. 29 (1981), 73–77.

Presented to the semester
Spectral Theory
September 23–December 16, 1977

SPECTRAL THEORY
BANACH CENTER PUBLICATIONS, VOLUME 8
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

DILATIONS TO SYSTEMS OF MATRIX UNITS

M. D. CHOI and CHANDLER DAVIS

Department of Mathematics, University of Toronto, Toronto, Canada

0. The context

The dilation theorem of M. A. Naĭmark ([9], Thm. I.8.2) concerns dilations to a spectral measure. The object dilated has some but not all of the defining properties of a spectral measure. The dilation theorem of W. F. Stinespring ([8], [2]), and that of the present paper, are of analogous nature. To show what we are about, we begin by restating Naĭmark's theorem in the case which gives a spectral measure of finite support.

THEOREM 0.1 (Naĭmark). *Assume the operators A_j (j running over a finite index set) satisfy $0 \leqslant A_j \in \mathscr{B}(\mathscr{H})$, $\sum_j A_j = 1$. Then there exists an isometric injection ι of $\mathscr{H}$ into a larger Hilbert space $\mathscr{K}$, and there exist commuting orthoprojectors $E_j \in \mathscr{B}(\mathscr{K})$ with $\sum_j E_j = 1$, such that $A_j = \iota^* E_j \iota$.*

The last equation holding for all j is what is meant by saying the E_j are a simultaneous dilation of the A_j; so the theorem may be phrased briefly thus: a finite family of positive operators adding to the identity have a simultaneous dilation to complementary orthoprojectors.

One way of proving Naĭmark's theorem [6] begins with a simple explicit construction for the case where there are only two A_j, and handles more numerous families by iterating this construction in nested fashion. Infinite families can be handled in the same way and the full force of Naĭmark's theorem recovered.

Stinespring's theorem concerns linear mappings on a C^*-algebra into $\mathscr{B}(\mathscr{H})$. We will restate it in the special case where the given algebra is that of all $n \times n$ complex matrices. This algebra is the linear span of the e_{jk} (this denotes the matrix having entry 1 in the j, k-place and all other entries zero), so any linear mapping of it is determined by the images A_{jk} of the e_{jk}. It is known ([4], Remark 1.8, or [5], Lemma 2.1) that the mapping is completely positive if and only if the A_{jk} form a positive operator-matrix; and that the mapping is a *-homomorphism if and only if, in addition, $A_{ij}A_{kl} = \delta_{jk}A_{il}$. The following is therefore a variant of this case of Stinespring's theorem (cf. [3], Lemma 3.2):

THEOREM 0.2. *Assume the operators $A_{jk} \in \mathscr{B}(\mathscr{H})$ (j, k running over a finite index set) form a positive operator-matrix with $\sum_j A_{jj} = 1$. Then there exists an isometric injection ι of $\mathscr{H}$ into a larger Hilbert space $\mathscr{K}$, and there exist $E_{jk} \in \mathscr{B}(\mathscr{K})$ forming a positive operator-matrix with $\sum_j E_{jj} = 1$, $E_{ij}E_{kl} = \delta_{jk}E_{il}$, such that $A_{jk} = \iota^* E_{jk}\iota$.*

That is, a finite normalized positive operator-matrix has a dilation to a system of matrix units, in the terminology of the present paper.

Now the A_{jj} in the hypothesis of Theorem 0.2 obey the conditions imposed on the A_j in Theorem 0.1, and the E_j found there go part way toward the conclusions of Theorem 0.2. Can the construction of the E_j be enriched to yield the whole matrix of the E_{jk}, then, providing an explicit proof of Theorem 0.2? Can the induction on the size of the index set be continued countably, and what then results?

We will carry out this program in Sections 1 and 2 of the paper. The resulting theorems, finite and countable, fail to encompass the full variety of cases. In Section 3 we give a continuous analogue. Section 4 is a discussion of the relationship between our results and Stinespring's theorem.

1. Finite operator-matrices

The main theorems deal with matrices whose entries A_{jk} are elements of $\mathscr{B}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$. The dilations E_{jk} are in $\mathscr{B}(\mathscr{K})$ for a different Hilbert space $\mathscr{K}$ but are indexed the same. The index set is finite in this section, countably infinite in the following one. Let us adopt the following notation for indices, which will permit us to state hypotheses, conclusions, and constructions in a consistent way.

Notations. Symbols $\mu, \nu, \ldots$ take the values 0, 1. Indices $j, k, \ldots$ will be strings $\mu\nu\ldots$ The length of a string j, denoted $|j|$, may have any value 0, 1, ... In case $|j| = 0$, the string is empty: $j = \emptyset$.

DEFINITION 1.1. Let an operator $A_{jk} \in \mathscr{B}(\mathscr{H})$ be given for every j, k with $|j| = |k| = m$. The indexed collection will be called a *normalized positive* operator-matrix provided

(i) the operator-matrix $(A_{jk})_{j,k}$ is $\geqslant 0$;

(ii) $\sum_j A_{jj} = 1$.

In this case we define A_{hi} for strings h, i with $|h| = |i| < m$ by applying (iteratively) the rule

(iii) $A_{hi} = A_{h0,i0} + A_{h1,i1}$.

One verifies that the matrix of lesser size produced by rule (iii) is again normalized positive. Note that automatically $A_{\emptyset\emptyset} = 1$. Note also that, given any normalized positive $2^m \times 2^m$ operator-matrix, there are various easy ways to produce a $2^{m+1} \times 2^{m+1}$ normalized positive operator-matrix which is related to it by (iii); for one, say $A_{j0,k0} = A_{jk}$ and otherwise $A_{j\mu,k\nu} = 0$.

The restriction to $2^m \times 2^m$ matrices which is apparently imposed by the notation, is inessential. Given an $n \times n$ operator-matrix satisfying (i) and (ii), with n not a power of 2, one could always border it with zeros and treat the resulting matrix, discarding zero rows and columns at the end. We will say no more about this point.

DEFINITION 1.2. A normalized positive operator-matrix (A_{jk}) will be said to comprise a *system of matrix units* if it satisfies

(iv) $A_{ij}A_{kl} = \delta_{jk}A_{il}$.

One computes in one line that in this case, the operator-matrices obtained from it by applying (iii) will themselves satisfy (iv).

An alternative characterization will come in handy for our proofs:

PROPOSITION 1.1. *An $n \times n$ normalized positive operator-matrix $(A_{jk})_{j,k}$ is a system of matrix units iff*

(v) $\frac{1}{n}(A_{jk})_{j,k}$ *is an orthoprojector.*

Proof. The implication in the forward direction being immediate, we assume (i), (ii), (v), and seek to prove (iv). Because $A_{ij} = A_{ji}^*$, the hypothesis

$$\sum_{i=1}^{n} A_{ji}A_{ik} = nA_{jk}$$

can be rewritten, in terms of the temporary definition

$$B_i = \frac{1}{\sqrt{n}}(A_{i1}\ A_{i2}\ \ldots\ A_{in}),$$

as $B_iB_j^* = \frac{1}{n}A_{ij}$. In particular, $B_iB_i^*$ is the positive contraction $\frac{1}{n}A_{ii}$. Hence $B_i^*B_i = \frac{1}{n}(A_{ji}A_{ik})_{j,k}$ is also a positive contraction. On the other hand, the average of these n operators is

$$\frac{1}{n}\sum_i B_i^*B_i = \frac{1}{n}(A_{jk})_{j,k},$$

which by assumption (v) is an orthoprojector. Any orthoprojector is extreme in the convex set of positive contractions, consequently each of the operators being averaged must equal $\frac{1}{n}(A_{jk})_{j,k}$. This is the desired conclusion

$$A_{ji}A_{ik} = A_{jk}. \tag{1.1}$$

To complete the proof of (iv), it remains only to prove $A_{ij}A_{kl}$ zero for $j \neq k$. By application of (1.1), it is enough to prove $A_{jj}A_{kk}$ zero for $j \neq k$. But by (1.1), (i), (ii), the A_{jj} are orthoprojectors whose sum is 1, hence they are mutually orthogonal. ∎

The main result of this section is

THEOREM 1.1. *Assume the operators $A_{jk} \in \mathscr{B}(\mathscr{H})$ ($|j| = |k| = m$) form a normalized positive operator-matrix. Then there exists an isometric injection ι of $\mathscr{H}$ into a Hilbert space $\mathscr{K}$, and there exist $E_{jk} \in \mathscr{B}(\mathscr{K})$ comprising a system of matrix units, such that $A_{jk} = \iota^* E_{jk} \iota$.*

Proof. In the case $m = 0$ we take $\mathscr{K} = \mathscr{H}$, $E_{\emptyset\emptyset} = 1$, and $\iota = 1$. By virtue of (iii) and the remarks above, we can proceed by induction on m.

Let us not start the induction at $m = 0$, but rather display the case $m = 1$. The general inductive step will thereby be better motivated.

Let $m = 1$ for now, so we are treating $(A_{\mu\nu})_{\mu,\nu=0}^{1}$. Define $F_\mu = (A_{\mu\mu})^{1/2}$; these are commuting positive operators with $F_0^2 + F_1^2 = 1$. The construction combines two devices, one confined to the diagonal and one to take care of the off-diagonal entries.

LEMMA 1.1 (E. A. Michael [7]). *Every positive contraction $\in \mathscr{B}(\mathscr{H})$ has a dilation to $\mathscr{H} \oplus \mathscr{H}$ which is an orthoprojector.*

In the notation of our problem, the given contraction might be F_0^2. Michael's construction gives as its dilation

$$\begin{pmatrix} F_0^2 & F_0 F_1 \\ F_1 F_0 & F_1^2 \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} (F_0 \ \ F_1),$$

which is easily seen to be an orthoprojector. Along with this we would want to dilate $1 - F_0^2 = F_1^2$ to the complementary orthoprojector, namely

$$\begin{pmatrix} F_1^2 & -F_0 F_1 \\ -F_1 F_0 & F_0^2 \end{pmatrix} = \begin{pmatrix} F_1 \\ -F_0 \end{pmatrix} (F_1 \ \ -F_0).$$

This is enough to handle the diagonal of our matrix, and it can be repeated to handle the diagonal of a $2^m \times 2^m$ matrix for any m, i.e., to prove Theorem 0.1. Such is the method of [6].

For off-diagonal entries we need the following result, a variant of one by Ju. L. Šmul'jan. It appears as Theorem I.1 of [1].

LEMMA 1.2. *Assume the operator-matrix*

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

is self-adjoint, and that the $A_{\mu\mu}$ are positive, say $F_\mu = A_{\mu\mu}^{1/2}$. In order that the operator-matrix be positive, it is necessary and sufficient that $A_{01} = F_0 T F_1$ for some contraction T. This T is uniquely determined if one insists that $T^* \mathscr{N}(F_0) = T \mathscr{N}(F_1) = \{0\}$.

We omit the brief proof.

We will use this notation for $A_{\mu\nu}$. We will further set $Q = (1 - T^*T)^{1/2}$ and $Q_* = (1 - TT^*)^{1/2}$. It is well known that then $TQ = Q_* T$.

Our dilations $E_{\mu\nu} = \iota^* A_{\mu\nu} \iota$ necessarily admit an analogous expression

$$\text{(1.2)} \qquad \begin{pmatrix} E_{00} & E_{01} \\ E_{10} & E_{11} \end{pmatrix} = \begin{pmatrix} \tilde{F}_0^* \tilde{F}_0 & \tilde{F}_0^* \tilde{T} \tilde{F}_1 \\ \tilde{F}_1^* \tilde{T}^* \tilde{F}_0 & \tilde{F}_1^* \tilde{F}_1 \end{pmatrix}$$

with $\|\tilde{T}\| \leqslant 1$. A reformulation of the hoped-for properties of the $E_{\mu\nu}$ will now be made (it shows already why we prefer not to require the $\tilde{F}_\mu$ to be self-adjoint):

LEMMA 1.3. *In order for the operator-matrix* (1.2) *to be* 2 *times an orthoprojector, it is sufficient that $\tilde{T}$ be unitary and that*

$$\tilde{F}_0 \tilde{F}_0^* = \tilde{F}_1 \tilde{F}_1^* = 1.$$

Indeed, if one squares (1.2) and applies these conditions, the result reduces at once to twice matrix (1.2).

Remember that, by Proposition 1.1, this is just what we will need to know of our $E_{\mu\nu}$ to reach the conclusion of Theorem 1.1 for $m = 1$.

We use one more well known fact [7]:

LEMMA 1.4. *If T is a contraction then*

$$\begin{pmatrix} T & Q_* \\ Q & -T^* \end{pmatrix}$$

is unitary.

Now the construction can be written down. We prescribe $\mathscr{K}$ as the space of 4-component column-vectors whose entries are in $\mathscr{H}$, and ι as the mapping taking any $x \in \mathscr{H}$ to $\begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix}$. We specify the $E_{\mu\nu}$, in the notation (1.2), by

$$\text{(1.3)} \qquad \tilde{F}_0 = \begin{pmatrix} F_0 & F_1 & 0 & 0 \\ 0 & 0 & F_0 & F_1 \end{pmatrix}, \quad \tilde{F}_1 = \begin{pmatrix} F_1 & -F_0 & 0 & 0 \\ 0 & 0 & F_1 & -F_0 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} T & Q_* \\ Q & -T^* \end{pmatrix}.$$

From the lemmas, it is easy to see that we get a system of matrix units; also $\iota^* E_{\mu\nu} \iota = A_{\mu\nu}$ by definitions. This proves Theorem 1.1 in the case $m = 1$.

The inductive step applies all the same ideas. Assume given normalized positive $A_{j\mu,k\nu}$, where $|j| = |k| = m$, and define A_{jk} therefrom by (iii). The inductive hypothesis is that there is an injection $\iota^{(m)}$ into some $\mathscr{K}^{(m)}$ and a system of matrix units $(E_{jk}^{(m)})$ such that $A_{jk} = \iota^{(m)*} E_{jk}^{(m)} \iota^{(m)}$. We can condense all this by introducing some more notation. Let $B = 2^{-m}(A_{jk})_{j,k}$ and $B_{\mu\nu} = 2^{-m}(A_{j\mu,k\nu})_{j,k}$, operators on $\mathscr{H}^{2^m}$. Then we know that

$$\text{(1.4)} \qquad \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \geqslant 0, \quad B = B_{00} + B_{11}.$$

Further let $D = 2^{-m}(E_{jk}^{(m)})_{j,k}$. By Proposition 1.1, the inductive hypothesis says D is an orthoprojector. Our method will be, using this and (1.4), to put $B_{\mu\nu}$ in the role $A_{\mu\nu}$ played for $m = 1$. We will dilate $B_{\mu\nu}$ simultaneously to $D_{\mu\nu}$ on some $\mathscr{K}^{2^m}$ such that

$$\text{(1.5)} \qquad \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \quad \text{is twice an orthoprojector}$$

and then define the $E_{j\mu,k\nu}$ on this $\mathscr{K}$ by setting

$$(E_{j\mu,k\nu})_{j,k} = 2^{-m}D_{\mu\nu}.$$

Now we have an injection $\iota^{[m]}$ (the diagonal sum of 2^m copies of $\iota^{(m)}$) such that $B = \iota^{[m]*}D\iota^{[m]}$. In generalization of Lemma 1.2, we write

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} \iota^{[m]*}D & 0 \\ 0 & \iota^{[m]*}D \end{pmatrix} \begin{pmatrix} F_0^2 & F_0TF_1 \\ F_1T^*F_0 & F_1^2 \end{pmatrix} \begin{pmatrix} D\iota^{[m]} & 0 \\ 0 & D\iota^{[m]} \end{pmatrix},$$

where the F_μ are commuting positive operators such that $F_0^2+F_1^2$ is the orthoprojector onto range $(D\iota^{[m]})$, and where T is a contraction. Let us say instead that $F_0^2+F_1^2 = D$, by giving the F_μ arbitrary acceptable definitions on the complementary subspace. Then if we define $\tilde{F}_\mu$, $\tilde{T}$ by the same formulas (1.3), we see that $\tilde{T}$ is unitary and $\tilde{F}_0\tilde{F}_0^* = \tilde{F}_1\tilde{F}_1^* = D$. Defining the $D_{\mu\nu}$ by formula (1.2) used before for the $E_{\mu\nu}$, one computes as in that case that the resulting matrix is twice an orthoprojector, i.e., (1.5). (The space $\mathscr{K}$, it has turned out, is the direct sum of 4 copies of $\mathscr{K}^{(m)}$.) One also computes

$$D_{00}+D_{11} = \tilde{F}_0^*\tilde{F}_0+\tilde{F}_1^*\tilde{F}_1 = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}. \tag{1.6}$$

The $E_{j\mu,k\nu}$ have been defined so as to make up a plainly positive operator-matrix. That $\sum_{j,\mu} E_{j\mu,j\mu}$ is the identity on $\mathscr{K}$ follows directly from (1.6) together with the fact that $\sum_j E_{jj}$ is the identity on $\mathscr{K}^{(m)}$. Thus Proposition 1.1 applies; to show the $E_{j\mu,k\nu}$ comprise a system of matrix units, we need only prove $2^{-m-1}(E_{j\mu,k\nu})_{j,k,\mu\nu}$ idempotent. But

$$2^{-2m}\Big(\sum_{i,\varrho} E_{j\mu,i\varrho}E_{i\varrho,k\nu}\Big)_{j,k} = \sum_{\varrho} D_{\mu\varrho}D_{\varrho\nu} = 2D_{\mu\nu} = 2^{-m+1}(E_{j\mu,k\nu})_{j,k}$$

by definitions and (1.5).

Finally, it is clear how $B_{\mu\nu}$ is obtained as a compression of $D_{\mu\nu}$, and hence $A_{j\mu,k\nu}$ as a compression of $E_{j\mu,k\nu}$. The proof is complete.

2. Infinite operator-matrices

The definitions of the previous section extend at once.

DEFINITION 2.1. Let an operator $A_{jk} \in \mathscr{B}(\mathscr{H})$ be given for every pair of strings j, k of whatever (equal) finite length. The indexed collection will be called *normalized positive* provided

(i) for each m, the A_{jk} with $|j| = |k| = m$ form an operator-matrix $(A_{jk})_{j,k} \geq 0$;

(ii) $\sum_{|j|=m} A_{jj} = 1$;

(iii) $A_{hi} = A_{h0,i0}+A_{h1,i1}$.

DEFINITION 2.2. A normalized positive collection will be said to comprise a *system of matrix units* if it satisfies

(iv) $A_{ij}A_{kl} = \delta_{jk}A_{il}$,

i.e., if its elements for indices of each fixed length comprise a finite system of matrix units.

It is also easy to see what the dilation result ought to be.

THEOREM 2.1. *Assume the operators $A_{jk} \in \mathscr{B}(\mathscr{H})$ ($|j| = |k| = 0, 1, 2, \ldots$) form a normalized positive collection. Then there exists an isometric injection ι of $\mathscr{H}$ into a Hilbert space $\mathscr{K}$, and there exist $E_{jk} \in \mathscr{B}(\mathscr{K})$, comprising a system of matrix units, such that $A_{jk} = \iota^*E_{jk}\iota$.*

The induction in the last section, in this situation, provides a sequence of even larger dilation spaces. One may reasonably be discontented with this as an answer, and ask for a fairly explicit injection of $\mathscr{H}$ into a single space $\mathscr{K}$. We now give this.

The following conventions extend those of Section 1 in two directions.

Notations. Symbols $\varepsilon, \ldots$ take the values 0, 1, 2, 3. Indices $f, g, \ldots$ will be strings of such digits. The length of a string f, denoted $|f|$, may have any value 0, 1, ... In case $|f| = 0$, the string is empty: $f = \varnothing$.

The name of a string preceded by a point, as $.g$, denotes the corresponding quaternary (terminating) rational. In particular, $.\varnothing = .0$, and this rational will also be denoted by 0.

With these understandings, we describe our dilation space $\mathscr{K}$ as the direct sum of countably many copies of $\mathscr{H}$, indexed by all the distinct quaternary rationals in [0, 1]: $\mathscr{K} = \bigoplus_{.g}\mathscr{H}_{.g}$. We take ι to be the natural mapping of $\mathscr{H}$ onto $\mathscr{H}_0$.

In describing the dilations E_{jk} on this fixed $\mathscr{K}$, we will be doing hardly more than labelling constructions indicated above matricially. We want to follow the previous course of dilating for strings j, k of progressively greater lengths. We will always be bound by the

Rule: If $|j| = |k| = m$, then E_{jk} is determined by its action on the subspace $\mathscr{H}^{(m)} = \bigoplus_{|f|=m}\mathscr{H}_{.f}$. Specifically, E_{jk} leaves invariant (for each $.g$) the subspace $\mathscr{H}^{(m)}_{.g} = \bigoplus_{|f|=m}\mathscr{H}_{.fg}$, and its action there is got from its action on $\mathscr{H}^{(m)} = \mathscr{H}^{(m)}_{\varnothing}$ via the natural isomorphism of each $\mathscr{H}_{.f}$ to $\mathscr{H}_{.fg}$.

We will use the same symbol E_{jk} for the operator on all of $\mathscr{K}$ or for its restriction to one of these natural invariant subspaces.

Thus to begin the procedure, we give the $E_{\mu\nu}$ by the same expressions (1.2), (1.3) as above, acting on the space of 4-vectors from $\mathscr{H}$—but now we specify that these 4 components are regarded as belonging to $\mathscr{H}_{.0}$, $\mathscr{H}_{.1}$, $\mathscr{H}_{.2}$, $\mathscr{H}_{.3}$ respectively. By the Rule, $E_{\mu\nu}$ then must have (for each $.g$) the same matrix acting on the space of columns whose components belong respectively to $\mathscr{H}_{.0g}$, $\mathscr{H}_{.1g}$, $\mathscr{H}_{.2g}$, $\mathscr{H}_{.3g}$. This defines the $E_{\mu\nu}$, with the needed properties, on all of $\mathscr{K}$.

Now for the passage from length m to length $m+1$. We are able, because of the Rule, to confine attention to $\mathscr{H}^{(m+1)}$. We choose to regard it as the space of columns of 4 components, each from $\mathscr{H}^{(m)}$. That is, we choose to decompose

$$\mathscr{H}^{(m+1)} = \mathscr{H}^{(m)}_{.0}\oplus\mathscr{H}^{(m)}_{.1}\oplus\mathscr{H}^{(m)}_{.2}\oplus\mathscr{H}^{(m)}_{.3}. \tag{2.1}$$

The induction hypothesis gives all the E_{jk} for $|j| = |k| = m$ on each of these copies $\mathscr{H}^{(m)}_{.\varepsilon}$ of $\mathscr{H}^{(m)}$; we are to prescribe the $E_{j\mu,k\nu}$.

This is all consistent with Section 1. The only respect in which we have less freedom now is that we are committed, by the Rule, to choosing the $E_{j\mu,k\nu}$ so that

$$E_{j0,k0}+E_{j1,k1} = \begin{pmatrix} E_{jk} & 0 & 0 & 0\\ 0 & E_{jk} & 0 & 0\\ 0 & 0 & E_{jk} & 0\\ 0 & 0 & 0 & E_{jk}\end{pmatrix}$$

(the 4×4 matrix on the right has entries operators on $\mathscr{H}^{(m)}$, the operator on the left acts on $\mathscr{H}^{(m+1)}$, and these spaces are related according to (2.1)). But our construction led to (1.6); since, by definitions, $D = (E_{jk})_{j,k}$ and $D_{00}+D_{11} = (E_{j0,k0}+ +E_{j1,k1})_{j,k}$, this gives the relation we now require. This concludes the construction.

3. A continuous-parameter analogue

To pass from the finite version of Naĭmark's dilation theorem (Theorem 0.1) to the usual version, one replaces the finite collection of positive operators A_j adding to 1 by an increasing operator-valued function $A(\cdot)$ on $[0, 1]$ such that $A(0) = 0$, $A(1) = 1$. It is not necessary to require any kind of continuity of the function (though one may prefer, at a value t where it has a jump, to attribute significance only to $A(t-)$ and $A(t+)$); consequently the finite version is retained within the continuous-parameter version, as the special case where the function is constant except for finitely many jumps.

We have been dilating two-parameter families (A_{jk}) with $\sum_j A_{jj} = 1$. Analogy prompts the introduction of operator-valued functions on $[0, 1]\times[0, 1]$, increasing on the diagonal; one expects differences $A(s+r, t+r)-A(s, t)$ to correspond to the A_{jk}. In this section we prove a dilation theorem of this sort. Away from the diagonal, discontinuities become intractable because we are deprived of any hypothesis of monotonicity; thus it is not surprising that Theorem 3.1 deals only with the continuous case and hence fails to include the case of Theorem 1.1. Indeed, we wonder whether a good common generalization of the two would use our present definition at all, or would need a differently described system of operators. (But see the Remark at the end of the section.)

The section is organized as follows. First we make the new definitions, of "normalized increasing systems" and "standard systems"; we prove simple properties, and we show how, in the case of binary rational parameters, normalized increasing systems correspond to the infinite normalized positive systems of Section 2 and standard systems to infinite systems of matrix units. Then we impose the hypothesis of continuity, and prove the dilation theorem for systems with real parameters.

DEFINITION 3.1. A *normalized increasing system* is a function $A(\cdot,\cdot)$ on $[0, 1]\times[0, 1]$ to $\mathscr{B}(\mathscr{H})$ such that $A(0, 0) = 0$, $A(1, 1) = 1$, and for $r > 0$ and any n-tuple of reals $s_\varkappa$ we have

$$\big(A(s_\varkappa, s_\lambda)-A(s_\varkappa-r, s_\lambda-r)\big)^n_{\varkappa,\lambda=1} \geqslant 0. \tag{3.1}$$

(By convention, $A(s, t) = 0$ if either $s < 0$ or $t < 0$.) A *rational normalized increasing system* is a function satisfying the same condition but defined only for terminating binary arguments.

Our notation for rationals will be like that used for quaternary rationals in the last section: if j is a binary string (always of finite length $|j|$) then $.j$ denotes the rational $2^{-|j|}j$.

DEFINITION 3.2. A *standard system* is a function $E(\cdot,\cdot)$ on $[0, 1]\times[0, 1]$ to $\mathscr{B}(\mathscr{H})$ such that $E(0, 0) = 0$, $E(1, 1) = 1$, $E(s, t) = E(t, s)^*$, and for any $t\geqslant 0$

$$E(s, t)E(t, u) = E(s, u)-E(s-t, u-t).$$

(By convention, $E(s, t) = 0$ if either $s < 0$ or $t < 0$.) A *rational standard system* is a function satisfying the same conditions but defined only for terminating binary arguments.

We remark that the condition $E(0, 0) = 0$, and indeed $E(s, 0) = E(0, t) = 0$, follow from the other conditions, because $E(s, 0)E(s, 0)^* = E(s, 0)E(0, s) = E(s, s)-E(s, s) = 0$.

PROPOSITION 3.1. *Every [rational] standard system is a [rational] normalized increasing system.*

Proof. Given $r > 0$ and $s_1, \ldots, s_n$, we set for the moment $B = (E(r, s_1) \ldots \ldots E(r, s_n))$. Then

$$\big(E(s_\varkappa, s_\lambda)-E(s_\varkappa-r, s_\lambda-r)\big)_{\varkappa,\lambda} = \big(E(s_\varkappa, r)E(r, s_\lambda)\big)_{\varkappa,\lambda} = B^*B\geqslant 0. \blacksquare$$

PROPOSITION 3.2. *In any standard system the $E(s, s)$ are an increasing family of orthoprojectors.*

Proof. It follows at once from the definition that the $E(s, s)$ are self-adjoint and idempotent. But also by (3.2)

$$E(s, s)-E(s-t, s-t) = E(s, t)E(t, s)\geqslant 0$$

if $t > 0$. $\blacksquare$

PROPOSITION 3.3. *In any standard system, for $r > 0$,*

$$E(s, t)E(t+r, v) = E(s, t)E(t, v-r).$$

Proof. Setting $X = E(t,t)E(t+r,r)$, we have

$$XX^* = E(t,t)E(t+r,r)E(r,t+r)E(t,t)$$
$$= E(t,t)\big(E(t+r,t+r)-E(t,t)\big)E(t,t) = 0$$

by Proposition 3.2. Hence $X = 0$, and

$$0 = E(s,t)XE(r,v) = E(s,t)E(t,t)E(t+r,r)E(r,v)$$
$$= E(s,t)\big(E(t+r,v)-E(t,v-r)\big)$$

by (3.2). This is the desired equality.

This result allows us to replace (3.2) by a more general identity, using the popular notation $r_+ = \max(0,r)$ for reals r: if $t \geqslant 0$, $u \geqslant 0$ then

$$E(s,t)E(u,v) = E\big(s-(t-u)_+, v-(u-t)_+\big)-E(s-t, v-u). \tag{3.3}$$

This means in particular that the linear span of the $E(s,t)$ is closed under multiplication (as well as under taking adjoints).

Proposition 3.4. *Normalized positive collections $\{A_{jk}\}_{j,k}$ of Definition 2.1 correspond one-one to rational normalized increasing systems $A(\cdot,\cdot)$ under the following correspondence. For binary rationals $.j$, $.k$ (without loss of generality $|j| = |k| = m$), let $A(.i,.k)$ be the sum of all $A_{j'k'}$ with $|j'| = |k'| = m$, $j-j' = k-k' > 0$. This correspondence takes systems of matrix units to standard systems.*

The proof consists of verifications which need not all be written out. We call attention to a few points. The reason one is free to suppose $|j| = |k|$ is the fact that $.j0 = .j$ so strings of unequal length can be replaced by strings of equal length by adjoining zeros. But this brings one more thing to verify, namely independence of adjunction of superfluous zeros — which follows by using (iii) of Definition 2.1. This feature is also needed in checking (3.1): it follows neatly from (i) of Definition 2.1 once we have provided that r and the $s_\varkappa$ are all represented by binary strings of the same length.

Let us note also the inverse correspondence:

$$A_{jk} = A\big(.(j+1), .(k+1)\big)-A(.j,.k)$$

if it is agreed that adding 1 to a string j means adding it in the last ($|j|$-th) place.

Because of this proposition, there is a dilation theorem for rational normalized increasing systems; namely, Theorem 2.1 says they can be dilated to rational standard systems. This is related to the new dilation result which follows.

Theorem 3.1. *Assume $A(\cdot,\cdot)$ is a weakly continuous normalized increasing system of operators of $\mathscr{B}(\mathscr{H})$. Then there exists an isometric injection ι of $\mathscr{H}$ into a Hilbert space $\mathscr{K}$, and there exists a weakly continuous standard system $E(\cdot,\cdot)$ of operators of $\mathscr{B}(\mathscr{K})$, such that $A(s,t) = \iota^*E(s,t)\iota$.*

One reason for assuming continuity is this. We will naturally get our dilation by first restricting the arguments of $A(\cdot,\cdot)$ to be binary rationals, and dilating the rational system to a rational standard system by the theorem already known. The resulting operators $E(.j,.k)$ do not necessarily tell us all we want to know.

Example 3.1. The space $\mathscr{H}$ will be $L^2(0,1)$. On this space we will use the notation $\varphi(r,s)$ for the operator of multiplication by the characteristic function of $[r,s[$; the notation $\tau(r)$ will denote the partial isometry of translation, i.e.,

$$(\tau(r)f)(t) = \begin{cases} f(t+r) & (t \leqslant 1-r),\\ 0 & (\text{otherwise}).\end{cases}$$

Thus for example $\tau(r)\tau(r)^* = \varphi(0,r)$. Now define $A(t,t)$ to be $\varphi(0,t)$, while $A(s,t) = 0$ for all $s < t$ except for

$$A(s,s+\tfrac{2}{3}) = \varphi(0,s)\tau(\tfrac{2}{3}) = \tau(\tfrac{2}{3})\varphi(\tfrac{2}{3},s+\tfrac{2}{3}) \text{ for } 0 < s \leqslant \tfrac{1}{3}$$
$$(\text{and for } s > t,\ A(s,t) = A(t,s)^*).$$

It is easy to verify that this $A(\cdot,\cdot)$ is a normalized increasing system, even a standard system. On the diagonal, restricting to binary rational arguments would leave adequate information available. By contrast, for $s \neq t$ if we looked only at pairs (s,t) made of binary rationals we would lose sight of all the non-zero operators, for no such pair has $t = s+\frac{2}{3}$.

The hypothesis of continuity, as already pointed out, is very restrictive. We accordingly include an example to show it is satisfied by some reasonable systems.

Example 3.2. With the notations φ and τ as before, define $A(s,t) = \varphi(0,s)\tau(t-s) = \tau(t-s)\varphi(t,s+t)$ for all $s \leqslant t$ (and for $s > t$, $A(s,t) = A(t,s)^*$). It is easy to verify that this is a standard system, and that the dependence of $A(s,t)$ on (s,t) is strongly continuous.

Proof of Theorem 3.1. We are given the $A(s,t) \in \mathscr{B}(\mathscr{H})$. By Proposition 3.4 and Theorem 2.1, we have $\iota\colon \mathscr{H} \to \mathscr{K}$ and operators $E(.j,.k) \in \mathscr{B}(\mathscr{K})$ giving a rational standard system dilating the $A(.j,.k)$. It remains to define the $E(s,t)$ for other real values and to prove the asserted properties. Because the linear span of the $E(.j,.k)\iota x$ is closed under operation by the other $E(.j,.k)$ (Proposition 3.3), we may assume without loss of generality that it is dense in $\mathscr{K}$; and we will not need to go to a new dilation space. That is, we will simply provide a workable definition of $y^*\iota^*E(.i,.j)E(s,t)E(.k,.l)\iota x$ for arbitrary $x, y \in \mathscr{H}$ and arbitrary values of the parameters.

Lemma 3.1. *$E(.i,.j)E(.i',.j')E(.k,.l)$ is a weakly continuous function of its arguments.*

This is clear from (3.3).

It has the consequence that $y^*\iota^*E(.i,.j)E(s,t)E(.k,.l)\iota x$ can be defined as the limit of approximating expressions with s,t replaced by binary rational approximants. Furthermore, the properties in Definition 3.2 then carry over by continuity. And it is immediate that the resulting $E(\cdot,\cdot)$ dilates $A(\cdot,\cdot)$.

To prove weak continuity of $E(s,t)$ it is enough to confine attention to the dense subset of $\mathscr{K}$ afforded by linear spans of the $E(.j,.k)\iota x$. But this reduces to proving $y^*\iota^*E(.i,.j)E(s,t)E(.k,.l)\iota x$ continuous in s,t — which follows from the lemma. Theorem 3.1 is proved.

Remark. Here is one way of including the finite dilation result in the continuous one. Given $2^m \times 2^m$ normalized positive (A_{jk}) and its dilating system of matrix units (E_{jk}), there exist a weakly continuous normalized increasing $A(\cdot\,,\,\cdot)$ and a weakly continuous standard $E(\cdot\,,\,\cdot)$ dilating it, such that $A(.j,\,.k)$ is obtained from the A_{jk} as in Proposition 3.4, and so is $E(.j,\,.k)$ from the E_{jk}. Indeed, we just use the idea of Example 3.2 to produce $E(\cdot\,,\,\cdot)$, and then obtain $A(\cdot\,,\,\cdot)$ by compression.

4. Several kinds of dilation

Let us give more particulars of the relationship between our theorems and those of Naĭmark and Stinespring.

First and simplest—Naĭmark's theorem treats the diagonal of our matrices, but more fully. Far from having to make separate, analogous statements about discrete and continuous spectral measures, it treats general ones. Values of the argument where the given generalized spectral measure is absolutely continuous are values where the dilation is also absolutely continuous; by contrast, our proof of continuity of $E(s,t)$ at a point involves, via (3.3), the hypothesis of continuity at distant parameter values.

We have already stated the basis for the partial equivalence between Stinespring's dilation and ours: the fact that a linear map on a full matrix algebra is completely positive if and only if the images A_{jk} of the natural matrix units form a positive operator-matrix. This makes Theorem 1.1, the result of our construction in its finite version, the same as Theorem 0.2, a version of a special case of Stinespring's theorem.

This interpretation can be extended to our Theorem 2.1. Let $\mathscr{A}$ be a UHF algebra, so that there is an increasing sequence of subalgebras isomorphic to full matrix algebras, whose union is dense in $\mathscr{A}$. For convenience, suppose that these matrix algebras are $2^m \times 2^m$, the modifications involved in the general case being minor. The co-ordinates can be chosen so that the natural matrix units e_{jk} follow the notational conventions of Section 1.2, and so that $e_{j0,k0}+e_{j1,k1} = e_{jk}$. Now given any linear map on $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$, it is clear that it will be completely positive if and only if the same is true of its restrictions to all the matrix algebras. The test to be applied to the images A_{jk} of the e_{jk} is just the property of our definition of countable normalized positive system. That is, our Theorem 2.1 is also an explicit proof for a case of Stinespring's dilation.

In the commutative case this can be pushed farther. If the algebra is $\mathscr{C}$, the continuous functions on the Cantor set, then positivity of a map into $\mathscr{B}(\mathscr{H})$ is just positivity of the images of the characteristic functions of a sufficient family of closed open sets. By the sort of explicit construction under discussion (essentially, by Section 2 of [6]) we can dilate a positive unital map to one in which these characteristic functions go to orthoprojectors. But this takes care of an arbitrary

separable commutative C^*-algebra $\mathscr{A}$; for it can be imbedded as an expectation image in $\mathscr{C}$, and then we can deduce the Stinespring conclusion by completing this diagram.

$$\begin{tikzcd}
& \mathscr{C} \arrow[dr] \arrow[rrr, dashed] & & & \mathscr{B}(\mathscr{K}) \arrow[d, dashed, "\text{compression}"] \\
\mathscr{A} \arrow[ur, hook] \arrow[rr, phantom, "="] & & \mathscr{A} \arrow[rr] & & \mathscr{B}(\mathscr{H})
\end{tikzcd}$$

The reason the interpretation falls short of covering the non-commutative case is that a non-commutative C^*-algebra may well fail to have any generating set, with the sort of binary structure required, which can play the role of the matrix units in UHF algebras or the characteristic functions in $\mathscr{C}$. For more general C^*-algebras — say, for $\mathscr{B}(\mathscr{H})$ — we do not know any criterion for complete positivity which would fit the Stinespring dilation into the pattern. In particular, we do not see that the theory of Section 3 admits such an interpretation.

References

[1] T. Ando, *Topics on operator inequalities*, Ryukyu University Lecture Note Series No. 1, Okinawa, 1978.

[2] W. B. Arveson, *Subalgebras of C*-algebras, I*, Acta Math. 123 (1969), 141–224.

[3] —, *Notes on extensions of C*-algebras*, Duke Math. J. 44 (1977), 329–355.

[4] M. D. Choi, *Positive linear maps on C*-algebras*, Doctoral Dissertation, University of Toronto, 1972.

[5] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. 24 (1977), 156–209.

[6] Ch. Davis, *Geometric approach to a dilation theorem*, Linear Alg. Appl. 18 (1977), 33–43.

[7] P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. 2 (1950), 125–134.

[8] W. F. Stinespring, *Positive functions on C*-algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216.

[9] B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Budapest and Paris, 1967.

Presented to the semester
Spectral Theory
September 23–December 16, 1977