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SEVERAL VARIABLE SPECTRAL THEORY IN THE NONCOMMUTATIVE CASE

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1. Introduction

Let $x_1, \ldots, x_N$ be N commuting elements of a complex Banach algebra $\mathfrak{R}$ with unit $1_{\mathfrak{R}}$. Let $\mathscr{A}$ be a complex algebra of functions (or of equivalence classes of functions) on a set $G \subset \boldsymbol{C}^N$ endowed with (the algebraic operations induced by) the pointwise operations of addition, multiplication, and multiplication by scalars. Suppose that $\mathscr{A}$ contains the algebra of all polynomials in N variables. An $\mathscr{A}$-*functional calculus* Ψ for $x = (x_1, \ldots, x_N)$ is then an algebraic homomorphism Ψ: $\mathscr{A} \to \mathfrak{R}$ with $\Psi(1_{\mathscr{A}}) = 1_{\mathfrak{R}}$ and $\Psi(Z_j) = x_j$ $(j = 1, \ldots, N)$, where Z_j: $\boldsymbol{C}^N \to \boldsymbol{C}$ are the coordinate functions $(z_1, \ldots, z_N) \mapsto z_j$ $(j = 1, \ldots, N)$.

In the case of noncommuting elements $x_1, \ldots, x_N \in \mathfrak{R}$ this definition is of course no longer possible and we have to modify the concept of functional calculus. There are two possibilities how to attack this problem:

(A) In order to obtain a *homomorphism* Ψ: $\mathscr{A} \to \mathfrak{R}$ one has to consider functions of several "*noncommuting*" variables. This has been done by N. H. McCoy [16] (for quasicommuting matrices) and (more general for noncommuting elements of a Banach algebra) by R. E. Harte [14], [15] and J. L. Taylor [19]–[21].

(B) If one prefers to work with usual functions in several "*commuting*" variables, one has to give up the homomorphism property of Ψ: $\mathscr{A} \to \mathfrak{R}$. Then, Ψ can only be a linear mapping, but it may still have some *symmetry properties*. Such functional calculi have been first introduced for the (unbounded) coordinate and momentum operators of quantum mechanics by H. Weyl ([24], Kap. IV, § 14) and earlier with some other symmetry conditions in the case, where $\mathscr{A}$ is the algebra of polynomials, by M. Born and P. Jordan (in [11], § 4). For self-adjoint operators on Hilbert and on Banach spaces the Weyl functional calculus has been investigated by M. E. Taylor [22] and R. F. V. Anderson [5]–[7]. In [8] a more general approach, using Laplace transform methods, has been given. In the general case of noncommuting elements of a Banach algebra, E. Nelson [17] has given a new concept by means of the *algebra of operants* (see 3.5 for the definition). E. Nelson showed that in the case of bounded self-adjoint operators this algebra of operants is a convenient tool for the description of the Weyl functional calculus. This concept has been further investigated by the author in [1], Kap. III. Especially a noncommutative version of the Šilov idempotent theorem has been obtained in this framework.

In this note we propose a generalization of the algebra of operants (cf. Def. 3.1) and generalize and improve the results of Kap. III in [1]. In the following section we introduce some algebraic notations and prove two algebraic lemmas which will be needed for the proof of the noncommutative Šilov idempotent theorem. In Section 3 we introduce the notion of an operating algebra and give some examples. In the fourth part we prove the noncommutative version of the Šilov idempotent theorem, and in the last section we study spectra, numerical ranges, and non-analytic functional calculi in operating algebras.

This note is a detailed and complete version of the second part of a series of lectures held by the author during the "Spectral Theory Semester" at the Stefan Banach International Mathematical Center. The first part of these lectures was concerned with the theory of non-analytic functional calculi and joint spectra of commuting operators on a Banach space. Part of this has been done in joint work with Şt. Frunză. For the details see [4] and [3].

The author wants to express his gratitude to Professor B. Gramsch for suggesting the topic of [1] and for valuable discussions. He also would like to thank the organizers of the "Spectral Theory Semester", especially Professor W. Żelazko, for creating such a stimulating atmosphere. Many thanks also to the participants of the "Spectral Theory Semester", especially to the Professors V. Pták and L. Waelbroeck for their interest, their suggestions, and discussions concerning the results presented here.

2. Algebraic preliminaries

If E is a vector space over $\boldsymbol{C}$, we denote by $S_0(E)$ the symmetric tensor algebra over E. Let $\{e_i\}_{i\in I}$ be a Hamel basis for E. It is well known that $S_0(E)$ may be identified with the polynomial algebra $\boldsymbol{C}[X_i\colon i\in I]$. Then, the canonical linear injection $\hat{}\colon E\to S_0(E)$ is given by

$$\hat{x} := \sum_{j=1}^{k} c_j X_{i_j} \quad \text{for} \quad x = \sum_{j=1}^{k} c_j e_{i_j} \in E.$$

Let now E be a linear subspace of a complex algebra $\mathfrak{R}$ with unit $1_{\mathfrak{R}}$. There is a unique linear mapping $T_0\colon S_0(E)\to\mathfrak{R}$ such that $T_0(1) = 1_{\mathfrak{R}}$ and

$$T_0(\hat{x}_1 \dots \hat{x}_n) = \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n} x_{\pi(1)}\dots x_{\pi(n)}$$

for all $n\in\boldsymbol{N}$, $x_1,\dots,x_n\in E$. Here, $\mathfrak{S}_n$ denotes the set of all permutations of $(1,\dots,n)$.

Let now $f\colon D_f\to\boldsymbol{C}$ be an arbitrary function defined on a set $D_f\subset\boldsymbol{C}^N$. For $r,s,t\in\boldsymbol{C}$ we introduce functions $I_r f$ and $J_{s,t}f$ by

$$(I_r f)(z) := f(rz) \quad \text{for } z\in\boldsymbol{C}^N \text{ with } rz\in D_f,$$

and

$$(J_{s,t}f)(z,w) := f(sz+tw) \quad \text{for } z,w\in\boldsymbol{C}^N \text{ with } sz+tw\in D_f.$$

Especially, this defines linear mappings

$$I_r\colon \boldsymbol{C}[Z_1,\dots,Z_N]\to\boldsymbol{C}[Z_1,\dots,Z_N]$$

and

$$J_{s,t}\colon \boldsymbol{C}[Z_1,\dots,Z_N]\to\boldsymbol{C}[Z_1,\dots,Z_N]\otimes\boldsymbol{C}[W_1,\dots,W_N],$$

where we identify the algebras $\boldsymbol{C}[Z_1,\dots,Z_N,W_1,\dots,W_N]$ and $\boldsymbol{C}[Z_1,\dots,Z_N]\otimes\otimes\boldsymbol{C}[W_1,\dots,W_N]$ in the natural way.

Let now $x_1,\dots,x_N$ be N elements of a complex algebra $\mathfrak{R}$ with unit $1_{\mathfrak{R}}$ and put $E := \mathrm{LH}\{x_1,\dots,x_N\}$ (= linear hull of $\{x_1,\dots,x_N\}$). Denote by $H\colon \boldsymbol{C}[Z_1,\dots\dots,Z_N]\to S_0(E)$ the canonical homomorphism given by $H(p) := p(\hat{x}_1,\dots,\hat{x}_N)$ for $p\in\boldsymbol{C}[Z_1,\dots,Z_N]$ and write

$$\Psi := T_0\circ H. \tag{2.0}$$

Finally, we define linear mappings $A_j\colon \mathfrak{R}\otimes\mathfrak{R}\to\mathfrak{R}$ $(j=1,2)$ by

$$A_1 u := \sum_{k=1}^{m} a_k b_k \quad \text{and} \quad A_2 u := \sum_{k=1}^{m} b_k a_k$$

for $u = \sum_{k=1}^{m} a_k\otimes b_k\in\mathfrak{R}\otimes\mathfrak{R}$. With these notations we have:

LEMMA 2.1. *For all $s,t\in\boldsymbol{C}$ and $j=1,2$ the diagram*

$$\begin{tikzcd}
\boldsymbol{C}[Z_1,\dots,Z_N] \arrow[rr, "J_{s,t}"] \arrow[d, "I_{s+t}"'] & & \boldsymbol{C}[Z_1,\dots,Z_N]\otimes\boldsymbol{C}[W_1,\dots,W_N] \arrow[d, "\Psi^{(Z)}\otimes\Psi^{(W)}"] \\
\boldsymbol{C}[Z_1,\dots,Z_N] \arrow[dr, "\Psi"'] & & \mathfrak{R}\otimes\mathfrak{R} \arrow[dl, "A_j"] \\
& \mathfrak{R} &
\end{tikzcd}$$

is commutative, i.e.

$$\Psi\circ I_{s+t} = A_j\circ(\Psi^{(Z)}\otimes\Psi^{(W)})\circ J_{s,t}, \tag{2.1}$$

where $\Psi^{(Z)}$ (resp. $\Psi^{(W)}$) means that Ψ has to be applied with respect to $Z_1,\dots,Z_N$ (resp. $W_1,\dots,W_N$).

Proof. It is sufficient to prove that (2.1) holds on the set of all monomials. Hence, let p be an arbitrary monomial,

$$p = \prod_{k=1}^{N} Z_k^{n_k} \quad \text{with } n_1,\dots,n_N\in\boldsymbol{N}_0.$$

Put $n := n_1 + \ldots + n_N$. If $n = 0$ then $p = 1$, and (2.1) applied to $p = 1$ is trivially fulfilled. Hence we may suppose that $n \geqslant 1$. Let $\nu\colon \{1, \ldots, n\} \to \{1, \ldots, N\}$ be a mapping such that $|\nu^{-1}(k)| = n_k$ for $k = 1, \ldots, N$. Then we may write p in the form

$$p = \prod_{k=1}^{n} Z_{\nu(k)}. \tag{2.2}$$

In the following, $\sum\limits_{|\Lambda|=r}'$ means that the sum is over all r-tuples $\Lambda = (\lambda_1, \ldots, \lambda_r) \in \boldsymbol{N}^r$ (if $r = 0$ then there is only the void tuple) such that $1 \leqslant \lambda_1 < \ldots < \lambda_r \leqslant n$ $(0 \leqslant r \leqslant n)$. If $\Lambda = (\lambda_1, \ldots, \lambda_r)$ then $\Lambda' = (\lambda'_1, \ldots, \lambda'_{n-r})$ denotes the $(n-r)$-tuple given by $\{\lambda_1, \ldots, \lambda_r\} \cup \{\lambda'_1, \ldots, \lambda'_{n-r}\} = \{1, \ldots, n\}$ and $1 \leqslant \lambda'_1 < \ldots < \lambda'_{n-r} \leqslant n$. With these notations we obtain by applying $J_{s,t}$ to equation (2.2):

$$J_{s,t}p = \prod_{k=1}^{n} (sZ_{\nu(k)} + tW_{\nu(k)}) = \sum_{r=0}^{n} \sum_{|\Lambda|=r}' \Big(\prod_{k=1}^{r} sZ_{\nu(\lambda_k)}\Big)\Big(\prod_{i=1}^{n-r} tW_{\nu(\lambda'_i)}\Big).$$

Hence,

$$\begin{aligned}
&(A_1 \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{s,t})(p)\\
&= \sum_{r=0}^{n} \sum_{|\Lambda|=r}' \Psi^{(Z)}\Big(\prod_{k=1}^{r} sZ_{\nu(\lambda_k)}\Big)\Psi^{(W)}\Big(\prod_{i=1}^{n-r} tW_{\nu(\lambda'_i)}\Big)\\
&= \sum_{r=0}^{n} \sum_{|\Lambda|=r}' \frac{s^r}{r!} \sum_{\sigma\in\mathfrak{S}_r} x_{\nu(\lambda_{\sigma(1)})} \ldots x_{\nu(\lambda_{\sigma(r)})} \frac{t^{n-r}}{(n-r)!} \sum_{\tau\in\mathfrak{S}_{n-r}} x_{\nu(\lambda'_{\tau(1)})} \ldots x_{\nu(\lambda'_{\tau(n-r)})}\\
&= \sum_{r=0}^{n} \sum_{|\Lambda|=r}' \binom{n}{r} s^r t^{n-r} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_r} \sum_{\tau\in\mathfrak{S}_{n-r}} x_{\nu(\lambda_{\sigma(1)})} \ldots x_{\nu(\lambda_{\sigma(r)})} x_{\nu(\lambda'_{\tau(1)})} \ldots x_{\nu(\lambda'_{\tau(n-r)})}\\
&= \sum_{r=0}^{n} \binom{n}{r} s^r t^{n-r} \frac{1}{n!} \sum_{|\Lambda|=r}' \sum_{\sigma\in\mathfrak{S}_r} \sum_{\tau\in\mathfrak{S}_{n-r}} x_{\nu(\pi_{\Lambda,\sigma,\tau}(1))} \ldots x_{\nu(\pi_{\Lambda,\sigma,\tau}(n))},
\end{aligned}$$

where $\pi_{\Lambda,\sigma,\tau} \in \mathfrak{S}_n$ is the permutation given by $\pi_{\Lambda,\sigma,\tau}(k) := \lambda_{\sigma(k)}$ for $k = 1, \ldots, r$ and $\pi_{\Lambda,\sigma,\tau}(r+i) := \lambda'_{\tau(i)}$ for $i = 1, \ldots, n-r$. If $\Lambda = (\lambda_1, \ldots, \lambda_r)$ and $M = (\mu_1, \ldots, \mu_r)$ are r-tuples with $1 \leqslant \lambda_1 < \ldots < \lambda_r \leqslant n$ and $1 \leqslant \mu_1 < \ldots < \mu_r \leqslant n$ and if permutations $\sigma, \varphi \in \mathfrak{S}_r$ and $\tau, \psi \in \mathfrak{S}_{n-r}$ are given, then obviously

$$\pi_{\Lambda,\sigma,\tau} = \pi_{M,\varphi,\psi} \quad \text{iff} \quad \Lambda = M,\ \sigma = \varphi, \text{ and } \tau = \psi. \tag{2.3}$$

The number of all r-tuples $\Lambda = (\lambda_1, \ldots, \lambda_r)$ with $1 \leqslant \lambda_1 < \ldots < \lambda_r \leqslant n$ is $\binom{n}{r}$. Thus,

$$S_r := \sum_{|\Lambda|=r}' \sum_{\sigma\in\mathfrak{S}_r} \sum_{\tau\in\mathfrak{S}_{n-r}} x_{\nu(\pi_{\Lambda,\sigma,\tau}(1))} \ldots x_{\nu(\pi_{\Lambda,\sigma,\tau}(n))}$$

is a sum over $\binom{n}{r} \cdot r! \cdot (n-r)! = n!$ summands. From this and (2.3) we conclude that

$$S_r = \sum_{\pi\in\mathfrak{S}_n} x_{\nu(\pi(1))} \ldots x_{\nu(\pi(n))}.$$

Thus,

$$\begin{aligned}
(A_1 \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{s,t})(p) &= \sum_{r=0}^{n} \binom{n}{r} s^r t^{n-r} \frac{1}{n!} S_r\\
&= \sum_{r=0}^{n} \binom{n}{r} s^r t^{n-r} \Psi\Big(\prod_{k=1}^{n} Z_{\nu(k)}\Big)\\
&= (s+t)^n \Psi\Big(\prod_{k=1}^{n} Z_{\nu(k)}\Big)\\
&= \Psi\Big(\prod_{k=1}^{n} (s+t) Z_{\nu(k)}\Big) = (\Psi \circ I_{s+t})(p).
\end{aligned}$$

Because of $A_1 \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{s,t} = A_2 \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{t,s}$ we also obtain

$$(A_2 \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{s,t})(p) = (\Psi \circ I_{s+t})(p)$$

for all $s, t \in \boldsymbol{C}$ and the proof is complete.

For $p \in \boldsymbol{C}[X_1, \ldots, X_n]$ $(n \in \boldsymbol{N})$ we define the linear mapping

$$M_p\colon \boldsymbol{C}[X_1, \ldots, X_n] \to \boldsymbol{C}[X_1, \ldots, X_n]$$

by

$$M_p q := pq \quad \text{for} \quad q \in \boldsymbol{C}[X_1, \ldots, X_n].$$

With this notation we can prove:

LEMMA 2.2. *If $s, t \in \boldsymbol{C}$ with $s+t \neq 0$ and $p \in \boldsymbol{C}[Z_1, \ldots, Z_N]$, then*

$$\Psi \circ M_p \circ I_{t+s} = A_j \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ M_{p_{s,t}} \circ J_{s,t} \qquad (j = 1, 2),$$

where $p_{s,t} \in \boldsymbol{C}[Z_1, \ldots, Z_N, W_1, \ldots, W_N] = \boldsymbol{C}[Z_1, \ldots, Z_N] \otimes \boldsymbol{C}[W_1, \ldots, W_N]$ is the polynomial $p_{s,t} := J_{\frac{s}{s+t}, \frac{t}{s+t}}(p)$.

Proof. Indeed, if $q \in \boldsymbol{C}[Z_1, \ldots, Z_N]$ we conclude by means of Lemma 2.1:

$$\begin{aligned}
(\Psi \circ M_p \circ I_{s+t})(q) &= (\Psi \circ I_{s+t})\big((I_{1/(s+t)}p)q\big)\\
&= (A_j \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ J_{s,t})\big((I_{1/(s+t)}p)q\big)\\
&= (A_j \circ (\Psi^{(Z)} \otimes \Psi^{(W)}) \circ M_{p_{s,t}} \circ J_{s,t})(q).
\end{aligned}$$

3. Operating algebras

In the following, let $\mathfrak{R}$ be a complex Banach algebra with unit $1_{\mathfrak{R}}$ and let E be a linear subspace of $\mathfrak{R}$. In order to have the possibility to apply usual commutative spectral theory we introduce the notion of an operating algebra with respect to E and $\mathfrak{R}$.

DEFINITION 3.1. A commutative complex Banach algebra $\mathfrak{A}$ with unit $1_{\mathfrak{A}}$ is called an *operating algebra with respect to E and $\mathfrak{R}$* if there exist a linear mapping $\tilde{\ }\colon E \to \mathfrak{A}$ and a continuous linear mapping $T\colon \mathfrak{A} \to \mathfrak{R}$ such that the following three conditions are satisfied:

(3.1) The subalgebra of $\mathfrak{A}$ generated by $1_{\mathfrak{A}}$ and the range of $\tilde{\ }$ is dense in $\mathfrak{A}$.

(3.2) $T(1_{\mathfrak{A}}) = 1_{\mathfrak{R}}$.

(3.3) If $x_1, \ldots, x_n \in E$, $n \in N$, then

$$T(\tilde{x}_1 \ldots \tilde{x}_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} x_{\pi(1)} \ldots x_{\pi(n)}.$$

$\mathfrak{A}$ is called a *faithfully operating algebra with respect to E and $\mathfrak{R}$* if in addition to (3.1)–(3.3) the following condition (3.4) is fulfilled:

(3.4) If $\alpha \in \mathfrak{A}$ such that $T(\alpha\beta) = 0$ for all $\beta \in \mathfrak{A}$ then $\alpha = 0$.

Remarks 3.2. (a) The mapping $\tilde{\ }$ in Definition 3.1 is injective and we have $T \circ \tilde{\ } = \mathrm{id}_E$.

(b) If $\tilde{\ }$ is continuous with $\|\tilde{\ }\| \leqslant 1$ and if $\|T\| \leqslant 1$ then $\tilde{\ }$ is a linear isometry.

(c) $\mathfrak{N} := \{\alpha \in \mathfrak{A}\colon T(\alpha\beta) = 0 \text{ for all } \beta \in \mathfrak{A}\}$ is a closed ideal in $\mathfrak{A}$. $\mathfrak{A}_0 := \mathfrak{A}/\mathfrak{N}$, endowed with the obvious mappings induced by $\tilde{\ }\colon E \to \mathfrak{A}$ and $T\colon \mathfrak{A} \to \mathfrak{R}$, is then a faithfully operating algebra with respect to E and $\mathfrak{R}$ (notice, that $\mathfrak{N} \subset \subset \ker(T)$).

(d) If $a \in E$ commutes with all $x \in E$, then $T(\tilde{a}\beta) = aT(\beta) = T(\beta)a$ for all $\beta \in \mathfrak{A}$.

(e) If $1_{\mathfrak{R}} \in E$ and if $\mathfrak{A}$ is a faithfully operating algebra with respect to E and $\mathfrak{R}$, then $\tilde{1}_{\mathfrak{R}} = 1_{\mathfrak{A}}$.

Proof. (a)–(c) are obvious. (d): By (a) and (3.2) $T(\tilde{a}1_{\mathfrak{A}}) = a = aT(1_{\mathfrak{A}}) = T(1_{\mathfrak{A}})a$. If $x_1, \ldots, x_n \in E$, then with $x_{n+1} := a$ we have

$$T(\tilde{x}_1 \ldots \tilde{x}_n\tilde{a}) = \frac{1}{(n+1)!} \sum_{\pi \in \mathfrak{S}_{n+1}} x_{\pi(1)} \ldots x_{\pi(n+1)}$$

$$= a\frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} x_{\pi(1)} \ldots x_{\pi(n)} = aT(\tilde{x}_1 \ldots \tilde{x}_n)$$

and also $T(\tilde{x}_1 \ldots \tilde{x}_n\tilde{a}) = T(\tilde{x}_1 \ldots \tilde{x}_n) \cdot a$. As the set of finite linear combinations of $1_{\mathfrak{A}}$ and finite products of elements in the range of $\tilde{\ }$ are dense in $\mathfrak{A}$ we obtain the statement of (d).

(e): By (d) we have $T\big((1_{\mathfrak{A}} - \tilde{1}_{\mathfrak{R}})\beta\big) = T(\beta) - 1_{\mathfrak{R}}T(\beta) = 0$ for all $\beta \in \mathfrak{A}$. Hence, $\tilde{1}_{\mathfrak{A}} = 1_{\mathfrak{R}}$ as $\mathfrak{A}$ is faithfully operating.

Let now $S_0(E)$ be the symmetric tensor algebra over E and let $\mathfrak{A}$ be an operating algebra with respect to E and $\mathfrak{R}$. By the universal property of the symmetric tensor algebra there exists a unital algebraic homomorphism $h_{\mathfrak{A}}\colon S_0(E) \to \mathfrak{A}$ such that the diagram

$$\begin{tikzcd}
 & S_0(E) \arrow[dr, "T_0"] \arrow[dd, "h_{\mathfrak{A}}"] & \\
E \arrow[ur, "\wedge"] \arrow[dr, "\sim"'] & & \mathfrak{R} \\
 & \mathfrak{A} \arrow[ur, "T"'] &
\end{tikzcd}$$

is commutative. By putting $p_{\mathfrak{A}}(\alpha) := \|h_{\mathfrak{A}}(\alpha)\|_{\mathfrak{A}}$ for $\alpha \in S_0(E)$ we obtain a submultiplicative seminorm on $S_0(E)$. $p_{\mathfrak{A}}$ induces an algebra norm on $S_{\mathfrak{A}} := S_0(E)/p_{\mathfrak{A}}^{-1}(0)$. Endowed with this norm, $S_{\mathfrak{A}}$ is isometrically isomorphic to the normed subalgebra of $\mathfrak{A}$ algebraically generated by $1_{\mathfrak{A}}$ and the range of $\tilde{\ }$. Hence, by (3.1), the completion of $S_{\mathfrak{A}}$ is isometrically isomorphic to $\mathfrak{A}$ and we have proved the first part of the following proposition. Part (b) is obvious.

PROPOSITION 3.3. (a) *Every operating algebra with respect to E and $\mathfrak{R}$ is isometrically isomorphic to the completion of a quotient algebra $S_{\mathfrak{A}}$ of $S_0(E)$ endowed with some algebra norm.*

(b) *Conversely, if p is a submultiplicative seminorm on $S_0(E)$ such that the mapping $T_0\colon S_0(E) \to \mathfrak{R}$ is p-continuous, then the completion of $S_0(E)/p^{-1}(0)$ with respect to the norm induced by p is in the natural way an operating algebra with respect to E and $\mathfrak{R}$.*

EXAMPLE 3.4. *The complete symmetric tensor algebra* (cf. [17]). We may endow $S_0(E)$ with the algebra norm $\|\cdot\|$ given by

$$\|\alpha\| := \inf\Big\{|a| + \sum_{j=1}^{n} \|x_{j,1}\|_{\mathfrak{R}} \ldots \|x_{j,m_j}\|_{\mathfrak{R}}\Big\}$$

where the infimum is taken over all representations of $\alpha \in S_0(E)$ of the form

$$\alpha = a + \sum_{j=1}^{n} \hat{x}_{j,1} \ldots \hat{x}_{j,m_j} \tag{3.5}$$

with $a \in C$, $n, m_1, \ldots, m_n \in N$, and $x_{j,k} \in E$ ($j = 1, \ldots, n$; $k = 1, \ldots, m_j$). Denote by $S(E)$ the completion of $S_0(E)$ with respect to this norm. Endowed with the mappings $\tilde{\ }\colon E \to S(E)$ and $T\colon S(E) \to \mathfrak{R}$ induced by $\hat{\ }\colon E \to S_0(E)$ and $T_0\colon S_0(E) \to \mathfrak{R}$, $S(E)$ is an operating algebra with respect to E and $\mathfrak{R}$ (notice, that T_0 is continuous with $\|T_0\| = 1$). We have $\|\tilde{\ }\| = 1 = \|T\|$. $S(E)$ is not very useful for obtaining "good" functional calculi. This may be illustrated by the following fact which is due to E. Nelson [17]: There is a homeomorphism H from the space Φ of all characters of $S(E)$ onto the unit ball $B(E^*)$ of the dual E^* of E (where Φ resp. $B(E^*)$ is endowed with the topology $\sigma(S(E)^*, S(E))$ resp. $\sigma(E^*, E)$) such that for all $\varphi \in \Phi$ we have

$$\varphi(\tilde{x}) = H(\varphi)(x) \quad \text{for all } x \in E.$$

Thus we have for example for the spectrum $\sigma_{S(E)}(\tilde{x})$ of $\tilde{x}$, where x is an arbitrary element of E:

$$\sigma_{S(E)}(\tilde{x}) = \{z \in \boldsymbol{C}\colon |z| \leqslant \|x\|_{\mathfrak{R}}\},$$

and the joint spectrum of $(\tilde{x}_1, \ldots, \tilde{x}_N)$ in $S(E)$ for arbitrary $x_1, \ldots, x_N \in E$ is the polydisc

$$\{z \in \boldsymbol{C}^N\colon |z_j| \leqslant \|x_j\|_{\mathfrak{R}} \text{ for } j = 1, \ldots, N\}.$$

EXAMPLE 3.5. *The algebra $\mathfrak{A}(E)$ of operants of E. Nelson* [17]. Instead of $S(E)$ we may consider the algebra $\mathfrak{A}(E) := S(E)/\mathfrak{N}(E)$ where $\mathfrak{N}(E) := \{\alpha \in S(E)\colon T(\alpha\beta) = 0 \text{ for all } \beta \in S(E)\}$. By Remark 3.2(c), $\mathfrak{A}(E)$ is a faithfully operating algebra with respect to E and $\mathfrak{R}$. Notice, that we still have $\|\tilde{\ }\| \leqslant 1$ and $\|T\| \leqslant 1$, so that $\tilde{\ }\colon E \to \mathfrak{A}(E)$ is a linear isometry (by 3.2(b)).

This algebra has been introduced and investigated by E. Nelson in [17]. There it has been shown that in the case where $\mathfrak{R}$ is the algebra $\mathfrak{L}(\mathfrak{H})$ of all continuous linear operators on a Hilbert space $\mathfrak{H}$ and where E is the linear span of $\mathrm{id}_{\mathfrak{H}}$ and a finite family $A_1, \ldots, A_N$ of self-adjoint operators in $\mathfrak{L}(\mathfrak{H})$, the algebra $\mathfrak{A}(E)$ of operants over E is the adequate tool to describe the Weyl functional calculus for $A = (A_1, \ldots, A_N)$ (see Section 5 for some details). However, there are natural examples where different constructions are more useful.

EXAMPLE 3.6. Put $\mathfrak{H} := L^2([0, 1])$ (with the Lebesgue measure) and let E be the one dimensional subspace of $\mathfrak{R} := \mathfrak{L}(\mathfrak{H})$ spanned by the Volterra integral operator V given by

$$(Vf)(t) := \int_0^t f(s)\,ds \quad (f \in \mathfrak{H}).$$

In this case the representations of elements $\alpha \in S_0(E)$ in the form (3.5) are unique. Thus,

$$\|p\| = \sum_{j=0}^{n} |c_j| \quad \text{if} \quad p = \sum_{j=0}^{n} c_j \hat{V}^j = \sum_{j=0}^{n} c_j X^j \in \boldsymbol{C}[X] = S_0(E).$$

Hence, $S(E)$ is isometrically isomorphic to the Banach algebra of all power series

$$F = \sum_{j=0}^{\infty} c_j X^j \quad \text{with} \quad \|F\| := \sum_{j=0}^{\infty} |c_j| < \infty.$$

The mapping $T\colon S(E) \to \mathfrak{R}$ is given by $T(F) := \sum_{j=0}^{\infty} c_j V^j$. If $T(F) = 0$ we have $T(F)1 = 0$ in $\mathfrak{H}$ and therefore,

$$\big(T(F)1\big)(t) = \sum_{j=0}^{\infty} c_j \frac{1}{j!} t^j = 0 \text{ (a.e.)}.$$

Since this function is analytic, this is only possible if $c_j = 0$ for all $j \in N_0$, i.e. if $F = 0$. Thus T is injective and we obtain in this case $\mathfrak{A}(E) = S(E)$. Hence (cf. 3.4):

$$\sigma_{\mathfrak{A}(E)}(\tilde{V}) = \sigma_{S(E)}(\tilde{V}) = \{z \in \boldsymbol{C}\colon |z| \leqslant 1\},$$

but $\sigma_{\mathfrak{R}}(V) = \{0\}$.

This example shows that in general one has to look either for a more interesting subspace E of $\mathfrak{R}$ (in the present case this would be for example a commutative subalgebra of $\mathfrak{R}$ containing V and $\mathrm{id}_{\mathfrak{H}}$) or for better operating algebras (in our example, the closed commutative subalgebra $\mathfrak{A}$ of $\mathfrak{R}$ generated by V and $\mathrm{id}_{\mathfrak{H}}$ would be much better, $\tilde{\ }\colon E \to \mathfrak{A}$ and $T\colon \mathfrak{A} \to \mathfrak{R}$ being the canonical inclusion mappings).

EXAMPLES 3.7. Let us indicate some good choices for E in the case that we want to consider N elements $x_1, \ldots, x_N$ of a given unital Banach algebra $\mathfrak{R}$.

(i) $E((x)) := \mathrm{LH}\{1_{\mathfrak{R}}, x_1, \ldots, x_N\}$.

(ii) $E((x)'') := \mathrm{LH}(\{x_1, \ldots, x_N\} \cup ((x)'' \cap (x)'))$, where $(x)'$ and $(x)''$ are the commutant and bicommutant algebra for $x_1, \ldots, x_N$.

(iii) $E(\mathfrak{B}) := \mathrm{LH}(\{x_1, \ldots, x_N\} \cup \mathfrak{B})$, where $\mathfrak{B}$ is a commutative subset of $(x)'$ with $1_{\mathfrak{R}} \in \mathfrak{B}$.

If $\mathfrak{B}$ is a maximal commutative subalgebra of $(x)'$ we obtain

$$E((x)) \subset E((x)'') \subset E(\mathfrak{B}).$$

Therefore,

$$S(E((x))) \subset S(E((x)'')) \subset S(E(\mathfrak{B})) \tag{3.6}$$

and

$$\mathfrak{N}(E((x))) \subset \mathfrak{N}(E((x)'')) \subset \mathfrak{N}(E(\mathfrak{B})), \tag{3.7}$$

where the inclusions in (3.7) follow by means of 3.2(d). By (3.6) and (3.7) we have continuous monomorphisms

$$\mathfrak{A}(E((x))) \to \mathfrak{A}(E((x)'')) \to \mathfrak{A}(E(\mathfrak{B})).$$

Thus, we obtain for the joint spectra of $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_N)$:

$$\sigma_{\mathfrak{A}(E(\mathfrak{B}))}(\tilde{x}) \subset \sigma_{\mathfrak{A}(E((x)''))}(\tilde{x}) \subset \sigma_{\mathfrak{A}(E((x)))}(\tilde{x}). \tag{3.8}$$

In the special case that $x_1, \ldots, x_N$ are mutually commuting we have $E(\mathfrak{B}) = \mathfrak{B}$ and $E((x)'') = (x)''$. Consequently, $\mathfrak{A}(E(\mathfrak{B}))$ is isometrically isomorphic to $\mathfrak{B}$ and $\mathfrak{A}(E((x)''))$ is isometrically isomorphic ~~to (x)~~$''$. Moreover, in this case $\sigma_{\mathfrak{A}(E((x)))}(\tilde{x})$

contains the joint spectrum of $x = (x_1, \dots, x_N)$ in the closed commutative subalgebra (x) of $\mathfrak{R}$ generated by $1_{\mathfrak{R}}$ and $x_1, \dots, x_N$. Since it is well known that the inclusions $\sigma_{\mathfrak{P}}(x) \subset \sigma_{(x)''}(x) \subset \sigma_{(x)}(x)$ may be all proper, we see that the inclusions in (3.8) may be proper too. Moreover, it follows that in the commutative case spectral analysis in "good" algebras of operants will be as good as that in commutative closed subalgebras of $\mathfrak{R}$ containing $x_1, \dots, x_N$.

Let us now construct a very natural operating algebra in the case of Volterra and multiplication operators. Denote by K^∞ the space of all (equivalence classes modulo zero functions of) Lebesgue measurable functions k: $[0,1]\times[0,1] \to C$ such that for a.e. $s \in [0,1]$ the function $k(s,\cdot)$ is in $L^\infty[0,1]$ and $s \to \|k(s,\cdot)\|_\infty$ is in $L^\infty[0,1]$ too. K^∞ is a normed linear space by means of

$$|||k|||_\infty := \operatorname*{ess\,sup}_{s\in[0,1]} \|k(s,\cdot)\|_\infty \qquad (k \in K^\infty).$$

For $k \in K^\infty$ let V_k be the Volterra integral operator on $L^p([0,1])$ $(1 \le p \le \infty)$ defined by

$$(V_k f)(s) := \int_0^s k(s,t) f(t)\,dt \qquad (f \in L^p([0,1])).$$

For $h \in L^\infty([0,1])$ we introduce the multiplication operator $M_h \in \mathfrak{L}(L^p([0,1]))$ by $(M_h f)(s) := h(s)f(s)$ $(f \in L^p([0,1]))$. Let now $E \subset \mathfrak{L}(L^p([0,1]))$ be the linear space

$$E := \{M_h + V_k\colon\ h \in L^\infty([0,1]),\ k \in K^\infty\}.$$

We define a submultiplicative seminorm ν on $S_0(E)$ by

$$\nu(\alpha) := \inf\Big\{|a| + \sum_{j=1}^{n}\Big(\prod_{i=1}^{n_j}\|h_{j,i}\|_\infty\Big)\Big(\prod_{r=1}^{m_j}|||k_{j,r}|||_\infty\Big)\big((m_j-1)!\big)^{-1}\Big\},$$

where the infimum is over all representations of $\alpha \in S_0(E)$ in the form

$$\alpha = a + \sum_{j=1}^{n}\Big(\prod_{i=1}^{n_j}\hat{M}_{h_{j,i}}\Big)\Big(\prod_{r=1}^{m_j}\hat{V}_{k_{j,r}}\Big) \tag{3.9}$$

with $n, n_j, m_j \in N_0$, $h_{j,i} \in L^\infty([0,1])$, $k_{j,r} \in K^\infty$. For the proof of the ν-continuity of T_0: $S_0(E) \to \mathfrak{L}(L^p([0,1]))$ we need two elementary lemmas.

Lemma 3.8. *If $k_1, \dots, k_n \in K^\infty$ then*

$$\|V_{k_n}\dots V_{k_1}\| \le \frac{1}{(n-1)!}\prod_{j=1}^{n}|||k_j|||_\infty.$$

Proof. If $g \in L^p([0,1])$ and $0 \le s \le 1$ then (with $s_{n+1} := s$)

$$(V_{k_n}\dots V_{k_1}g)(s) = \int_0^s\int_0^{s_n}\dots\int_0^{s_2} g(s_1)\prod_{j=1}^{n}k(s_{j+1}, s_j)\,ds_1\dots ds_{n-1}\,ds_n.$$

Hence,

$$|(V_{k_n}\dots V_{k_1}g)(s)| \le \Big(\prod_{j=1}^{n}|||k_j|||_\infty\Big)\int_0^s\int_0^{s_n}\dots\int_0^{s_2}|g(s_1)|\,ds_1\dots ds_{n-1}\,ds_n$$
$$\le \Big(\prod_{j=1}^{n}|||k_j|||_\infty\Big)\frac{s^{n-1}}{(n-1)!}\|g\|_p$$

which implies

$$\|V_{k_n}\dots V_{k_1}g\|_p \le \frac{1}{(n-1)!}\Big(\prod_{j=1}^{n}|||k_j|||_\infty\Big)\|g\|_p.$$

Lemma 3.9. *If $k_1, \dots, k_n \in K^\infty$, $h_1, \dots, h_m \in L^\infty([0,1])$ and if P is a product of the operators $V_{k_1}, \dots, V_{k_n}, M_{h_1}, \dots, M_{h_m}$ in an arbitrary order, then the following estimate holds*:

$$\|P\| \le \frac{1}{(n-1)!}\Big(\prod_{j=1}^{n}|||k_j|||_\infty\Big)\Big(\prod_{i=1}^{m}\|h_i\|_\infty\Big). \tag{3.10}$$

Proof. If $g \in L^p([0,1])$ we have a.e.

$$|(Pg)(t)| \le \Big(\prod_{i=1}^{m}\|h_i\|_\infty\Big)\big(V_{|k_{\pi(1)}|}\dots V_{|k_{\pi(n)}|}|g|\big)(t),$$

where π is a permutation of $(1, \dots, n)$. The estimate (3.10) now follows by Lemma 3.8.

Corollary 3.10. *The mapping T_0: $S_0(E) \to \mathfrak{L}(L^p([0,1]))$ is ν-continuous.*

Proof. For every $\alpha \in S_0(E)$ there is a representation of α in the form (3.9) such that for a given $\varepsilon > 0$

$$|a| + \sum_{j=1}^{n}\Big(\prod_{i=1}^{n_j}\|h_{j,i}\|_\infty\Big)\frac{1}{(m_j-1)!}\Big(\prod_{r=1}^{m_j}|||k_{j,r}|||_\infty\Big) \le \nu(\alpha) + \varepsilon.$$

By the preceding lemma

$$\|T_0\alpha\| \le |a| + \sum_{j=1}^{n}\Big(\prod_{i=1}^{n_j}\|h_{j,i}\|_\infty\Big)\frac{1}{(m_j-1)!}\Big(\prod_{r=1}^{m_j}|||k_{j,r}|||_\infty\Big) \le \nu(\alpha) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the ν-continuity of T_0 follows.

Thus T_0 induces a continuous linear mapping T_ν: $S_\nu \to \mathfrak{L}(L^p([0,1]))$ with $\|T_\nu\| \le 1$, where S_ν is the completion of $S_0(E)/\ker(\nu)$ with respect to the norm induced by ν. Put

$$\mathfrak{N} := \{\alpha \in S_\nu\colon\ T_\nu(\alpha\beta) = 0 \text{ for all } \beta \in S_\nu\}$$

and

$$\mathfrak{A} := S_\nu/\mathfrak{N}$$

endowed with the induced norm. Let $T\colon \mathfrak{A} \to \mathfrak{L}(L^p([0,1]))$ and $\tilde{}\colon E \to \mathfrak{A}$ be the canonical mappings induced by T_v and $\hat{}$. By our construction, $\mathfrak{A}$ is a faithfully operating algebra with respect to E and $\mathfrak{L}(L^p([0,1]))$. Obviously $\|T\| \leqslant 1$ and $\|\tilde{}\,|E_0\| \leqslant 1$ where $E_0 := \{M_h\colon h \in L^\infty([0,1])\}$. This shows that $\tilde{}\,|E_0$ is a linear isometry. For arbitrary $k \in K^\infty$ we have by Corollary 3.10:

$$\|\tilde{V}_k^n\| \leqslant \nu(\hat{V}_k^n) \leqslant \||k|\|_\infty^n \frac{1}{(n-1)!}$$

which shows that $\tilde{V}_k$ is a quasinilpotent element of $\mathfrak{A}$. We have proved:

PROPOSITION 3.11. *$\mathfrak{A}$ is a faithfully operating algebra with respect to $E = \{M_h + V_k\colon h \in L^\infty([0,1]),\ k \in K^\infty\}$ and $\mathfrak{L}(L^p([0,1]))$ $(1 \leqslant p \leqslant \infty)$ with the properties:*

(i) $\|T\| \leqslant 1$.

(ii) $\tilde{}\,|\{M_h\colon h \in L^\infty([0,1])\}$ *is a linear isometry.*

(iii) *For every $k \in K^\infty$, $\tilde{V}_k$ is quasinilpotent in $\mathfrak{A}$.*

4. The analytic functional calculus

First we have to introduce some notations. For an open set $G \subset C^N$ we denote by $H(G)$ the algebra of all locally analytic functions on G, endowed with the topology of uniform convergence on all compact subsets of G. For $g \in H(G)$ let $M_g\colon H(G) \to H(G)$ be the multiplication operator $f \to fg$. We shall use the fact that $H(G\times G)$ is topologically isomorphic to the complete projective tensor product $H(G)\hat{\otimes} H(G)$ and shall identify these spaces (cf. for example [23], Th. 51.6). If $K \subset C^N$ is compact we denote by $H(K)$ the algebra of all germs of complex valued functions, which are analytic in some neighbourhood of K, endowed with the usual inductive limit topology. If f is locally analytic in a neighbourhood of K we write $[f]_K$ or simply $[f]$ for the germ of f in $H(K)$. The mapping $p \to [p]$ gives us a canonical monomorphism from the algebra $C[Z_1, \ldots, Z_N]$ into $H(K)$. Thus we may consider $C[Z_1, \ldots, Z_N]$ as a subalgebra of $H(K)$. We shall use the fact that for polynomial convex compact sets $K \subset C^N$ the algebra $C[Z_1, \ldots, Z_N]$ is dense in $H(K)$ (cf. [12], Chap. I, § 4, n° 5, Prop. 3). If K is compact and $K = K_1 \cup K_2$ with $\emptyset \neq K_j = \overline{K_j}$ and $K_1 \cap K_2 = \emptyset$, and if $S\colon H(K) \to X$ is a linear mapping from $H(K)$ to some vector space X, we say that S *vanishes on* K_1 if $S([f]) = 0$ for all f which are locally analytic in a neighbourhood of K and vanish in a neighbourhood of K_2. For compact $K \subset C^N$ and open $G \subset C^N$, $r, s, t > 0$, with $rG \subset K$ and $sG + tG \subset K$ we may define $I_r\colon H(K) \to H(G)$ and $J_{s,t}\colon H(K) \to H(G\times G)$ by

$$(I_r[f])(z) := f(rz) \quad \text{and} \quad (J_{s,t}[f])(z, w) := f(sz + tw)$$

for $f \in [f] \in H(K)$ and $z, w \in G$. Obviously these mappings are well defined and continuous. Their restrictions to $C[Z_1, \ldots, Z_N]$ coincide with the corresponding mappings introduced in Section 2.

In this section $\mathfrak{R}$ is a complex unital Banach algebra. The mappings $A_j\colon \mathfrak{R}\otimes\mathfrak{R} \to \mathfrak{R}$ $(j = 1, 2)$, introduced in part 2, are continuous with respect to the projective tensor product topology on $\mathfrak{R}\otimes\mathfrak{R}$ and hence have continuous extensions (again denoted by A_j) $A_j\colon \mathfrak{R}\hat{\otimes}\mathfrak{R} \to \mathfrak{R}$ to the complete projective tensor product.

The following theorem is a first version of the noncommutative Šilov idempotent theorem.

THEOREM 4.1. *Let $x_1, \ldots, x_N$ be N elements of $\mathfrak{R}$ and let $K \subset C^N$ be a polynomial convex compact set such that there exists a continuous linear mapping Φ: $H(K) \to \mathfrak{R}$ with*

$$\Phi|C[Z_1, \ldots, Z_N] = \Psi, \tag{4.1}$$

where $\Psi\colon C[Z_1, \ldots, Z_N] \to \mathfrak{R}$ is the mapping introduced in (2.0). *Suppose that $K = K_1 \cup K_2$ with $\emptyset \neq K_j = \overline{K_j}$ $(j = 1, 2)$ and $K_1 \cap K_2 = \emptyset$. Let h be a function such that $h \equiv 1$ in a neighbourhood of K_1 and $h \equiv 0$ in a neighbourhood of K_2. We put $m := \Phi([h])$. Then:*

(a) *$m^2 = m$.*

(b) *$\Phi([f][h]) = \Phi([f])m = m\Phi([f])$ for all $[f] \in H(K)$.*

Especially (because of (4.1)),

$$x_j m = m x_j = \Phi([Z_j][h]) \quad \textit{for} \quad j = 1, \ldots, N.$$

(c) *If Φ does not vanish on K_1 then $m \neq 0$.*

(d) *If Φ does not vanish on K_2 then $m \neq 1_{\mathfrak{R}}$.*

Proof. Let $U_1, U_2 \subset C^N$ be open such that $K_j \subset U_j$ $(j = 1, 2)$ and $U_1 \cap U_2 = \emptyset$. As K has a neighbourhood base of polynomial convex compact sets (cf. [12], Chap. I, Appendice, Lemme 2), there are a polynomial convex compact set $W \subset C^N$ and an open set $V \subset C^N$ with $K \subset V \subset \overline{V} \subset \mathring{W} \subset W \subset U_1 \cup U_2$. Without loss of generality we may suppose that $0 \in K_1$. If not, then fix $z \in K_1$ and consider $x_j - z_j 1_{\mathfrak{R}}$ instead of x_j $(j = 1, \ldots, N)$, $K - z$ instead of K, and $\Phi_0\colon H(K-z) \to \mathfrak{R}$ with $\Phi_0([f]) := \Phi([f(\cdot - z)])$ for $f \in [f] \in H(K-z)$ instead of Φ. We put now

$$D := \sup_{w\in W}(1+|w|) \quad \text{and} \quad d := \frac{1}{3D}\operatorname{dist}(V, C^N \setminus W).$$

Let now s and t be positive real numbers with $s + t \leqslant d$ and $t \leqslant (s+t)d$. Consider the bilinear mappings

$$B_{s,t}, C_{s,t}\colon H(W)\times H(dW) \to \mathfrak{R}$$

defined by

$$B_{s,t}([g], [f]) := (\Phi_V M_g I_{s+t})([f]) \tag{4.2}$$

and

$$C_{s,t}([g],[f]) := \left(A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ M_{J_{\frac{s}{s+t},\frac{t}{s+t}}[g]} \circ J_{s,t}\right)([f]) \tag{4.3}$$

for $g \in [g] \in H(W)$ and $[f] \in H(dW)$, where $\Phi_V\colon H(V) \to \mathfrak{R}$ is the mapping induced by Φ. By our choice of s and t, the mappings $B_{s,t}$ and $C_{s,t}$ are obviously well defined and continuous. By (4.1) and Lemma 2.2 we obtain $B_{s,t}([p],[q]) = C_{s,t}([p],[q])$ for all polynomials $p, q \in \boldsymbol{C}[Z_1, \ldots, Z_N]$. As W and dW are polynomial convex compact sets, the set of germs of polynomials is dense in $H(W)$ and in $H(dW)$. Using the continuity of $B_{s,t}$ and $C_{s,t}$ we conclude that

$$B_{s,t} = C_{s,t} \quad \text{on } H(W) \times H(dW) \tag{4.4}$$

for all $s, t > 0$ with $s+t \leqslant d$ and $t \leqslant (s+t)d$. Let now $g\colon U_1 \cup U_2 \to \boldsymbol{C}$ be the function defined by $g|U_1 \equiv 1$ and $g|U_2 \equiv 0$. Then $[g] = [h]$ in $H(K)$. Notice, that by our choice of s and t

$$(J_{\frac{s}{s+t},\frac{t}{s+t}}[g])(z,w) = g\left(\frac{sz+tw}{s+t}\right) = g\left(z + \frac{t}{s+t}(w-z)\right) = g(z)$$

for all $z, w \in V$. Thus,

$$M_{J_{\frac{s}{s+t},\frac{t}{s+t}}[g]} = M_g^{(z)} \otimes \mathrm{id}_{H(V)}. \tag{4.5}$$

By (4.2)–(4.5) we obtain

$$\Phi_V \circ M_g \circ I_{s+t} = A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}_{H(V)}) \circ J_{s,t} \quad \text{for } s, t > 0,\ s+t \leqslant d,\ t \leqslant (s+t)d. \tag{4.6}$$

In our next step we show that this equation holds for all $s \in (0, d/2]$ and $t \in (0, \delta]$ with $\delta := \min\{d^2, d/2\}$, so that the choice of t becomes independent of s. We proceed by induction. Fix $s \in (0, d/2]$. It is sufficient to show that the equation in (4.6) is true for all $n \in \boldsymbol{N}$ and $t \in (0, nsd] \cap (0, \delta]$. For $n = 1$ this follows by (4.6). Suppose now that for some $n \in \boldsymbol{N}$

$$\Phi_V \circ M_g \circ I_{s+t} = A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}_{H(V)}) \circ J \quad \text{for all } t \in (0, nsd] \cap (0, \delta]. \tag{4.6$_n$}$$

If $(0, \delta] \subset (0, nsd]$ then (4.6_n) implies (4.6_{n+1}). If $t \in (nsd, (n+1)sd] \cap (0, \delta]$ then $t = u+v$ with $u = nsd$ and $v = t-u$. As $(s+u)+v = s+t \leqslant d$ and $v = t - nsd \leqslant sd \leqslant ((s+u)+v)d$, we may apply (4.6) and obtain for an arbitrary polynomial $p \in \boldsymbol{C}[Z_1, \ldots, Z_N]$

$$\begin{aligned}
(\Phi_V \circ M_g \circ I_{s+t})[p] &= (\Phi_v \circ M_g \circ I_{(s+u)+v})[p] \\
&= \left(A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}) \circ J_{s+u,v}\right)[p] \\
&= \left(A_j \circ ((\Phi_V^{(z)} \circ M_g^{(z)} \circ I_{s+u}^{(z)}) \otimes \Phi_V^{(w)}) \circ J_{1,v}\right)[p] \\
&= \left(A_j \circ \left(\left(A_j \circ (\Phi_V^{(x)} \otimes \Phi_V^{(y)}) \circ (M_g^{(x)} \otimes \mathrm{id}) \circ J_{s,u}^{(z)}\right) \otimes \Phi_V^{(w)}\right) \circ J_{1,v}\right)[p],
\end{aligned}$$

where the last equality follows by (4.6_n). Using the facts that

$$A_j \circ (A_j \otimes \mathrm{id}_{\mathfrak{R}}) = A_j \circ (\mathrm{id}_{\mathfrak{R}} \otimes A_j) \quad \text{on} \quad \mathfrak{R} \otimes \mathfrak{R} \otimes \mathfrak{R}$$

and that

$$(J_{s,u} \otimes \mathrm{id}) \circ J_{1,v} = (\mathrm{id} \otimes J_{u,v}) \circ J_{s,1} \quad \text{on} \quad \boldsymbol{C}[Z_1, \ldots, Z_N]$$

we conclude with (4.1) and Lemma 2.1:

$$\begin{aligned}
&(\Phi_V \circ M_g \circ I_{s+t})[p] \\
&\quad = \left(A_j \circ (\mathrm{id}_{\mathfrak{R}} \otimes A_j) \circ ((\Phi_V^{(x)} \circ M_g^{(x)}) \otimes (\Phi_V^{(y)} \otimes \Phi_V^{(w)})) \circ (\mathrm{id} \otimes J_{u,v}) \circ J_{s,1}\right)[p] \\
&\quad = \left(A_j \circ ((\Phi_V^{(\tilde z)} \circ M_g^{(\tilde z)}) \otimes (\Phi_V^{(\tilde w)} \circ I_{u+v}^{(\tilde w)})) \circ J_{s,1}\right)[p] \\
&\quad = \left(A_j \circ (\Phi^{(\tilde z)} \otimes \Phi^{(\tilde w)}) \circ (M_g^{(\tilde z)} \otimes \mathrm{id}_{H(V)}) \circ J_{s,t}\right)[p],
\end{aligned}$$

where we introduced new names for the variables. As the polynomials are dense in $H(dW)$ we obtain by the continuity of the mappings

$$\Phi_V \circ M_g \circ I_{s+t}\colon H(dW) \to \mathfrak{R} \quad \text{and} \quad A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}_{H(V)}) \circ J_{s,t}$$

on $H(dW)$ that these mappings coincide on $H(dW)$. Thus, we proved (4.6_{n+1}) and we conclude that

$$\Phi_V \circ M_g \circ I_{s+t} = A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}_{H(V)}) \circ J_{s,t} \quad \text{on} \quad H(dW) \quad \text{for all } s \in (0, d/2] \text{ and } t \in (0, \delta],\ j = 1, 2. \tag{4.7}$$

Let now p be an arbitrary polynomial in $\boldsymbol{C}[Z_1, \ldots, Z_N]$ and fix $t \in (0, \delta]$. Then

$$I_{s+t}p|V \to I_t p|V \quad \text{in } H(V) \quad \text{for } s \to 0$$

and

$$J_{s,t}p|V \times V \to 1 \otimes (I_t p|V) \quad \text{in } H(V \times V) = H(V) \tilde{\otimes} H(V) \quad \text{for } s \to 0.$$

Hence, by (4.7),

$$\begin{aligned}
\Phi_V(g \cdot (I_t p)) &= \lim_{s \to 0} (\Phi_V \circ M_g \circ I_{s+t})[p] \\
&= \lim_{s \to 0} \left(A_j \circ (\Phi_V^{(z)} \otimes \Phi_V^{(w)}) \circ (M_g^{(z)} \otimes \mathrm{id}) \circ J_{s,t}\right)[p] \\
&= \left(A_j \circ ((\Phi_V^{(z)} \circ M_g^{(z)}) \otimes \Phi_V^{(w)})\right)(1 \otimes (I_t p|V)) \\
&= A_j\left(\Phi_V(g \cdot 1) \otimes \Phi_V(I_t p|V)\right).
\end{aligned}$$

Thus, by the definition of A_1 and A_2,

$$\Phi_V(g \cdot (I_t p)) = \Phi_V(g)\Phi_V(I_t p) = \Phi_V(I_t p)\Phi_V(g).$$

As $I_t\colon \boldsymbol{C}[Z_1, \ldots, Z_N] \to \boldsymbol{C}[Z_1, \ldots, Z_N]$ is an isomorphism for $t \neq 0$, we obtain for every polynomial $q \in \boldsymbol{C}[Z_1, \ldots, Z_N]$

$$\Phi_V(gq) = \Phi_V(g)\Phi_V(q) = \Phi_V(q)\Phi_V(g),$$

and thus (using $[h] = [g]$ in $H(K)$),

$$\Phi([h][q]) = m\Phi([q]) = \Phi([q])m.$$

The germs of polynomials being dense in $H(K)$ we conclude

$$\Phi([h][f]) = m\Phi([f]) = \Phi([f])m \quad \text{for every } [f] \in H(K) \tag{4.8}$$

and (b) is proved. Taking $[f] := [h]$ and observing $[h]^2 = [h]$ we obtain (a).

If Φ does not vanish on K_1 then there is a function f which is locally analytic in some neighbourhood of K and vanishes in a neighbourhood of K_2 such that $\Phi([f]) \neq 0$. Now $[f] = [h][f]$ in $H(K)$ so that by (4.8)

$$0 \neq \Phi([f]) = \Phi([h][f]) = m\Phi([f])$$

which implies $m \neq 0$ and thus (c).

If Φ does not vanish on K_2 then there is a function f which is locally analytic in some neighbourhood of K and vanishes in a neighbourhood of K_1 such that $\Phi([f]) \neq 0$. Thus $[h][f] = 0$ in $H(K)$ and we obtain from (4.8)

$$0 \neq \Phi([f]) = \Phi([f]) - \Phi([h][f]) = \Phi([f])(1_{\mathfrak{R}} - m)$$

which implies $m \neq 1_{\mathfrak{R}}$ and therefore (d).

We are now able to prove

THEOREM 4.2. *Let E be a linear subspace of $\mathfrak{R}$ and let $\mathfrak{A}$ be a faithfully operating algebra with respect to E and $\mathfrak{R}$. If there are $x_1, \ldots, x_N \in E$ such that $\sigma_{\mathfrak{A}}(\tilde{x}_1, \ldots, \tilde{x}_N)$ is not connected then there exists a nontrivial idempotent element m in $\mathfrak{R}$ which commutes with every element of E.*

Proof. We have $\sigma_{\mathfrak{A}}(\tilde{x}_1, \ldots, \tilde{x}_N) = S_1 \cup S_2$ with $\emptyset \neq S_j = \overline{S}_j$ $(j = 1, 2)$ and $S_1 \cap S_2 = \emptyset$. Let $[h_0]$ be the germ of a function h_0 with $h_0 \equiv 1$ in a neighbourhood of S_1 and $h_0 \equiv 0$ in a neighbourhood of S_2. Put $m := T(\Theta_{\tilde{x}}([h_0]))$, where $\Theta_{\tilde{x}}$: $H(\sigma_{\mathfrak{A}}(\tilde{x})) \to \mathfrak{A}$ is the analytic functional calculus for $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_N)$ in the commutative Banach algebra $\mathfrak{A}$ in the sense of Theorem 1 in Chapter I, § 4, n° 1, of [12]. Let y be an arbitrary element of E and let U_j be open neighbourhoods of S_j $(j = 1, 2)$ such that $U_1 \cap U_2 = \emptyset$. By the proof of Lemma 10 in [12], Chapter I, § 4, n° 6, and by the fact that the algebra generated by $1_{\mathfrak{A}}$ and the range of $\tilde{}$ is dense in $\mathfrak{A}$, there exist elements $y_1, \ldots, y_k \in E$ such that we have for the polynomial convex hull K of $\sigma_{\mathfrak{A}}(\tilde{x}_1, \ldots, \tilde{x}_N, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_k)$ $(\subset \boldsymbol{C}^N \times \boldsymbol{C}^{k+1})$

$$\sigma_{\mathfrak{A}}(\tilde{x}) \subset \pi_N(K) \subset U_1 \cup U_2,$$

where π_N: $\boldsymbol{C}^N \times \boldsymbol{C}^{k+1} \to \boldsymbol{C}^N$ denotes the canonical projection. Hence, K is not connected, $K = K_1 \cup K_2$ with $K_j := \pi_N^{-1}(U_j) \cap K \neq \emptyset$ and $K_1 \cap K_2 = \emptyset$. Put $\tilde{u} := (\tilde{x}_1, \ldots, \tilde{x}_N, \tilde{y}, y_1, \ldots, \tilde{y}_k)$ and let h be a function with $h \equiv 1$ in a neighbourhood of K_1 and $h \equiv 0$ in a neighbourhood of K_2. Then we obtain by Theorem 1 (ii) in [12], Chapter I, § 4, n° 1,

$$m = T(\Theta_{\tilde{x}}([h_0])) = T(\Theta_{\tilde{u}}([h])). \tag{4.9}$$

We define Φ: $H(K) \to \mathfrak{R}$ by $\Phi([f]) := T(\Theta^K([f]))$ for $[f] \in H(K)$, where Θ^K: $H(K) \to \mathfrak{A}$ is the continuous homomorphism induced by $\Theta_{\tilde{u}}$: $H(\sigma_{\mathfrak{A}}(\tilde{u})) \to \mathfrak{A}$. By the properties of T and $\Theta_{\tilde{u}}$ we see that (4.1) in Theorem 4.1 is fulfilled. Hence, we obtain by Theorem 4.1 and (4.9)

$$m^2 = m, \quad mx_j = x_j m \ (j = 1, \ldots, N),$$

and

$$T(\Theta_{\tilde{u}}([h])\tilde{y}) = my = ym. \tag{4.10}$$

Now, y was an arbitrary element of E, so that (4.10) is valid for all $y \in E$. Therefore, one concludes easily that

$$T(\Theta_{\tilde{u}}([h])\alpha) = mT(\alpha) = T(\alpha)m \tag{4.11}$$

for all α in the algebra $\mathfrak{A}_0$ generated by $1_{\mathfrak{A}}$ and the range of $\tilde{}$. As $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, (4.11) holds for all $\alpha \in \mathfrak{A}$.

Assume now that $m = 0$. Then, by (4.11) $T(\Theta_{\tilde{u}}([h])\alpha) = 0$ for all $\alpha \in \mathfrak{A}$. As $\mathfrak{A}$ is a faithfully operating algebra, we must have $\Theta_{\tilde{u}}([h]) = 0$ in contradiction to the fact that $\sigma_{\mathfrak{A}}(\Theta_{\tilde{u}}([h])) = \{0, 1\}$ by the spectral mapping theorem. Hence, $m \neq 0$.

In the same way we show that $m \neq 1_{\mathfrak{R}}$: If $m = 1_{\mathfrak{R}}$ then we would have (by (4.11)) $T((\Theta_{\tilde{u}}([h]) - 1_{\mathfrak{A}})\alpha) = 0$ for all $\alpha \in \mathfrak{A}$, thus $\Theta_{\tilde{u}}([h]) = 1_{\mathfrak{A}}$, in contradiction to $\sigma_{\mathfrak{A}}(\Theta_{\tilde{u}}([h])) = \{0, 1\}$.

COROLLARY 4.3. *Let E be a linear subspace of $\mathfrak{R}$ and let $\mathfrak{A}$ be a faithfully operating algebra with respect to E and $\mathfrak{R}$. If the maximal ideal space $\mathfrak{M}(\mathfrak{A})$ of $\mathfrak{A}$ is not connected, then there exists a nontrivial idempotent element m in $\mathfrak{R}$ which commutes with all elements of E.*

Proof. As the algebra generated by $1_{\mathfrak{A}}$ and the range of $\tilde{}$ is dense in $\mathfrak{A}$, $\mathfrak{M}(\mathfrak{A})$ is homeomorphic to $\sigma_{\mathfrak{A}}((\tilde{a})_{a \in E})$ $(\subset \boldsymbol{C}^E)$ (cf. [12], Chap. I, § 3, n° 5, Prop. 9). Then there exist $x_1, \ldots, x_N \in E$ such that $\sigma_{\mathfrak{A}}(\tilde{x}_1, \ldots, \tilde{x}_N) = \pi_N(\sigma_{\mathfrak{A}}((\tilde{a})_{a \in E}))$ is not connected in $\boldsymbol{C}^N$, where π_N: $\boldsymbol{C}^E \to \boldsymbol{C}^N$ is the canonical projection. Hence, we may apply Theorem 4.2 and the proof is complete.

In the converse direction we have:

PROPOSITION 4.4. *Let $x_1, \ldots, x_N$ be N elements of $\mathfrak{R}$. If there exists a nontrivial idempotent element m in $\mathfrak{R}$ which commutes with $x_1, \ldots, x_N$, then the maximal ideal space $\mathfrak{M}(\mathfrak{A}(E))$ of the Nelson algebra of operants over $E := \mathrm{LH}\{x_1, \ldots, x_N, 1_{\mathfrak{R}}, m\}$ is not connected.*

Proof. Obviously, we have $\tilde{m}^2 = \tilde{m}$ in $\mathfrak{A}(E)$ and $0 \neq \tilde{m} \neq 1_{\mathfrak{A}}$ (as m is nontrivial in $\mathfrak{R}$). Hence, $\sigma_{\mathfrak{A}}(\tilde{m}) = \{0, 1\}$ which implies that $\mathfrak{M}(\mathfrak{A}(E))$ is not connected.

5. Spectra, numerical ranges, and nonanalytic functional calculi in operating algebras

In the sequel let E be a linear subspace of a Banach algebra $\mathfrak{R}$ with unit $1_{\mathfrak{R}}$ and let $\mathfrak{A}$ be an operating algebra with respect to E and $\mathfrak{R}$. In general, it is difficult to compute $\sigma_{\mathfrak{A}}(\tilde{x}_1, \ldots, \tilde{x}_N)$ for given elements $x_1, \ldots, x_N \in E$. However, we shall obtain some estimates by means of numerical range techniques.

If $x = (x_1, \ldots, x_N) \in E^N$, we shall write (x) (resp. $(\tilde{x})$) for the closed subalgebra of $\mathfrak{R}$ (resp. $\mathfrak{A}$) generated by $1_{\mathfrak{R}}, x_1, \ldots, x_N$ (resp. $1_{\mathfrak{A}}, \tilde{x}_1, \ldots, \tilde{x}_N$). If the elements $x_1, \ldots, x_N$ are mutually commuting, then $T|(\tilde{x})\colon (\tilde{x}) \to (x)$ is obviously an algebraic homomorphism. Thus, we have

LEMMA 5.1. *If $x = (x_1, \ldots, x_N) \in E^N$ is a commuting N-tuple, then $\sigma_{(x)}(x) \subset \sigma_{(\tilde{x})}(\tilde{x})$.*

COROLLARY 5.2. *If $x \in E$, then $\sigma_{(x)}(x) \subset \sigma_{(\tilde{x})}(\tilde{x})$.*

Recall that the *joint (algebra) numerical range $V(\mathfrak{R}, x)$ of* $x = (x_1, \ldots, x_N) \in \mathfrak{R}^N$ may be defined as

$$V(\mathfrak{R}, x) := \{(f(x_1), \ldots, f(x_N))\colon f \in D(\mathfrak{R}, 1)\},$$

where

$$D(\mathfrak{R}, 1) := \{f \in \mathfrak{R}^*\colon f(1_{\mathfrak{R}}) = 1 = \|f\|\}.$$

For the theory of numerical ranges we refer to [9] and [10].

LEMMA 5.3. *Let $\mathfrak{D}$ be a Banach algebra with unit $1_{\mathfrak{D}}$, $x = (x_1, \ldots, x_N) \in \mathfrak{R}^N$ and let $\varphi\colon E_0 := \mathrm{LH}\{1_{\mathfrak{R}}, x_1, \ldots, x_N\} \to \mathfrak{D}$ be a linear mapping with $\|\varphi\| = 1$ and $\varphi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$. Then we have (with $\varphi(x) := (\varphi(x_1), \ldots, \varphi(x_N))$):*

$$V(\mathfrak{D}, \varphi(x)) \subset V(\mathfrak{R}, x). \tag{5.1}$$

Moreover, if φ is an isometry then

$$V(\mathfrak{D}, \varphi(x)) = V(\mathfrak{R}, x). \tag{5.2}$$

Proof. If $f \in D(\mathfrak{D}, 1)$ then $f \circ \varphi \in E_0^*$ with

$$(f \circ \varphi)(1_{\mathfrak{R}}) = 1 = \|f \circ \varphi\|.$$

By the Hahn–Banach theorem there exists $g \in D(\mathfrak{R}, 1)$ such that $g|E_0 = f \circ \varphi$. This implies (5.1).

As an immediate consequence we obtain the well known

COROLLARY 5.4. *Let $\mathfrak{D}$ be a subalgebra of $\mathfrak{R}$ endowed with the norm of $\mathfrak{R}$ and containing $1_{\mathfrak{R}}, x_1, \ldots, x_N$. Then*

$$V(\mathfrak{D}, (x_1, \ldots, x_N)) = V(\mathfrak{R}, (x_1, \ldots, x_N)).$$

This allows us to write $V(x)$ instead of $V(\mathfrak{R}, x)$ for $x = (x_1, \ldots, x_N) \in \mathfrak{R}^N$

COROLLARY 5.5. *Suppose that $\mathfrak{R}$ is a unital C^*-algebra and let $x = (x_1, \ldots, x_N)$ be a N-tuple of mutually commuting normal elements $x_1, \ldots, x_N \in \mathfrak{R}$. Let $\mathfrak{D}$ be a unital Banach algebra and let $\varphi\colon E_0 := \mathrm{LH}\{1_{\mathfrak{R}}, x_1, \ldots, x_N\} \to \mathfrak{D}$ be a linear mapping with $\varphi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$ and $\|\varphi\| = 1$. Then*

$$\operatorname{co}\sigma(\varphi(x)) \subset V(\varphi(x)) \subset V(x) = \operatorname{co}\sigma_{\mathfrak{B}}(x) = \operatorname{co}\sigma(x),$$

where $\sigma(\varphi(x))$ (resp. $\sigma(x)$) denotes the intersection of the left and the right spectrum of $\varphi(x) := (\varphi(x_1), \ldots, \varphi(x_N))$ (resp. x) in $\mathfrak{D}$ (resp. $\mathfrak{R}$) and $\mathfrak{B}$ is any commutative C^*-subalgebra of $\mathfrak{R}$ containing $x_1, \ldots, x_N, 1_{\mathfrak{R}}$.

Proof. The first inclusion follows by the convexity of $V(\varphi(x))$ and Theorem 12 in [9], p. 24. The second inclusion is a consequence of Lemma 5.3 and $\sigma_{\mathfrak{B}}(x) = \sigma(x)$ by Theorem 6 in [4]. Moreover, $V(x) = V(\mathfrak{B}, x)$ by Corollary 5.4. By the fact that the norm $\|\cdot\|$ of $\mathfrak{B}$ coincides with the spectral radius and is thus minimal in the set of all algebra norms p on $\mathfrak{B}$ which are equivalent to $\|\cdot\|$ and satisfy $p(1_{\mathfrak{R}}) = 1$, we conclude $V(x) = \operatorname{co}\sigma_{\mathfrak{B}}(x)$ by means of Theorem 13 in [9], p. 24.

As a first application we obtain a multi-variable variant of a result of Brown and Halmos (cf. [13], p. 182) for Toeplitz operators.

COROLLARY 5.6. *Let $\varphi_1, \ldots, \varphi_N$ be in $L^\infty(\Pi)$ and let $T_{\varphi_1}, \ldots, T_{\varphi_N}$ be the corresponding Toeplitz operators. Then*

$$\operatorname{co}\sigma(T_{\varphi_1}, \ldots, T_{\varphi_N}) \subset \operatorname{co}\sigma_{L^\infty(\Pi)}(\varphi_1, \ldots, \varphi_N).$$

THEOREM 5.7. *Let E be a linear subspace of $\mathfrak{R}$ with $1_{\mathfrak{R}} \in E$, let $\mathfrak{A}$ be an operating algebra with respect to E and $\mathfrak{R}$.*

(a) *If $x = (x_1, \ldots, x_N) \in E^N$, $E_0 := \mathrm{LH}\{1_{\mathfrak{R}}, x_1, \ldots, x_N\}$, and $\|{}^\sim|E_0\| = 1$*

$$\operatorname{co}\sigma_{\mathfrak{A}}(\tilde{x}) \subset V(\tilde{x}) \subset V(x).$$

Moreover, if ${}^\sim|E_0$ is an isometry then $V(\tilde{x}) = V(x)$.

(b) *If $\mathfrak{R}$ is a C^*-algebra and if $x_1, \ldots, x_N \in \mathfrak{R}$ are mutually commuting and normal then*

$$\operatorname{co}\sigma_{(\tilde{x})}(\tilde{x}) = V(\tilde{x}) = V(x) = \operatorname{co}\sigma(x) = \operatorname{co}\sigma_{\mathfrak{B}}(x),$$

where $\mathfrak{B}$ is an arbitrary commutative C^*-subalgebra of $\mathfrak{R}$ containing $1_{\mathfrak{R}}, x_1, \ldots, x_N$.

Proof. (a) follows by Lemma 5.3, the convexity of $V(\tilde{x})$, and Theorem 12 in [9], p. 24.

As $V(\tilde{x}) = V((\tilde{x}), \tilde{x})$ (cf. Corollary 5.4) we obtain also $\operatorname{co}\sigma_{(\tilde{x})}(\tilde{x}) \subset V(\tilde{x})$. Hence, by Lemma 5.1 and Corollary 5.5,

$$\operatorname{co}\sigma_{(x)}(x) \subset \operatorname{co}\sigma_{(\tilde{x})}(\tilde{x}) \subset V(\tilde{x}) \subset V(x) = \operatorname{co}\sigma(x) = \operatorname{co}\sigma_{\mathfrak{B}}(x) \subset \operatorname{co}\sigma_{(x)}(x)$$

which proves (b).

Recall that $x \in \mathfrak{R}$ is called hermitean if $V(x) \subset \boldsymbol{R}$ or equivalently if $\|\exp(itx)\| = 1$ for all $t \in \boldsymbol{R}$. The following has been proved by E. Nelson in [17] for algebras of operants.

COROLLARY 5.8. *Let E be a linear subspace of a unital Banach algebra $\mathfrak{R}$ with $1_{\mathfrak{R}} \in E$. Let $\mathfrak{A}$ be an operating algebra with respect to E and $\mathfrak{R}$. If $x \in E$ is hermitean and $\|{}^\sim|\mathrm{LH}\{1_{\mathfrak{R}}, x\}\| = 1$ then $\tilde{x}$ is hermitean in $\mathfrak{A}$.*

Let us now return to the example discussed at the end of Section 3.

PROPOSITION 5.9. *Let $\mathfrak{A}$ be the faithfully operating algebra with respect to $E := \{M_h + V_k\colon h \in L^\infty([0, 1]),\ k \in K^\infty\}$ and $\mathfrak{L}(L^2([0, 1]))$ of Proposition 3.11.*

(a) *If $h \in L^\infty([0, 1])$ and $k \in K^\infty$ then $\sigma_{\mathfrak{A}}(\tilde{M}_h + \tilde{V}_k) = \sigma_{\mathfrak{A}}(\tilde{M}_h)$ is connected and*

$$\operatorname{co}\sigma_{\mathfrak{A}}(\tilde{M}_h + \tilde{V}_k) = \operatorname{co}\sigma_{\mathfrak{A}}(\tilde{M}_h) = \operatorname{co}R_e(h), \tag{5.3}$$

where $R_e(h)$ denotes the essential range of h.

(b) *If $h \in L^\infty([0,1])$ is real valued, then $\tilde{M}_h$ is hermitean in $\mathfrak{A}$ and*

$$\sigma_{\mathfrak{A}}(\tilde{M}_h+\tilde{V}_k) = \sigma_{\mathfrak{A}}(\tilde{M}_h) = [\operatorname{ess\,inf} h,\ \operatorname{ess\,sup} h]$$

for every $k \in K^\infty$.

Proof. By Proposition 3.11, $\tilde{V}_k$ is a quasi-nilpotent element of the commutative Banach algebra $\mathfrak{A}$ so that $\sigma_{\mathfrak{A}}(\tilde{M}_h) = \sigma(\tilde{M}_h+\tilde{V}_k)$. As $\operatorname{co}\sigma_{(\tilde{M}_h)}(\tilde{M}_h) = \operatorname{co}\sigma_{\mathfrak{A}}(\tilde{M}_h)$ and $\sigma(M_h) = R_e(h)$ in $\mathfrak{L}(L^2([0,1]))$ we obtain (5.3) by Theorem 5.7(b). Assume that $\sigma_{\mathfrak{A}}(M_h)$ is not connected for some $h \in L^\infty([0,1])$. Then, by Theorem 4.2, there exists a nontrivial idempotent $P \in \mathfrak{L}(L^2([0,1]))$ commuting with all $S \in E$. Hence, $P(\mathfrak{H})$ and $(\mathrm{id}_{\mathfrak{H}}-P)(\mathfrak{H})$ are nontrivial closed invariant subspaces for V_1 (where $\mathfrak{H} := L^2([0,1])$) and we obtain

$$P(\mathfrak{H}) = \{f \in \mathfrak{H}\colon f = 0 \text{ a.e. on } [0,\alpha]\}$$

and

$$(\mathrm{id}_{\mathfrak{H}}-P)(\mathfrak{H}) = \{f \in \mathfrak{H}\colon f = 0 \text{ a.e. on } [0,\beta]\}$$

for some $0 < \alpha, \beta < 1$ (cf. Th. 4.14 in [18]) which is a contradiction. Thus, (a) is proved.

(b) is an immediate consequence of (a).

We prove now a support theorem for nonanalytic functional calculi which can be factorized over a faithfully operating algebra. For the theory of nonanalytic functional calculi for commuting N-tuples in Banach algebras see [2], [4].

THEOREM 5.10. *Let E be a linear subspace of $\mathfrak{R}$ of the type $E = \mathrm{LH}(\{1_{\mathfrak{R}}, x_1, \ldots, x_N\}\cup\mathfrak{B})$ where $\mathfrak{B}$ is a commutative subset of $\mathfrak{R}$ commuting with $x_1, \ldots, x_N$. Let $\mathfrak{A}$ be a faithfully operating algebra with respect to E and $\mathfrak{R}$. If $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_N)$ is an $\mathscr{A}$-scalar N-tuple (in the sense of [4]) in $\mathfrak{A}$ and if $\Lambda\colon \mathscr{A} \to \mathfrak{A}$ is an $\mathscr{A}$-functional calculus for $\tilde{x}$, then*

$$\operatorname{supp}(T\circ\Lambda) = \operatorname{supp}(\Lambda) = \sigma_{\mathfrak{A}}(\tilde{x}).$$

Proof. By Theorem 6 in [4] we have $\operatorname{supp}(\Lambda) = \sigma_{\mathfrak{A}}(\tilde{x})$. Obviously, $T\circ\Lambda$ vanishes on $C^N\setminus\operatorname{supp}(\Lambda)$. Let now $f \in \mathscr{A}$ be a function with $\operatorname{supp}(f) \subset G$, where G is an arbitrary open subset of C^N such that $T\circ\Lambda$ vanishes on G. If p is any polynomial in N variables then

$$T(p(\tilde{x})\Lambda(f)) = T(\Lambda(pf)) = 0.$$

For $y \in \mathfrak{B}$ we have by 3.2 (d),

$$T(\tilde{\ }\Lambda(f)) = yT(\Lambda(f)) = 0.$$

Hence, if $y_1, \ldots, y_k \in \mathfrak{B}$ and if p is any polynomial in $N+k$ variables, then

$$T(p(\tilde{x}_1, \ldots, \tilde{x}_N, \tilde{y}_1, \ldots, \tilde{y}_k)\Lambda(f)) = 0.$$

As the subalgebra of $\mathfrak{A}$ generated by $1_{\mathfrak{A}}$ and the range of $\tilde{\ }$ is dense in $\mathfrak{A}$ we obtain $T(\Lambda(f)\beta) = 0$ for every $\beta \in \mathfrak{A}$ and therefore, $\Lambda(f) = 0$. Thus, Λ vanishes on G too and we obtain $\operatorname{supp}(T\circ\Lambda) = \operatorname{supp}(\Lambda)$.

As an application we obtain a result which is essentially due to [5] and [17].

THEOREM 5.11. *Let $x_1, \ldots, x_N$ be N hermitean elements of a unital Banach algebra $\mathfrak{R}$. Put $E := \mathrm{LH}\{1_{\mathfrak{R}}, x_1, \ldots, x_N\}$ and denote by $\mathfrak{A}(E)$ the Nelson algebra of operants over E. Then $z \to \exp(i\langle z, \tilde{x}\rangle)$ (where $\langle z, \tilde{x}\rangle := z_1\tilde{x}_1+ \ldots +z_N\tilde{x}_N$) is the Fourier–Laplace transform of a $\mathfrak{R}$-valued distribution of order $k \leqslant (N+2)/2$ with compact support.*

$$\varphi \to \Lambda_W(\varphi) := (2\pi)^{-N/2}\int_{R^N} \exp(i\langle t, \tilde{x}\rangle)\hat{\varphi}(t)dt \tag{5.4}$$

defines a $C^k(R^N)$-functional calculus for $x = (x_1, \ldots, x_N)$. Moreover,

$$(T\circ\Lambda_W)(\varphi) = (2\pi)^{-N/2}\int_{R^N} \exp(i\langle t, x\rangle)\hat{\varphi}(t)dt \tag{5.5}$$

and

$$\operatorname{supp}(T\circ\Lambda_W) = \operatorname{supp}(\Lambda_W) = \sigma_{\mathfrak{A}(E)}(\tilde{x}). \tag{5.6}$$

Proof. By Corollary 5.8, $\tilde{x}_1, \ldots, \tilde{x}_N$ are hermitean in $\mathfrak{A}(E)$. Therefore,

$$\|\exp(i\langle t, \tilde{x}\rangle)\| = 1 = \|\exp(i\langle t, x\rangle)\| \quad \text{for all } t \in R^N \tag{5.7}$$

(as $\langle t, \tilde{x}\rangle$ and $\langle t, x\rangle$ are again hermitean in $\mathfrak{A}(E)$ resp. in $\mathfrak{R}$). By [5] (cf. also [2], Satz 3.3) $\tilde{x}$ is a $C^k(R^N)$-scalar N-tuple for some $k \leqslant (n+2)/2$ and the $C^k(R^N)$-functional calculus for $\tilde{x}$ is given by (5.4). (5.5) follows by (5.7) and the continuity of T, and (5.6) is now a consequence of Theorem 5.10.

Remark 5.12. $T\circ\Lambda_W$ is the so-called *Weyl functional calculus*. In [6], R. F. V. Anderson showed: If $\operatorname{supp}(T\circ\Lambda_W)$ is not connected then there exists a nontrivial idempotent element in $\mathfrak{R}$ which commutes with $x_1, \ldots, x_N$. This is now a consequence of Theorem 4.2 and (5.6).

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SOME USES OF SUBHARMONICITY IN FUNCTIONAL ANALYSIS

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The aim of this paper is to present a summary of the eight lectures I gave at the Banach Center in Warsaw on this subject. I shall speak only about applications in the theory of Banach algebras and in the theory of polynomial approximation in several complex variables. All the details and other results will be found in my book *Propriétés Spectrales des algèbres de Banach*.

For the definition and the main properties of subharmonic functions see [5], [10], [11], [18], [22].

1. Banach algebras theory

In the following pages $\mathrm{Sp}\,x$ denotes the *spectrum of* x, $\sigma(x)$ denotes the *full spectrum*, i.e. the union of $\mathrm{Sp}\,x$ with its holes, $\varrho(x)$, $\delta(x)$, $c(x)$ denote respectively the *radius*, the *diameter* and the *capacity* of the spectrum of x. $\mathrm{Rad}\,A$ is the Jacobson radical of the algebra A.

The fundamental starting point is

THEOREM 1 (Vesentini). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\mathbf{C}$ into a complex Banach algebra A, then $\lambda \to \varrho(f(\lambda))$ and $\lambda \to \log\varrho(f(\lambda))$ are subharmonic.*

For the proof, see [19], [20]. A more elementary proof not using Radó's theorem is given in [5]. With that result the well-known theorem of Kleinecke and Shirokov and related results are coming more naturally.

COROLLARY 1 (Kleinecke–Shirokov). *Let A be a Banach algebra and a, b elements of A verifying $a(ab-ba) = (ab-ba)a$, then $ab-ba$ is quasi-nilpotent.*

COROLLARY 2. *Let a, b be elements of A verifying $a(ab-ba) = 0$ or $(ab-ba)a = 0$ and suppose that 0 is on the exterior boundary of the spectrum of a (i.e. the boundary of the full spectrum), then $ab-ba$ is quasi-nilpotent.*

COROLLARY 3. (*Principle of maximum for full spectrum*) (Vesentini). *Let $\lambda \to f(\lambda)$ be an analytic function from a domain D in $\mathbf{C}$ into a complex Banach algebra*