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SHEAF CONSTRUCTIONS IN UNIVERSAL ALGEBRA AND MODEL THEORY

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In model theory the notions of reduced products, limit powers, limit reduced powers, Boolean powers, and bounded Boolean powers have been widely accepted and appreciated. These notions can be put under a common roof by the notion of the structure of all sections of a Hausdorff-sheaf over a Boolean space. This notion is very cumbersome and for the uninitiated very hard to understand. Several attempts have been made to get an easier definition of these objects, which then may pupularize the use of sheaf-representations. In the more general setting of sheaves over arbitrary spaces such attempts have been made by V. Weisspfenning and D. Clark & P. Krauss. The applications in universal algebra and model theory have proved to require the special case of sheaves over a Boolean space and there exist basically two easy definitions. One of them is the notion of a Boolean valued structure, which was investigated in this connection by H. Volger. Another is the notion of a Boolean product, which we shall adopt here. We define the Boolean products as a straightforward generalization of bounded Boolean powers.

All the above notions turn out to be special subdirect products with certain additional properties. Likewise our definition of a Boolean product will be one of a special subdirect product whose subdirect factors we shall refer to as *stalks*, having in mind the sheaf-representation of these structures. In our approach there will be hardly any need to talk about topology at all, which will make it easy for the algebraist to follows our reasoning.

So far we have only dwelled on the fact that Boolean products provide a common generalization for the notions above. The second aspect is the fact that for Boolean products there holds a Feferman–Vaught-type theorem which has enormous consequences for decidability and $\aleph_0$-categoricity-problems. The third aspect of Boolean products is based on the fact that for many classes of structures the algebraically closed members can be described in terms of Boolean products. This again gives rise to several results about model companions. The last aspect we investigate here is the special rôle played by injectivity and its generalizations in a class of Boolean products. In this framework we are able to describe atomic compact (= pure injective) structures, absolute subretracts, and (weak) injectives in many classes of structures.

Most of the results presented here stem from the joint work of the present author and S. Burris and B. Davey. Here we give a unified approach to all those different aspects of Boolean products. Several of the results are presented in a new form, more general than in the original papers.

For these lecture notes the reader is assumed to be familiar with ordinary first-order languages. A *structure* for a language L is a set A together with an n-ary function f^A for each n-ary fundamental function-symbol f in L and an m-ary relation R^A for each m-ary fundamental relation symbol R in L. Constants are considered to be 0-ary functions or 1-ary relations whichever is more appropriate. We do not distinguish notationally between a structure and its carrier-set, and if there is no danger of confusion we also omit the superscripts A above f and R. A *morphism* between two structures A and B is a map $g\colon A \to B$ which preserves each n-ary fundamental operation f and strictly preserves each m-ary fundamental relation R, i.e.

(1) $$a_1, \ldots, a_n \in A \Rightarrow g\big(f^A(a_1, \ldots, a_n)\big) = f^B(ga_1, \ldots, ga_n),$$

(2) $$a_1, \ldots, a_m \in A \;\&\; (a_1, \ldots, a_m) \in R^A \Rightarrow (ga_1, \ldots, ga_m) \in R^B,$$

(3) $$a_1, \ldots, a_m \in A \;\&\; (ga_1, \ldots, ga_m) \in R^B \Rightarrow \exists c_1 \ldots \exists c_m\, (c_1, \ldots, c_m) \in R^A \;\&\; gc_i = ga_i.$$

If g only satisfies (1) and (2), we speak about a *weak* morphism. A *substructure* B of A is a subset B of A which is closed under the operations on A (i.e. B is a *subalgebra*) and the relations on B are just the restrictions of the corresponding relations on A (i.e. the inclusion $B \rightarrowtail A$ is a morphism).

A one-to-one morphism is denoted by $A \rightarrowtail B$ and is referred to as an embedding whereas an onto morphism is called a *quotient map* and denoted by $A \twoheadrightarrow B$. A *congruence* on a structure A is an equivalence relation θ on A such that $(a_1, b_1) \in \theta, \ldots, (a_n, b_n) \in \theta \Rightarrow \big(f^A(a_1, \ldots, a_n), f^A(b_1, \ldots, b_n)\big) \in \theta$ for each fundamental operation f. Clearly, for each morphism $f\colon A \to B$ its *kernel* $\operatorname{Ker} f = \{(a, b) \mid fa = fb\}$ is a congruence on A. For each congruence θ on A we have the *quotient structure* $A/_\theta$ together with the quotient map $p_\theta\colon A \twoheadrightarrow A/_\theta$.

We say that a morphism $f\colon A \to B$ *preserves* a formula $\varphi(x_1, \ldots, x_n)$ if for each $a_1, \ldots, a_n \in A$ we have $A \models \varphi(a_1, \ldots, a_n)$ provided $B \models \varphi(fa_1, \ldots, fa_n)$. An embedding clearly preserves each universal formula. $f\colon A \rightarrowtail B$ is called *pure* if it preserves each positive existential formula; *existential* if it preserves each existential formula; *elementary* if it preserves each formula; in this case A is called a *pure* (existential, elementary) *substructure* of B.

A formula $\pi(x_1, \ldots, x_n)$ is said to be *primitive* if it is an existential conjunct of atomic and negatomic formulas. Clearly, for an embedding $f\colon A \rightarrowtail B$ to be pure (existential) it suffices that it should preserve all positive-primitive (primitive) formulas, as disjunctions commute with the existential quantifier.

Chapter 1

BOOLEAN PRODUCTS

We are going to define Boolean products as generalization of bounded Boolean powers, and so we start with the definition and investigation of bounded Boolean powers.

1.1. Bounded Boolean powers

Let A be a structure and B a Boolean algebra, B^* denotes the Boolean space of all ultrafilters on B with the sets $S_b := \{u \in B^* \mid b \in u\}$ as clopen subsets. The structure $A[B]^*$ of all continuous maps from B^* into the discrete space A under the pointwise definition of operations and relations is called a *bounded Boolean power of* A or the *Boolean extension of* A *by* B.

As B^* is a compact space, a continuous map $f\colon B^* \to A$ can only take finitely many values in A, and so the set $\{f^{-1}a \mid a \in A\}$ is a finite partition of B^* into clopen sets which in turn corresponds to a decomposition of 1 in B ($b_1, \ldots, b_n \in B$, $1 = b_1 \vee \ldots \vee b_n, \mathop{\&}\limits_{i<j} b_i \wedge b_j = 0$). Using this observation, one can easily see the equivalence of our definition with Foster's original one as stated in Grätzer's book.

It is clear that all the constant maps belong to $A[B]^*$, and hence we have a subdirect representation $A[B]^* \to A^{B^*}$.

For any two elements $f, g \in A[B]^*$ the *equalizer* $E(f, g) := \{x \in B^* \mid fx = gx\}$ equals $\bigcup\limits_{a \in A} f^{-1}a \cap g^{-1}a$. By what we have just said it is a finite union and hence all equalizers are clopen subsets of B.

If S is a clopen subset of B^* and $f, g \in A[B]^*$, then, clearly, the map $h = f|_S \cup g|_{B^* \sim S}$ is continuous and thus belongs to $A[B]^*$ and satisfies

$$S \subseteq E(f, h) \;\&\; B^* \sim S \subseteq E(g, h).$$

Just as for the equalizers, we can see that for any formula $\varphi(x_1, \ldots, x_n)$ and $f_1, \ldots, f_n \in A[B]^*$ the set $\{x \in B^* | \ A \models \varphi(f_1 x, \ldots, f_n x)\}$ is clopen in B because this set is

$$\bigcup_{A \models \varphi(a_1, \ldots, a_n)} f_1^{-1} a_1 \cap \ldots \cap f_n^{-1} a_n,$$

which is in fact a finite union.

1.2. Solution sets

We want to define Boolean products by the properties we have just mentioned for the bounded Boolean powers. In order to do that we need a general notion for equalizers and their generalizations. Let $f\colon A \rightarrowtail \prod_{i \in I} A_i$ be a subdirect representation of A, i.e. let A be a substructure of $\prod_{i \in I} A_i$ such that the projections restricted to A are still quotient maps. For a formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$ we define the *solution-set*

$$[\varphi(a_1, \ldots, a_n)]_f := \{i \in I | \ A_i \models \varphi(f(a_1)_i, \ldots, f(a_n)_i)\}.$$

If there is no danger of confusion, we omit the subscript f.

We first note some easy facts about solution sets:

(1) $[a = b] = [b = a]$ & $[a = b] \cap [b = c] \subseteq [a = c]$.

(2) For each fundamental operation f:

$$[a_1 = b_1] \cap \ldots \cap [a_n = b_n] \subseteq [f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)].$$

(3) $[a = b] = I \Leftrightarrow A \models a = b$.

(4) For each fundamental relation R:

$$[a_1 = b_1] \cap \ldots \cap [a_m = b_m] \cap [R(a_1, \ldots, a_m)] \subseteq [R(b_1, \ldots, b_m)],$$

$$[R(a_1, \ldots, a_m)] = I \Leftrightarrow A \models R(a_1, \ldots, a_m),$$

$$[R(a_1, \ldots, a_m)] = \bigcup \{[a_1 = b_1] \cap \ldots \cap [a_m = b_m] | \ (b_1, \ldots, b_m) \in R^A\}.$$

(5) $[\neg \varphi] = I \sim [\varphi]$, $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$, $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.

(6) $[\exists x \ \varphi(x)] = \bigcup \{[\varphi(a)] | \ a \in A\}$.

We define a field of subsets B_0 of I by

$B_0 := \{[\varphi] | \ \varphi$ open sentence in the language of A without relations$\}$, which by condition (5) is clearly a Boolean subalgebra of the power set of I.

1.3. Boolean product

A structure A is called a *Boolean product* of the structures A_i $(i \in I)$ if there is a field B of subsets of I such that:

(1) A has a subdirect representation $f\colon A \rightarrowtail \prod_{i \in I} A_i$;

(2) for $a, b \in A$ the solution set $[a = b]$ belongs to B;

(3) for $S \in B$, $a, b \in A$ ex. $c \in A$ such that $S \subseteq [a = c]$ & $I \sim S \subseteq [b = c]$.

The structures A_i $(i \in I)$ are called the *factors* or *stalks* of the Boolean product and the set I together with the topology defined by B is called the *base-space* of the Boolean product. Clearly, the base-space is 0-dimensional (i.e. it has a basis of clopen sets), but in general it is neither Hausdorff nor compact. We call the Boolean product *full* if the base-space is Boolean (i.e. a 0-dimensional compact Hausdorf space).

Sometimes we are interested in more special Boolean products, which also satisfy the following condition:

(4) For each sentence φ over A the solution set $[\varphi]$ belongs to B.

For full Boolean products this condition is very restrictive and turns out to be equivalent to the following *maximum property*.

(MP) For each formula $\varphi(x)$ over A the set $\{[\varphi(a)] | \ a \in A\}$ has a maximum.

By an induction over the complexity of φ we shall see that (MP) implies (4), and for the converse we need the compactness of the base-space.

1.4. Patchwork property

In our definition of Boolean products the crucial condition is condition (3), whose consequences we are now going to investigate. Clearly, condition (2) just states that $B_0 \subseteq B$, and so we have in particular (3) for B_0 instead of B. An even weaker form of (3) is the *patchwork property*:

(PP) $\quad a, b, c, d \in A \Rightarrow \exists e \in A \, [a = b] \subseteq [c = e]$ & $[a \neq b] \subseteq [d = e]$.

An important consequence of (PP) is

(1) B consists of the sets $[a = b]$ and $[a \neq b]$ for $a, b \in A$ only.

To see this we first fix a and show for $b, c \in A$ the existence of some $d \in A$ such that

(i) $[a = b] \cap [a = c] = [a = d]$,

(ii) $[a = b] \cup [a = c] = [a = d]$,

(iii) $[a = b] \cup [a \neq c] = [a = d]$,

(iv) $[b = c] = [a = d]$, respectively.

(i) pick $d \in A$ such that $[a = b] \subseteq [c = d]$ & $[a \neq b] \subseteq [b = d]$,

(ii) pick $d \in A$ such that $[a = b] \subseteq [a = d]$ & $[a \neq b] \subseteq [c = d]$,

(iii) pick $d \in A$ such that $[b = c] \subseteq [a = d]$ & $[b \neq c] \subseteq [b = d]$,

(iv) first pick $e \in A$ as d in (i) and then pick $d \in A$ such that

$$[b = c] \subseteq [a = d] \,\&\, [b \neq c] \subseteq [e = d].$$

We have even proved more. If we fix any $a \in A$, we get

$$B_0 = \{[a = b] \mid b \in A\} \cup \{a \neq b] \mid b \in A\}.$$

Clearly, by induction the patchwork property (or condition (3) for that matter) implies the following stronger version:

(2) $S_1, \ldots, S_n \in B_0$, $\vec{a}_1, \ldots, \vec{a}_n \in A^k$,

$$\mathop{\&}_{i<j} S_i \cap S_j \subseteq [\vec{a}_i = \vec{a}_j] \Rightarrow \exists \vec{b} \in A^k \mathop{\&}_{i \leqslant n} S_i \subseteq [\vec{a}_i = \vec{b}].$$

As the last consequence we wish to show that (MP) $\Rightarrow$ (4) in 1.3 and that the converse holds for a compact base-space.

Let us assume (MP). First, we have to show $[R(a_1, \ldots, a_m)] \in B_0$. Consider the formula $\varphi(x) \equiv x = a_0 \rightarrow R(a_1, \ldots, a_m)$. If $b \in A$ is such that $[\varphi(b)]$ is the maximum of $\{[\varphi(a)] \mid a \in A\}$, then clearly it is the greatest element of B below $[R(a_1, \ldots, a_m)]$, but by 1.2 (4) $[R(a_1, \ldots, a_m)]$ is a union of members of B_0 and hence it belongs to B_0 itself. Now by 1.2 (5) and (6) an easy induction on the complexity of φ shows that $[\varphi] \in B_0$ for each sentence φ over A.

To see the converse we need the following easy consequence of 1.3 (3):

(3) If $[\exists x \varphi(x)]$ is compact and all $[\varphi(a)] (a \in A)$ are open, then there exists $a \in A$ such that $[\exists x \varphi(x)] = [\varphi(a)]$.

By 1.3 (3) the sets $[\varphi(a)]$ form a directed set whose union by 1.2 (6) is $[\exists x \varphi(x)]$, whose compactness now yields (3).

Now, clearly, if the base-space is compact, each clopen set is compact too, and thus by (3), clearly, 1.3 (4) implies (MP). Shortly we shall see an example of a Boolean product over a non-compact space which does satisfy 1.3 (4) but does not have the maximum property.

Finally, observe that the patchwork property is a special case of the maximum property, because the maximum of

$$\{(a = b \rightarrow c = x) \wedge (a \neq b \rightarrow d = x) \mid x \in A\}$$

shows the patchwork property.

1.5. Reductions

For some of the applications it is important that we have a full Boolean product rather than an arbitrary one. We therefore wish to construct a full Boolean product representation for each Boolean product. This will be achieved in three steps. In the first step we "standardize" the topology of the base-space by taking the coarsets topology possible, and we observe that this topology (defined by B_0) is defined in purely algebraic

terms. In the second step we make the base-space Hausdorff, which is done by just omitting "redundancies" in the subdirect representation. The main step then is the compactification of the base-space, and in that construction we have to put new points into the base-space and therefore have to add new stalks to the old ones. After these three steps we end up with a full Boolean product representation of the structure we started with.

Sometimes we want to do just the opposite, namely to get rid of superfluous stalks in a Boolean product. We give a procedure for that in step four.

Step 1. Replace B by B_0. As we have seen in 1.4, also B_0 is a field of subsets of I and still the properties (1)–(3) of 1.3 hold for B_0. Clearly, the maximum property remains valid, since it does not refer to B at all.

Step 2. In general, the base-space is not Hausdorff and we define $i \sim j$ iff, for all $a, b \in A$, $i \in [a = b] \Leftrightarrow j \in [a = b]$ and we let X be a system of representatives of $I/\sim$. We consider the canonical morphism

$$f\colon A \rightarrowtail \prod_{i \in X} A_i.$$

Clearly, this morphism f still satisfies 1.3 (2) and (3) and also (MP) provided the original Boolean product $A \rightarrowtail \prod_{i \in I} A_i$ satisfied (MP). We only have to show that f is a subdirect representation of A, i.e. that the projections p_i $(i \in I)$ still separate A. For $i, j \in I$ we have

$$\begin{aligned} i \sim j &\Leftrightarrow \forall a \forall b \;\; i \in [a = b] \Leftrightarrow j \in [a = b] \Leftrightarrow \forall a \forall b \;\; p_i a = p_i b \\ &\Leftrightarrow p_j a = p_j b \\ &\Leftrightarrow \ker p_i = \ker p_j; \end{aligned}$$

hence

$$\bigcap_{i \in X} \ker p_i = \bigcap_{i \in I} \ker p_i = \Delta.$$

Step 3. We now start with a Boolean product representation $A \rightarrowtail \prod A_i$ with a 0-dimensional Hausdorff base-space with B_0 as basis. Suppose that $\hat{X}$ is the Stone–Čech compactification of X, i.e. the points of $\hat{X}$ are the ultrafilters of B_0 and the clopen subsets of $\hat{X}$ are those of the form $S_b = \{\mathfrak{u} \in \hat{X} \mid b \in \mathfrak{u}\}$, and for each $\mathfrak{u} \in \hat{X}$ we let $A_{\mathfrak{u}}$ be the image of A under the morphism $A \rightarrowtail \prod A_i \twoheadrightarrow \prod A_i/\mathfrak{u}$.

Now we consider the resulting subdirect representation $g\colon A \rightarrowtail \prod_{\mathfrak{u} \in \hat{X}} A_{\mathfrak{u}}$.

By construction the base-space is Boolean and we have to show 1.3 (2) and (3). For $a, b \in A$ we have $[a = b]_g = \{\mathfrak{u} \mid a \equiv b \bmod \mathfrak{u}\} = S_{[a \neq b]_f}$.

This shows both (2) and (3) and moreover that the topology on X also is defined by B_0. We postpone a closer investigation of the new stalks and maximum property to a later moment.

Step 4. Let $f\colon A \rightarrowtail \prod_{i\in I} A_i$ be a Boolean product representation and $X \subseteq I$. $g\colon A \rightarrowtail \prod_{i\in X} A_i$ is a Boolean product representation iff, for $a, b \in A$, $X \subseteq [a = b]$ implies $a = b$.

To see this observe $ga = gb \Leftrightarrow \forall i \in X\ p_i a = p_i b \Leftrightarrow X \subseteq [a = b]$. Clearly, conditions 1.3 (2), (3) and (MP) hold for the field $\{S \cap X \mid S \in B\}$ of subsets of X provided they hold for the field B of subsets of I.

1.6. The new stalks

In the third reduction step of 1.5 we had to create new stalks in order to turn a Boolean product $A \rightarrowtail \prod_{i\in X} A_i$ into a full one $A \rightarrowtail \prod_{\mathfrak{u}\in\hat{X}} A_{\mathfrak{u}}$, and we do not have much information about the new stalks except that they are quotient structures of A. We first investigate what else can be said about the new stalks $A_{\mathfrak{u}}$ and about the solution sets with respect to g.

$\prod_{i\in X} A_i/_{\mathfrak{u}}$ is a reduced product, and an atomic formula α over A holds in this reduced product iff $[\alpha]$ contains some member of $\mathfrak{u}$. Particularly, this means that for $a_1, \ldots, a_m \in A$ we have

$$\prod_{i\in X} A_i/_{\mathfrak{u}} \models R(a_1, \ldots, a_m) \quad \text{iff for some } (b_1, \ldots, b_m) \in R^A$$
$$[a_1 = b_1] \cap \ldots \cap [a_m = b_m] \in \mathfrak{u}.$$

Thus we know that $A_{\mathfrak{u}}$ is a substructure of the reduced product $\prod_{i\in X} A_i/_{\mathfrak{u}}$. Reduced products in general are not very easy to deal with, but since $\mathfrak{u}$ is an ultrafilter on B_0, this particular reduced product is not very far from being an ultraproduct. If we pick an ultrafilter $\mathfrak{b}$ on X which extends $\mathfrak{u}$, then, by Łoś's theorem, for each sentence φ over A we have $\prod_{i\in X} A_i/_{\mathfrak{b}} \models \varphi \Leftrightarrow [\varphi] \in \mathfrak{u}$, and if $[\varphi]$ belongs to B_0, this in turn is equivalent to $[\varphi] \in \mathfrak{u}$. For $a, b \in A$ we have in particular $\prod_{i\in X} A_i/_{\mathfrak{b}} = a/_{\mathfrak{b}} = b/_{\mathfrak{b}} \Leftrightarrow [a = b] \in \mathfrak{u} \Leftrightarrow A_{\mathfrak{u}} \models a = b$, and hence there is a weak embedding $A \mapsto A_{\mathfrak{u}}$.

In general this is not an embedding, but if $[R(a_1, \ldots, a_n)]$ is always in B_0, we infer by the same reasoning that it is an embedding. Thus we see that the new stalks are (weak) substructures of ultraproducts of the old stalks.

If a given ultraproduct satisfies (MP) then, by induction on the complexity of φ: we can show $A_{\mathfrak{u}} \models \varphi \Leftrightarrow [\varphi] \in \mathfrak{u}$, which proves that $A_{\mathfrak{u}}$ is even an elementary substructure of the ultraproduct $\prod_{i\in\hat{X}} A_i/_{\mathfrak{b}}$. For atomic formulas we have done that above, and for $\wedge, \vee, \neg$ the induction steps are easy. Assume $\varphi \equiv \exists x\, \psi(x)$. By (MP) we find $a \in A$ such that $[\varphi] = [\psi(a)]$ and by the induction hypothesis $A_{\mathfrak{u}} = \psi(b) \Leftrightarrow [\psi(b)] \in \mathfrak{u}$. Thus

$$A_{\mathfrak{u}} \models \varphi \Leftrightarrow A_{\mathfrak{u}} \models \psi(a) \Leftrightarrow [\psi(a)] = [\varphi] \in \mathfrak{u}.$$

This proof also tells us that the new Boolean product representation g also satisfies (MP) provided f did.

1.7. Reduced Boolean products

Assume that $f\colon A \rightarrowtail \prod_{i\in I} A_i$ is a Boolean product and $\mathfrak{D}$ is a filter on B_0. Defining $a \equiv b \,(\mathrm{mod}\, \mathfrak{D}) \Leftrightarrow [a = b] \in \mathfrak{D}$, we get a congruence on A by 1.2(1)–(3). The structure $A/\mathfrak{D}$ is called a *reduced Boolean product*, and we shall prove that it has itself a Boolean product representation. Let $g\colon A \rightarrowtail \prod_{x\in\hat{X}} A_x$ be the corresponding full Boolean product and $D = \bigcap \mathfrak{D}$. As $[a = b]$ is compact in $\hat{X}$, we have $D \subseteq [a = b]_g \Rightarrow [a = b]_g \in \mathfrak{D}$, and so we have an embedding $A/\mathfrak{D} \rightarrowtail \prod_{x\in D} A_x$, which is clearly a full Boolean product representation. Moreover, $A/\mathfrak{D}$ has (MP) provided A does.

1.8. The Boolean product operators

For a class $\mathfrak{A}$ of structures and a theory T of Boolean algebras we denote by $\Gamma_T \mathfrak{A}$ the class of all Boolean products with stalks in $\mathfrak{A}$ and $B_0 \models T$:

$$\Gamma_T^a \mathfrak{A} = \{A \in \Gamma_T \mathfrak{A} \mid \forall a_1 \ldots \forall a_m\ [R(a_1, \ldots, a_m)] \in B_0 \textit{ for each atomic } R\},$$

$$\Gamma_T^e \mathfrak{A} = \{A \in \Gamma_T \mathfrak{A} \mid A \models (\mathrm{MP})\}.$$

If T is the theory of all Boolean algebras, we omit the subscript T. The Boolean products in $\Gamma^a \mathfrak{A}$ and $\Gamma^e \mathfrak{A}$ are called *atomic* and *elementary* respectively because the solution sets of all atomic (elementary) sentences are clopen.

1.9. The ternary discriminator

On a structure A the ternary discriminator t is defined by

$$t(x, y, z) := \begin{cases} x, & x \neq y, \\ z, & x = y \end{cases}$$

and A^t denotes the structure A augmented by t as a new ternary operation.

THEOREM. *A subdirect representation* $A \rightarrowtail \prod_{i\in I} A_i$ *is a Boolean product iff* A *is a substructure of* $\prod_{i\in I} A_i^t$.

Proof. We have to show that A has (PP) iff it is closed under the pointwise discriminator. Let $a, b, c, d \in A$ and define

$$e = t\big(t(a, b, c),\ t(a, b, d),\ d\big);$$

then clearly $[a = b] \subseteq [e = c]$ as $t(c, d, d) = c$ and $[a \neq b] \subseteq [a = d]$ as $t(a, a, d) = d$.

If $a, b, c \in A$, then $t(a, b, c)$ is the $d \in A$ with $[a = b] \subseteq [a = d)$ and $[a \neq b] \subseteq [c = d]$.

A *discriminator variety* is a variety of algebras with a ternary term t such that the subdirectly irreducibles are characterized by t being the discriminator on them. Examples are: Boolean algebras, monadic algebras, cylindric algebras, relation algebras, Baer *-rings, and $(x^n = x)$-rings.

COROLLARY. *In each discriminator variety each subdirect representation by subdirectly irreducibles is a Boolean product representation.*

1.10. Comer's theorem

A subdirect representation of an algebra is given by a set of congruences separating the algebra. For the case of Boolean products S. Comer has given a characterization by such sets of congruences. This characterization has been generalized to a wider class of sheaves by A. Wolf, and we shall give here a refinement of his method, which is even better, in so far as the stalks are smaller and can be visualized much more easily.

THEOREM 1. *Let* A *be an algebra and* $\mathscr{L}$ *a set of congruences on* A *such that* $\mathscr{L}$ *is an arithmetical* (= *permutable* & *distributive*) *sublattice of* $\mathrm{Con}(A)$ *containing* $\Delta = \{(x, x) \mid x \in A\}$. *For each prime-ideal* i *define* $A_i = A/_{\cup i}$. *Then the canonical embedding* $A \rightarrowtail \prod_i A_i$ *satisfies* 1.4(2). *Moreover, if* $\mathscr{L}$ *is relatively complemented and, for every* $a, b \in A$, *there is the smallest* Φ *in* $\mathscr{L}$ *containing* (a, b), *then this representation is a Boolean product.*

Proof. The complement of each prime ideal i is a prime-filter, and hence we can pick the space $\mathscr{L}^*$ of all prime-filters on $\mathscr{L}$ with the basic open sets $S_\theta = \{u \in \mathscr{L}^* \mid \theta \notin u\}$ as the base space. Clearly, $A_i \models a = b$ iff a certain $\Phi \in \mathscr{L}$ with $(a, b) \in \Phi$ belongs to i iff $\mathscr{L} \setminus i \in \bigcup_{(a,b)\in\Phi} S_\Phi$, and thus $[a = b] = \bigcup_{(a,b)\in\Phi} S_\Phi$ is an open set. It is immediately clear that $S_\Phi \cap S_0 = S_{\Phi\vee\theta} \cdot S_\Phi \subseteq [a = b]$ iff $(a, b) \in \Phi$ because for $(a, b) \notin \Phi$ there exists a prime-filter not containing Φ but extending the filter $\{\theta \in \mathscr{L} \mid (a, b) \in \theta\}$.

Now condition 1.4(2) translates into $\Phi_1, \ldots, \Phi_n \in \mathscr{L}$, $a_1, \ldots, a_n \in A$, $\mathop{\&}_{i<j} (a_i, a_j) \in \Phi_i \vee \Phi_j \Rightarrow \exists\, b \in A \mathop{\&}_{i<n} (a_i, b) \in \Phi_i$, which is the Chinese remainder theorem, which in turn is equivalent to the arithmeticity of $\mathscr{L}$. If $\mathscr{L}$ is relatively complemented, then $\mathscr{L}^*$ is Boolean, and if $\{\Phi \in \mathscr{L} \mid (a, b) \in \Phi\}$ has a minimum Φ_0 then $[a = b] = S_{\Phi_0}$ is clopen.

In order to generalize this theorem to arbitrary structures we have to investigate which sets of congruence define subdirect representations.

If $f\colon A \mapsto \prod_{i\in I} A_i$ is an embedding such that the projections are onto and $\Phi_i \in \mathrm{Con}(A)$ are the kernels of the projections $(i \in I)$, then f is a subdirect product provided for each fundamental relation R we have $\vec{a} \in R \Leftrightarrow \forall i \in I\ R \cap [(\vec{a})]\Phi_i \neq \emptyset$.

If we are now in the situation of Theorem 1, this condition reads as follows: $\vec{a} \in R$ iff for each prime-ideal i there exists an $\theta \in i$ such that $R \cap [\vec{a}]\theta \neq \emptyset$. In other words: $\vec{a} \in R \Leftrightarrow [\vec{a}R] := \{\Phi \in \mathscr{L} \mid R \cap [a]\Phi \neq \emptyset\}$ is not contained in any prime-filter of $\mathscr{L}$. Clearly, if $[\vec{a}R]$ is always a filter, then $[\vec{a}R]$ is contained in a prime-filter unless $[\vec{a}R] = \mathscr{L}$ and hence $R \cap \cap [\vec{a}]\Delta \neq \emptyset$, which implies $\vec{a} \in R$.

The condition of $[\vec{a}R]$, always being a filter, is:
$\vec{b}, \vec{c} \in R$ and $\Phi, \theta \in \mathrm{Con}(A)$ such that $\vec{b}\,\Phi \cdot \theta\vec{c}$; then

$$\vec{d} \in R\vec{b}\,\Phi\vec{d}\,\theta\vec{c},$$

which implies that the congruences on A also permute on $R \subseteq A^k$ and moreover the join on R is the same as the join on A for all congruences on A. We can now reformulate Theorem 1 in the extended form.

THEOREM 2. *Let* A *be a structure and* $\mathscr{L} \subseteq \mathrm{Con}(A)$ *such that*

(1) $\mathscr{L}$ *is a distributive sublattice of* $\mathrm{Con}(A)$ *containing* Δ;

(2) $\theta, \Phi \in \mathscr{L} \Rightarrow \theta \circ \Phi \in \mathscr{L}$;

(3) *For each* n*-ary fundamental relation* R *on* A *and* $\theta, \Phi \in \mathscr{L}$, $\theta_R \circ \Phi_R = (\theta \circ \Phi)_R$, *where* θ_R *denotes the congruence on* R *induced by* θ;

(4) $\mathscr{L}$ *is relatively complemented*;

(5) $\{\Phi \in \mathscr{L} \mid (a, b) \in \Phi\}$ *has a minimum for all* $a, b \in A$.

For each prime ideal i *define* $A_i = A/_{\cup i}$; *then the canonical morphism* $A \to \prod_{i\in I} A_i$ *is a Boolean product, which is atomic iff* (3) *is replaced by the stronger condition*

(3$^+$) $\{\Phi \in \mathscr{L} \mid R \cap [\vec{a}]\Phi \neq \emptyset\}$ *has a minimum for all* n*-ary fundamental relations* R *and* $\vec{a} \in A^n$.

The above theorem does not give very much information about the stalks A_i. We just know that all stalks are quotients of A but we shall see that they have a special property with respect to the lattice $\mathscr{L}$ in question. As $\mathscr{L}$ is a relatively complemented distributive lattice, all prime ideals are maximal ideals and hence the congruences $\Phi \in \mathscr{L}$ induce either Δ or ∇ on A_i depending on whether $\Phi \in i$ or not. Observe that $\{(a/_{\cup i}, b/_{\cup i}) \mid (a, b) \in \Phi\}$ is always a congruence on $A/_{\cup i}$ since the congruences in $\mathscr{L}$ permute with each other. So if we regard A as a Boolean product, each member of $\mathscr{L}$ is a congruence of the form $\{(a, b) \mid N \subseteq [a = b]\}$ where N is the clopen set $\{i \mid \Phi \in i\}$. Observe that for each Boolean product and each basis $\mathfrak{B}$ of the topology of this base-space the set $\mathscr{L} = \{\{(a, b) \mid N \subseteq [a = b]\} \mid N \in \mathfrak{B}\}$ satisfies the assumptions of Theorem 2, so that in this sense Theorem 2 characterizes Boolean products.

We formulate the results in the following theorems:

THEOREM 3. *Let A be structure and let $\mathscr{L} \subseteq \mathrm{Con}(A)$ satisfy the assumptions* (1)–(5) *of Theorem* 2. *Let $A \rightarrowtail \prod_i A_i$ be the Boolean product given by Theorem* 2.

(a) *For each $\Phi \in \mathscr{L}$ the set $\{(p_i a, p_i b) \mid (a, b) \in \Phi\}$, where $p_i\colon A \twoheadrightarrow A_i$ is the projection, is one of the trivial congruences Δ or ∇ on A_i depending on whether $\Phi \in i$ or $\Phi \notin i$.*

(b) *For each $\Phi \in \mathscr{L}$ there is a clopen set N such that $(a, b) \in \Phi \Leftrightarrow N \subseteq [a = b]$.*

THEOREM 4. *Let $A \to \prod_{i\in I} A_i$ be a Boolean product and $\mathfrak{B}$ a basis of the topology on I closed under finite intersections and unions and relative complements. Then $\mathscr{L} = \{\Phi_N \mid N \in \mathfrak{B}\}$, where $\Phi_N = \{(a, b) \mid N \subseteq [a = b]\}$ satisfies the assumptions* (1)–(5) *of Theorem* 2. *If the Boolean product is atomic, then $\mathscr{L}$ satisfies assumption* (3^+) *as well.*

Before we go on with the theory of Boolean products, we wish to give some examples of Boolean products.

1.11. Direct products

Let $A = \prod_{i\in I} A_i$. Clearly, A is a Boolean product w.r.t. to 2^I satisfying (MP). The new stalks, after reduction 3, are precisely the ultraproducts $\prod A_i/_{\mathfrak{u}}$ for non-trivial ultrafilters $\mathfrak{u}$ on I.

1.12. Reduced products

Let $\mathfrak{D}$ be a filter on I. Then $\prod_{i\in I} A_i/_{\mathfrak{D}}$ has a Boolean product representation with ultraproducts $\prod_{i\in I} A_i/_{\mathfrak{u}}$ $(\mathfrak{D} \subseteq \mathfrak{u})$ as stalks. By 1.10 and 1.11 also this Boolean product representation has the maximum property.

1.13. Direct sums

Assume $\prod_{i\in I} A_i$ has a singleton retract $\{a\}$. The structure $A = \{b \in \prod_{i\in I} A_i \mid [a \neq b] < \aleph_0\}$ is called a *direct sum* of the A_i's. Obviously the embedding $A \rightarrowtail \prod_{i\in I} A_i$ is an atomic Boolean product with respect to the Boolean algebra B of all finite and cofinite subsets of I. B has only one non-trivial ultrafilter which defines a singleton stalk isomorphic to $\{a\}$. So, clearly, A does not satisfy the maximum property because e.g. $\exists x\; x \neq a$ does not have a maximal solution. Direct sums are special cases of sub-Boolean powers in example 1.19.

1.14. Limit powers

Let $\mathfrak{F}$ be a filter on $I \times I$ and A a structure. The *limit power* $A^I|_{\mathfrak{F}}$ is the substructure $\{a \in A^I \mid \mathrm{Ker}\, a \in \mathfrak{F}\}$ of A^I. (This subset is indeed a substructure because $\mathrm{Ker}(f(a_1, \ldots, a_n)) \subseteq \bigcap_{i\leqslant n} \mathrm{Ker}(a_i)$.) We claim that $A^I|_{\mathfrak{F}} \rightarrowtail A^I$ is a Boolean product w.r.t. to the Boolean subalgebra $2^I|_{\mathfrak{F}}$ of 2^I. As $A^I|_{\mathfrak{F}}$ contains all the constant maps, $A^I|_{\mathfrak{F}} \rightarrowtail A^I$ is clearly a subdirect representation. Note that $2^I|_{\mathfrak{F}} = \{M \subseteq I \mid M^2 \cup (I \setminus M)^2 \in \mathfrak{F}\}$.

Let $M = [a = b]$ for $a, b \in A^I|_{\mathfrak{F}}$. $a_i = a_j \wedge b_i = b_j$ implies either $a_i = b_i \wedge a_j = b_j$ or $a_i \neq b_i \wedge a_j \neq b_j$; so $\mathrm{Ker}\, a \cap \mathrm{Ker}\, b \subseteq M^2 \cup (I \setminus M)^2$, which puts $[a = b]$ into $2^I|_{\mathfrak{F}}$. For $N = [R(a_1, \ldots, a_m)]$ we have $\bigcap_{i\leqslant m} \mathrm{Ker}\, a_i \subseteq N^2 \cup (I \setminus N)^2$ for the same reason, and so $N \in 2^I|_{\mathfrak{F}}$. If $M \in 2^I|_{\mathfrak{F}}$ and $a, b \in A$, $a \neq b$, the map $u\colon I \to A$,

$$u(x) = \begin{cases} a & \text{if } i \in M, \\ b & \text{if } i \notin M, \end{cases}$$

belongs to $A^I|_{\mathfrak{F}}$, and so by proving (MP) we prove the maximum property and the patchwork property. Let $\varphi(x_0, \ldots, x_n)$ be a formula, $a_1, \ldots, a_n \in A^I|_{\mathfrak{F}}$, and $\theta = \bigcap \mathrm{Ker}\, a_i$.

$$i \in [\exists x\; \varphi(x, a_1, \ldots, a_n)] \Leftrightarrow [i]\theta \subseteq [\exists x\; \varphi(x, a_1, \ldots, a_n)],$$

and so we pick $a_0 \in A^I$, which is constant on the θ-Blocks and which satisfies $\varphi(a_0, \ldots, a_n)$ for each $i \in [\exists x\; \varphi(x, a_1, \ldots, a_n)]$. Then $\theta \Rightarrow \mathrm{Ker}\, a_0 \in \mathfrak{F}$, and so $a_0 \in A^I|_{\mathfrak{F}}$ and $[\varphi(a_0, \ldots, a_n)] = [\exists x\; \varphi(x, a_1, \ldots, a_n)]$. We have now seen that limit powers are Boolean products with the maximum property, and so by reduction 3 the new stalks are elementary substructures of ultrapowers $A^I/_{\mathfrak{u}}$ of A.

1.15. Limit reduced powers

Let $\mathfrak{F}$ be a filter on $I\times I$, $\mathfrak{D}$ a filter on I and A a structure. The *limit reduced power* $A^I_{\mathfrak{D}}|_{\mathfrak{F}}$ of A is defined as the substructure of $A^I/_{\mathfrak{D}}$ of all $a/_{\mathfrak{D}}$ where $a\in A^I|_{\mathfrak{F}}$ which is just $(A^I|_{\mathfrak{F}})/_{\mathfrak{D}\cap 2^I|\mathfrak{F}}$. By 1.10, $A^I_{\mathfrak{D}}|_{\mathfrak{F}}$ has a Boolean product representation with the maximum property and stalks which are elementary substructures of ultrapowers of A. In fact, the new stalks in 1.14, 1.15 are limit ultrapowers $A^I_{\mathfrak{u}}|_{\mathfrak{F}}$ for the ultrafilters $\mathfrak{u}$ on I (in 1.15: extending $\mathfrak{D}$).

1.16. (Unbounded) Boolean powers

Let I be a Boolean space, $B=I^*$ its Boolean algebra of clopen sets, and A a structure. $D(I,A)$ denotes the structure of all maps $I\to A$ which are continuous on a dense-open subset of I. For $u,v\in D(I,A)$ we define $u\sim v$ if u and v coincide on a dense-open subset of I. The *Boolean power* $A[B]$ is defined as the structure $D(I,A)/\sim$. Observe that this is a special case of 1.15 because picking

$$\mathfrak{F}=\{\bigcup_{j\in J} M_j^2|\ J \text{ set},\ M_j\in B,\ M_j\cap M_j=\emptyset \text{ for } i\neq j,\ \bigcup_{j\in J} M_j \text{ dense}\}$$

and $\mathfrak{D}:=\{M\subseteq I|\ M \text{ dense-open}\}$ we get

$$D(I,A)=A^I|_{\mathfrak{F}}\quad\text{and}\quad D(I,A)/\sim\,=A^I_{\mathfrak{D}}|_{\mathfrak{F}}.$$

1.17. Bounded Boolean powers

If I is a Boolean space, $B=I^*$ its Boolean algebra of clopen sets and A a structure, then the bounded Boolean power $A[B]^*$ is the structure of all continuous maps $I\to A$. The embedding $A[B]^*\rightarrowtail A^I$ is a full Boolean product w.r. to B. We have seen in 1.1 that the patchwork property holds and that for $a,b\in A[B]^*$ we have $[a=b]\in B$. Now assume that $\varphi(x_1,\ldots,x_n)$ is any formula and $a_1,\ldots,a_n\in A[B]^*$; as each a_i has only finitely many images, we can decompose I into finitely many disjoint clopen sets $M_1\cup\ldots\cup M_k=I$ such that $a_1,\ldots,a_n$ all are constant on each of the $M_1,\ldots,M_k$. Thus each of the M_i's is either contained in $[\varphi(a_1,\ldots,a_n)]$ or disjoint from it, and so $[\varphi(a_1,\ldots,a_n)]$ is clopen as a finite union of some of the M_i's. We have thus proved that $A[B]^*\rightarrowtail A^I$ is a full Boolean product w.r. to B having the maximum property. We have already seen in example 1.13 that not each Boolean product with stalks which are all the same structure A (and the max. prop.) is necessarily a bounded Boolean power, but we have:

THEOREM. *Each full Boolean product $A\rightarrowtail C^I$ containing all the constant maps is a bounded Boolean power of C.*

To see this observe that each $a\in A$ coincides at each point with one of the constant maps c and $[a=c]$ is clopen. I is disjointly covered by these clopen sets, and so by compactness there are only finitely many c which proves a to be continuous. Conversely, each continuous map can be patched together from finitely many constant maps by 1.4.

COROLLARY. *If C is finite of finite type, each full Boolean product $A\rightarrowtail C^I$ is a bounded Boolean power.*

To see this let $C=\{c_1,\ldots,c_n\}$ and let $\varphi(x_1,\ldots,x_n)$ be an open formula describing up to isomorphism $c_1,\ldots,c_n$ in C. Thus $[\exists x_1\ldots\exists x_n\ \varphi(x_1,\ldots,x_n)]=I$ and has a solution $a_1,\ldots,a_n\in A$ by 1.4 (3). These elements $a_1,\ldots,a_n$ serve as the constant maps in the proof of the theorem.

1.18. Boolean-valued structures

As the next example we give a type of Boolean products which in general does not satisfy the maximum property.

Let A be a structure and B a Boolean algebra. A map assigning to each atomic formula φ over A a value $[\varphi]$ in B is called a Boolean *valuation* of A iff it satisfies the following conditions:

(1) $[a=b]=[b=a]$, $[a=b]\cap[b=c]\subseteq[a=c]$;

(2) For each fundamental n-ary function f:

$$[a_1=b_1]\cap\ldots\cap[a_n=b_n]\subseteq[f(a_1,\ldots,a_n)=f(b_1,\ldots,b_n)];$$

(3) $[a=b]=1\Leftrightarrow A\models[a=b]$;

(4) For each fundamental m-ary relation R

$$[a_1=b_1]\cap\ldots\cap[a_m=b_m]\cap[R(a_1,\ldots,a_m)]\subseteq[R(b_1,\ldots,b_m)]$$

and

$$[R(a_1,\ldots,a_m)]=1\Leftrightarrow A\models R(a_1,\ldots,a_m).$$

We say that $[\ldots]$ is *finitely patching* iff, for $b\in B$ and $a_1,a_2\in A$, there exists an $a_3\in A$ such that $b\subseteq[a_1=a_3]$ and $1-b\subseteq[a_2=a_3]$. Clearly, by 1.4 each Boolean product has the solution set map as a valuation which is finitely patching.

In order to construct a Boolean product from a valuation we need the following

LEMMA. *Let $[\ ]$ be a valuation of A in B.*

(1) *For each filter $\mathfrak{F}$ on B, $\theta_{\mathfrak{F}}:=\{(a,b)|\ [a=b]\in\mathfrak{F}\}$ is a congruence on A.*

(2) *The canonical map $A\to\prod_{u\in B^*} A/_{\theta_u}$ is a subdirect representation.*

(3) *For each atomic sentence over A we have* $[\varphi] = \{u \in B^* \mid [\varphi] \in u\}$.

Proof. (1) is an immediate consequence of (1)–(3) in the definition of [ ].

(2) $\forall u \in B^*\ (a, b) \in \theta_u \Leftrightarrow [a = b] \in \bigcap_{u \in B^*} u = \{1\} \Leftrightarrow A \models a = b$.

(3) $u \in [\varphi] \Leftrightarrow A/_{\theta_u} \models \varphi \Leftrightarrow [\varphi] \in u$.

From this lemma we immediately infer that each finitely patching Boolean valuation of A gives rise to a full Boolean product $A \rightarrowtail \prod_{u \in B^*} A/_{\theta_u}$ and the solution-set-map coincides with the Boolean valuation.

We can also formulate the maximum property in terms of the valuation by stating that for each formula $\varphi(x)$ over A there exists an $a \in A$ such that $[\varphi(a)] = [\exists x\ \varphi(x)]$. As it stands, this statement needs some explanation. [ ] was only defined for atomic sentences, but we can extend this to arbitrary formulas by embedding B into a complete Boolean algebra $\hat{B}$ and then defining $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$, $[\neg \varphi] = 1 - [\varphi]$ and $[\exists x\ \varphi(x)] := \sup_{a \in A} [\varphi(a)]$; now (MP) states that $\sup_{a \in A} [\varphi(a)]$ exists for trivial reasons because the set $\{[\varphi(a)] \mid a \in A\}$ has a greatest element. So by induction on the complexity of φ we get $[\varphi] \in B$ and we do not need the completion $\hat{B}$ after all.

We have now seen that in some sense the Boolean products are just the same as the Boolean-valued structures. The main difference is that for the Boolean-valued structures we have no information about the stalks except that they are homomorphic images of A.

1.19. Sub-Boolean powers

Sub-Boolean powers have also been introduced as *filtered* Boolean powers and they are introduced for finite structures only. Let A be a finite structure and C a substructure of the (bounded) Boolean power $A[B]\,(= A[B]^*)$. We call C a *sub-Boolean power* of A if C has the patchwork property, i.e. $c_1, c_2 \in C$ and $b \in B$ implies $c_3 \in C$, $b \subseteq [c_1 = c_3]\ \&\ 1 - b \subseteq [c_2 = c_3]$.

For later applications we need the characterization of sub-Boolean powers as filtered Boolean powers. A *filtered* Boolean power of A is determined by a Boolean algebra B and a sequence $(F_i \mid i \leqslant n)$ of filters on B corresponding to a sequence $(A_i \mid i \leqslant n)$ of substructures of A. It is defined as the structure C of all continuous functions $f\colon B^* \to A$ such that for each ultrafilter $u \in B^*$ extending F_i the value $f(u)$ belongs to A_i for all $i \leqslant n$. Clearly, each filtered Boolean power is a sub-Boolean power because the filters on B correspond to the closed subsets of B^* and the condition says that each $f \in C$ takes only values in A_i on the closed subset I_i of B^* corresponding to F; now the patchwork property is obvious.

If we assume conversely that $C \leqslant A[B]$ is a sub-Boolean power, we can see that the set $I_i = \{u \in B^* \mid \forall f \in C\ f(u) \in A_i\}$ is a closed subset of B^* because for the constant maps $\hat{a} \in A[B]$ with value $a \in A$ we have $I_i = \bigcap_{f \in C} \bigcap_{a \notin A_i} [f \neq \hat{a}]$. Now, if $f \in A[B]$ satisfies $u \in I_i \Rightarrow f(u) \in A_i$, we claim that f belongs to C. For each $u \in B^*$ we find $g_u \in C$ such that $g_u(u) = f(u)$, indeed, if this were not so, there would be a substructure A_i of A containing all $g(u) \cdot (g \in C)$ but not containing $f(u)$, which contradicts $u \in I_i \Rightarrow f(u) \in A_i$. Thus the $[g_u = f]$ form a cover of B^*, and so we find a finite subcover $[g_i = f]$, $i = 1, \ldots, m$, which we can assume to be disjoint. Now, the patchwork property of C yields $f \in C$. This shows that C is the filtered Boolean product determined by the filters $F_i = \bigcap_{u \in I_i} u$ $(i \leqslant n)$ on the Boolean algebra B.

As the last example we have the Hausdorff-sheaves of structures over a Boolean space. In fact, the structures of all global sections of such sheaves are precisely the Boolean products.

1.20. Boolean sheaves

A *Boolean sheaf of structures* is a triple $\boldsymbol{S} = (S, \eta, X)$ such that:

(1) S is a Hausdorff-space and X is a Boolean space;

(2) $\eta\colon S \to X$ is a local homeomorphism;

(3) for each $x \in X$ the sets $S_x := \{s \in S \mid \eta s = x\}$ form structures of the same type;

(4) for each n-ary $(n \geqslant 0)$ fundamental operation f the induced map $f\colon S^{(n)} := \{(s_1, \ldots, s_n) \in S^n \mid \eta s_1 = \ldots = \eta s_n\} \to S$ is continuous;

(5) for each m-ary fundamental relation R the set

$$\boldsymbol{R} := \{(s_1, \ldots, s_m) \in S^{(m)} \mid (s_1, \ldots, s_m) \in R\}$$

is open in S^m.

The spaces S and X are called the *sheaf-space* and the *base-space* of S and the structures S_x $(x \in X)$ are called the *stalks* of S.

A continuous map $\sigma\colon X \to S$ is called a *section* of $\boldsymbol{S}$ if $\eta \circ \sigma = \mathrm{id}_X$ and $\boldsymbol{\Gamma S}$ denotes the set of all section of $\boldsymbol{S}$.

THEOREM 1. $\boldsymbol{\Gamma S} \to \prod_{x \in X} S_x$ *is a full Boolean product.*

We have to show (1)–(3) of 1.3. We first prove (2).

Let $\sigma, \tau \in \boldsymbol{\Gamma S}$ and $x \in [\sigma = \tau]$, i.e. $\sigma x = \tau x$. Let N be a neighbourhood of x on which η is a homeomorphism. Then $\sigma^{-1}N \cap \tau^{-1}N \subseteq [\sigma = \tau]$ is an open set containing x, and thus $[\sigma = \tau]$ is open. Now assume $x \in [\sigma \neq \tau]$, i.e. $\sigma x \neq \tau x$. As S is Hausdorff, we find disjoint neighbourhoods N, M of σx and τx on which η is a homeomorphism. The set $\eta^{-1}N \cap \eta^{-1}M$ is an open neighbourhood of x entirely contained in $[\sigma \neq \tau]$; thus also $[\sigma \neq \tau]$ is open and hence $[\sigma = \tau]$ is clopen. The proof of the patchwork property (3) is quite obvious since the disjoint union of continuous maps both defined on disjoint clopen subsets of X is clearly continuous again. In order to see (1) pick $s \in S_x$ and a neighbourhood N of S such that η is a homeomorphism on N. ηN is a neighbourhood of $\eta s = x$ and hence we find a clopen subset $M \subseteq \eta N$ with $x \in M$. Let σ be any section and $\tau := \sigma|_{X \setminus M} \cup \eta^{-1}|_M$; then $\tau(x) = s$, which shows that the projections $\Gamma S \to S_x$ are all onto $(x \in X)$. Now assume R is a fundamental relation and $(s_1, \dots, s_n) \in S_{x_0}^n$ satisfies $(s_1, \dots, s_n) \in R$. Then $(s_1, \dots, s_n) \in \boldsymbol{R}$ and we pick neighbourhoods $N_1, \dots, N_n$ of $s_1, \dots, s_n$ on which η is a homeomorphism. Pick a clopen neighbourhood M_0 of $x_0 = \eta s_1 = \dots = \eta s_n$ contained in $\eta N_1 \cap \dots \cap \eta N_n$. Thus we find a section $\sigma_1, \dots, \sigma_n \in \Gamma S$ such that $\sigma_1(x_0) = s_1, \dots, \sigma_n(x_0) = s_n$ and for $y \in M_0\,(\sigma_1(y), \dots, \sigma_n(y)) \in R$. We do this for every $x \in X$ and thereby cover X by clopen sets M. As X is Boolean, we can assume that the M's form a finite disjoint cover and by (3) we can patch the σ's together so that we find $\sigma_1, \dots, \sigma_n \in \Gamma S$ such that $(\sigma_1, \dots, \sigma_n) \in R$ and $(\sigma_1(x_0), \dots, \sigma_n(x_0)) = (s_1, \dots, s_n)$. Altogether we see that ΓS is a subdirect product of its stalks S_x $(x \in X)$.

Also the opposite of the above theorem is true.

THEOREM 2. *If* $A \to \prod_{i \in I} A_i$ *is a full Boolean product, then there is a* $\boldsymbol{S} = (S, \eta, I)$ *with stalks* $S_i \simeq A_i$ $(i \in I)$ *such that* $\Gamma \boldsymbol{S} \simeq A$.

A proof can be obtained by means of the standard construction of a sheaf from a subdirect product by taking for S the disjoint union of the A_i's $(i \in I)$, for η the obvious map $S \to I$ and the sets $\{a(i) \mid i \in N\}$ for $a \in A$, $N \subseteq I$ clopen. Easy checking yields $A \simeq \Gamma S$.

Chapter 2

FEFERMAN–VAUGHT-TECHNIQUES

The Feferman–Vaught theorem gives a procedure for determining whether or not a given sentence φ holds in a direct product. S. Comer generalized this theorem to Boolean products, Weinstein gave a similar theorem for reduced products, and B. Węglorz and B. Banaschewski & E. Nelson gave a proof for limit reduced powers and thus also for Boolean powers and bounded Boolean powers. The most streamlined proof was given by H. Volger for Boolean-valued structures, and we give his proof here in a slightly generalized version.

For a language L a sequence $(\varphi_0, \dots, \varphi_n)$ is called *acceptable*, if φ_0 is a formula in the language of Boolean algebras in at most n free variables and $\varphi_1, \dots, \varphi_n \in L$; the acceptable sequence $(\varphi_0, \dots, \varphi_n)$ is called *partitioning* (or *special*) if the formulas $\varphi_1 \vee \dots \vee \varphi_n$ and $\neg(\varphi_i \wedge \varphi_j)$ for $i \neq j$ are logical tautologies.

2.1. LEMMA. *There is an effective procedure for constructing for each acceptable sequence* $(\varphi_0, \dots, \varphi_n)$ *a partitioning sequence* $(\psi_0, \dots, \psi_k)$ *such that, for each structure* A *and each Boolean valuation* [ ] *of* A *in* B *and elements* $a_1, \dots, a_m \in A$, *we have*

$$B \models \varphi_0([\varphi_1(a_1, \dots, a_m)] \dots [\varphi_n(a_1, \dots, a_m)])$$

iff

$$B \models \psi_0([\psi_1(a_1, \dots, a_m)] \dots [\psi_k(a_1, \dots, a_m)]).$$

Proof. Let $\{S_i \mid i = 1, \dots, k\}$ be the set all subsets of $\{1, \dots, n\}$ and set

$$\psi_i := \bigcup_{j \in S_i} \varphi_j \wedge \bigcup_{j \notin S_i} \neg \varphi_j \qquad (i = 1, \dots, k)$$

and

$$\psi_0(x_1, \dots, x_k) := \varphi_0\Big(\bigcup_{1 \in S_j} x_j \dots \bigcup_{n \in S_j} x_j\Big).$$

Clearly, the sequence $(\psi_0, \dots, \psi_k)$ is partitioning,

$$\bigcup_{i \in S_j} [\psi_j(a_1, \dots, a_m)] = \bigcup_{i \in S_j} \Big[\bigcup_{k \in S_j} \varphi_k(a_1, \dots, a_m) \wedge \bigcup_{k \notin S_j} \neg \varphi_k(a_1, \dots, a_m)\Big]$$
$$= [\varphi_i(a_1, \dots, a_m)] \quad \text{for } i = 1, \dots, n,$$

which proves the lemma.

2.2. The companion sequence

We now give an effective procedure for constructing for each formula $\varphi \in L$ an acceptable sequence $\varphi^* = (\varphi_0, \dots, \varphi_n)$, which we call the *companion sequence* for φ.

(i) α atomic: $\alpha^* := (x_1 = 1, \alpha)$.

(ii) $\varphi^* = (\varphi_0, \dots, \varphi_n)$, $\psi^* = (\psi_0, \dots, \psi_m)$

$$\Rightarrow (\varphi \wedge \psi)^* = (\delta_0, \varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m),$$

$$\delta_0(x_1, \dots, x_{n+m}) = \varphi_0(x_1, \dots, x_n) \wedge \psi_0(x_{n+1}, \dots, x_{n+m}).$$

(iii) $\varphi^* = (\varphi_0(x_1, \dots, x_n), \varphi_1, \dots, \varphi_n)$

$$\Rightarrow (\neg \varphi)^* = (\neg \varphi_0(1 - x_1, \dots, 1 - x_n), \neg \varphi_1, \dots, \neg \varphi_n).$$

(iv) If $\varphi^* = (\varphi_0, \dots, \varphi_n)$ and $(\psi_0, \dots, \psi_m)$ is the corresponding partitioning sequence, then $(\exists x\, \varphi)^* = (\delta_0, \exists x\, \psi_1 \dots \exists x\, \psi_m)$ where $\delta_0(x_n, \dots, x_m)$ is the formula

$$\exists z_1 \dots \exists z_m\; z_1 \cup \dots \cup z_m = 1 \wedge \bigwedge_{i < j \leqslant m} z_i \cap z_j = \emptyset \wedge \bigwedge_{i \leqslant m} z_i \leqslant x_i \wedge \psi_0(z_1, \dots, z_m).$$

The important property of the companion sequence is the following *monotonicity property* of φ_0:

If $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$ and $B \models \varphi_0(x_1, \ldots, x_n)$, then $B \models \varphi_0(y_1, \ldots, y_n)$.

Clearly, $x_1 = 1$ is monotone in this sense, and the conjunct of two monotone formulas is monotone. The formula δ_0 in (iv) obviously is monotone. If $\varphi(x_1, \ldots, x_n)$, is monotone and $\psi(x_1, \ldots, x_n) = \neg\varphi(1-x_1, \ldots, 1-x_n)$, we have to show that $\psi(x_1, \ldots, x_n)$ is monotone; so assume $x_1 \leqslant y_1, \ldots$ $\ldots, x_n \leqslant y_n$ and $B \models \psi(x_1, \ldots, x_n)$; then $B \models \neg\varphi(1-x_1, \ldots, 1-x_n)$. If $B \models \varphi(1-y_1, \ldots, 1-y_n)$ were true, we would have $B \models \varphi(1-x_1, \ldots, 1-x_n)$ by the monotonicity of φ — a contradiction; thus $B \models \psi(y_1, \ldots, y_n)$. So an induction over the complexity of φ_0 proves the monotonicity of φ_0.

We are now ready to formulate the Fefermann–Vaught theorem in its generalized form for Boolean products (or Boolean-valued structures).

2.3. Theorem. *Let $A \rightarrowtail \prod_{i \in I} A_i$ be a Boolean product w.r. to $B \leqslant 2^I$ with the maximum property. If φ is a sentence over A with the companion sequence $\varphi^* = (\varphi_0, \ldots, \varphi_n)$, then $A \models \varphi \Leftrightarrow B \models \varphi_0([\varphi_1], \ldots, [\varphi_n])$.*

Proof. We proceed by induction on the complexity of φ. The proofs for atomic φ or $\neg\varphi$, $\varphi \wedge \psi$ are easy, and so assume $\varphi = \exists x\, \psi(x)$.

Assume that the partitioning sequence corresponding to $\psi(x)^* = (\psi_0, \psi_1(x), \ldots, \psi_n(x))$ is $(\mu_0, \mu_1(x), \ldots, \mu_m(x))$. It is clear from 2.1 that also μ_0 is monotone.

"$\Rightarrow$": Assume $A \models \varphi$; so for some $a \in A$ $A \models \psi(a)$, which implies $B \models \psi_0([\psi_1(a)], \ldots, [\psi_n(a)])$ by the induction hypothesis. By 2.1 $B \models \mu_0([\mu_1(a)], \ldots, [\mu_m(a)])$ and thus $B \models \delta_0([\exists x\, \mu_1(x)], \ldots, [\exists x\, \mu_m(x)])$ as $[\mu_i(a)] \subseteq [\exists x\, \mu_i(x)]$ (δ_0 from 2.2 (iv)).

"$\Leftarrow$": If $B \models \delta_0([x\mu_1(x)], \ldots, [x\mu_m(x)])$, we find $N_1, \ldots, N_m \in B$ forming a partition of I such that $N_i \subseteq [\exists x\, \mu_i(x)]$. By the maximum property we find $a_i \in A$ such that $N_i \subseteq [\mu_i(a_i)]$ and by the patchwork property we find $a \in A$ such that $N_i \subseteq [a = a_i]$ $(i = 1, \ldots, m)$. Because of $B \models \mu_0(N_1, \ldots, N_m)$ and the monotonicity of μ_0 we have $B \models \mu_0([\mu_1(a)], \ldots$ $\ldots, [\mu_m(a)])$, which by 2.1 implies $B \models \psi_0([\psi_1(a)], \ldots, [\psi_n(a)])$. Thus the induction hypothesis yields $A \models \psi(a)$, and thus $A \models \exists x\, \psi(x)$ or $A \models \varphi$.

2.4. Corollary. *Let $[\ldots]$ be a finitely patching Boolean valuation of A in B with the maximum property. If φ is a sentence over A with the companion sequence $\varphi^* = (\varphi_0, \ldots, \varphi_n)$, then $A \models \varphi \Leftrightarrow B \models \varphi_0([\varphi_1], \ldots, [\varphi_n])$.*

We can formulate similar corollaries for all the examples 1.11–1.17 and so cover all the special theorems mentioned at the beginning of this chapter. Now, that we have this theorem, we should ask just what it

might be used for. We shall investigate this question in the next part of this chapter.

2.5. Theorem. *For each theory T of Boolean algebras t.f. we have*

(1) $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B}) \Rightarrow \mathrm{Th}(\Gamma_T^e\mathfrak{A}) = \mathrm{Th}(\Gamma_T^e\mathfrak{B})$.

(2) $\mathrm{Th}(\mathfrak{A}) \models$ *complete*, $A \in \mathfrak{A} \Rightarrow \mathrm{Th}(\Gamma_T^e\mathfrak{A}) = \mathrm{Th}(\{A[B]^* \mid B \models T\})$.

(3) $\mathrm{Th}(\mathfrak{A})$ *complete*, T *complete*, $A \in \mathfrak{A}, B \models T \Rightarrow \mathrm{Th}(\Gamma_T^e\mathfrak{A}) = \mathrm{Th}(A[B]^*)$.

For the proof of this theorem we need the following auxiliary lemma:

2.6. Lemma. *Let $\varphi^* = (\varphi_0, \ldots, \varphi_n)$ be a partitioning companion sequence for φ and assume that $\varphi_{m+1}, \ldots, \varphi_n$ are those φ_i for which $\mathfrak{A} \models \neg\varphi_i$. Then for each theory T of Boolean algebras*

$\Gamma_T^e\mathfrak{A} \models \varphi$ *iff*

$$T \vdash \forall z_1 \ldots \forall z_m (z_1 \cup \ldots \cup z_m = 1 \wedge \bigwedge_{i<j} z_i \cap z_j = 0) \to \varphi_0(z_1, \ldots, z_m, 0, \ldots, 0).$$

Proof. Suppose the second condition holds. Then for any $A \in \Gamma_T^e\mathfrak{A}$ the sets $[\varphi_1], \ldots, [\varphi_m]$ form a disjoint cover of B^*, and so $B \models \varphi_0([\varphi_1], \ldots$ $\ldots, [\varphi_m], \emptyset, \ldots, \emptyset)$; hence $A \models \varphi$ and $\Gamma_T^e\mathfrak{A} \models \varphi$. For the converse suppose $\Gamma_T^e\mathfrak{A} \models \varphi$ and let $b_1, \ldots, b_m \in B \models T$ with $b_1 \vee \ldots \vee b_m = 1$ and $b_i \wedge b_j = 0$ for $1 \leqslant i < j \leqslant m$. $X_i := \{u \in B^* \mid b_i \in u\}$ $(i \leqslant n)$ are disjoint covers of B^*. For $i \in m$ pick $A_i \in \mathfrak{A}$ such that $A_i \models \varphi$ and put $A = \prod_{i \leqslant m} A_i[x_i^*]^*$. Then $A \in \Gamma_T^e\mathfrak{A}$ and $B \models \varphi([\varphi_1], \ldots, [\varphi_m], \emptyset, \ldots, \emptyset) = \varphi(b_1, \ldots, b_m, 0, \ldots, 0)$.

Now we proceed to the proof of Theorem 2.5:

Let $\varphi^* = (\varphi_0, \ldots, \varphi_n)$ be a partitioning companion sequence for φ in the language of $\mathfrak{A}, \mathfrak{B}$.

(1) As $\mathfrak{A} \models \neg\varphi_i \Leftrightarrow \mathfrak{B} \models \neg\varphi_i$, the conclusion follows from Lemma 2.6.

(2) The completeness of $\mathrm{Th}(\mathfrak{A})$ leads to $C \models \varphi \Leftrightarrow B \models \varphi_0(B, \emptyset, \ldots, \emptyset)$ for $C \in \Gamma_T^e\mathfrak{A}$ and hence $C \models \varphi \Leftrightarrow A[B^*] \models \varphi$.

(3) The completeness of T shows that in proof (2) the choice of B is unimportant.

We are now going to apply the Feferman–Vaught technique to problems concerning $\aleph_0$-categoricity and 1st order decidability.

2.7. Theorem. *Assume that the language of A is countable. If A is finite or $\mathrm{Th}(A)$ is $\aleph_0$-categorical and B is finite or $\mathrm{Th}(B)$ is $\aleph_0$-categorical, then $A[B]^*$ is finite or $\mathrm{Th}(A[B]^*)$ is $\aleph_0$-categorical.*

Proof. By a theorem of Ryll-Nardzewski a countable theory with no finite models is $\aleph_0$-categorical iff there are only finitely many (principal) n-types consistent with T for each $n < \omega$. If $A[B]^*$ is not finite, $\mathrm{Th}(A[B]^*)$ is complete and has no finite models. Fixing $n < \omega$, suppose there are k n-types $\tau_1, \ldots, \tau_k$ realized in A and m k-types $\mu_1, \ldots, \mu_m$ realized in B.

If $f_1, \ldots, f_n \in A[B]^*$ let $\sigma(\vec{f}) = i$ if $B \models \mu_i([\tau_1(\vec{f})], \ldots, [\tau_k(\vec{f})])$. By 2.6 (3) we can assume B to be countable. If $\sigma(\vec{f}) = \sigma(\vec{g})$, we have homeomorphisms $\lambda_i\colon [\tau_i(\vec{f})] \to [\tau_i(\vec{g})]$ $(1 \leqslant i \leqslant n)$ which induce an automorphism α of $A[B]^*$ such that, for $x \in B^*$, $\alpha(f)(x)$ has the same n-type in A as $g(x)$. Now, by Theorem 2.3, $\alpha(f)$ and g realize the same type in $A[B]^*$.

2.8. THEOREM. *If* $\mathrm{Th}\,\mathfrak{A}$ *is decidable and* T *is a finitely axiomatized theory of Boolean algebras, then* $\mathrm{Th}(\Gamma^e_T\mathfrak{A})$ *is decidable.*

Proof. Let β be an axiom for T. Let φ be a sentence in the language of $\mathfrak{A}$ and let $\varphi^* = (\varphi_0, \ldots, \varphi_n)$ be partitioning companion sequences of φ. Use the decidability of $\mathfrak{A}$ to determine for which $i \leqslant n$ $\mathfrak{A} \models \neg\varphi_i$; w.l.o.g. these are $\varphi_{m+1}, \ldots, \varphi_n$. The theory of Boolean algebras is decidable (Tarski: Bull. AMS 55, 63–64), and so it is decidable whether or not

$$\beta \to \forall z_1 \ldots \forall z_m \Big(z_1 \vee \ldots \vee z_m = 1 \wedge \bigwedge_{i<j} z_i \wedge z_j = 0\Big) \to \varphi_0(z_1, \ldots, z_m, 0, \ldots, 0)$$

holds in every Boolean algebra, which by 2.6 is equivalent to $\Gamma^e_T\mathfrak{A} \models \varphi$.

2.9. COROLLARY. *If* $\mathfrak{A}$ *is a finite set of finite structures, then* $\Gamma^e\mathfrak{A}$ *has a decidable theory.*

We also want to get some decidability results for the operator Γ^a.

2.10. THEOREM. *If* $\mathfrak{A}$ *is a class of structures with* $\mathrm{Th}(\mathfrak{A})$ *model complete and* $A \rightarrowtail \prod_{x \in X} A_x$ *is a Boolean product with* $A_x \in \mathfrak{A}$ $(x \in X)$, *then* $A \mapsto \prod_{x \in X} A_x$ *has the maximum property.*

Proof. For each sentence φ there is an existential sentence ψ such that $\mathfrak{A} \models \varphi \Leftrightarrow \psi$ as $\mathrm{Th}(\mathfrak{A})$ is model complete, and thus $[\varphi] = [\psi]$ because all A_x belong to $\mathfrak{A}$. For each existential sentence ψ the solution set $[\psi]$ is open, and thus, for each sentence φ, $[\varphi]$ and $[\neg\varphi]$ are open and hence clopen; thus $A \rightarrowtail \prod_{x \in X} A_x$ has the maximum property.

2.11. COROLLARY. *If* $\mathrm{Th}(\mathfrak{A})$ *is model complete, then* $\Gamma^a_T\mathfrak{A} = \Gamma^e_T\mathfrak{A}$.

2.12. COROLLARY. *If* $\mathfrak{A}$ *is a finite set of finite structures such that no two different members of* $\mathfrak{A}$ *are embeddable into each other, then* $\mathrm{Th}(\Gamma^a\mathfrak{A})$ *is decidable.*

Proof. $\Gamma\mathfrak{A}$ is an axiomatic class whose all embeddings are isomorphisms and hence elementary; thus $\mathrm{Th}(\mathfrak{A})$ is model complete. Thus by 2.9 $\Gamma^a\mathfrak{A} = \Gamma^e\mathfrak{A}$ has a decidable theory.

In order to get a much better result we need a certain refinement of the Feferman–Vaught technique. This refinement is based on an idea of Ershov, who found a much more direct translation for Boolean powers. We want to use this translation for filtered Boolean powers and we are going to translate each sentence in the language of the structures in question into the language of Boolean algebras with quantification over filters; in fact, we can even effect this translation for sentences with quantification over "closed" congruences in the filtered Boolean powers. The most important result we are going to use is Rabin's theorem stating the decidability of the theory of countable Boolean algebras with quantification over filters (Trans. AMS 141, 1–35). For a substructure C of $A[B]$ and a closed subset I of B^* (i.e. $I = \{\mathfrak{u} \in B^* \mid \mathfrak{F} \subseteq \mathfrak{u}\}$ for some filter $\mathfrak{F}$ on B) the relation $\theta_I := \{(f, g) \mid f, g \in C,\ I \subseteq [f = g]\}$ is a congruence on C and will be called a *closed* congruence. In many examples all congruences are closed, or at least the closed congruences can be characterized internally within C. For the remainder of the section A is a fixed finite structure with substructures $(A_I \mid i \leqslant n)$ and L is the language of the filtered Boolean powers of A with quantification over closed congruences.

2.13. The Ershov translation

To each formula $\varphi \in L$ we want to assign a formula $\hat{\varphi}$ in the language of Boolean algebras with quantification over filters. Assume that we have numbered the elements of A, $A = \{a_1, \ldots, a_m\}$:

(1) $\varphi \equiv f(x_1, \ldots, x_k) = x_{k+1}$,

$$\hat{\varphi} \equiv \Big[\bigwedge_{A \models f(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}}} (x_{1,i_1} \wedge \ldots \wedge x_{k,i_k}) \vee x'_{k+1,i_{k+1}}\Big] = 1,$$

(2) $\varphi \equiv R(x_1, \ldots, x_k)$,

$$\hat{\varphi} \equiv \Big[\bigvee_{A \models R(a_{1_1}, \ldots, a_{1_k})} x_{1,i_1} \wedge \ldots \wedge x_{k,i_k}\Big] = 1,$$

(3) $\varphi \equiv (x, y) \in \theta$,

$$\hat{\varphi} \equiv \Big[\bigvee_{i \leqslant m} x_1 \wedge y_i\Big] \in \mathfrak{F}_0,$$

(4) $\widehat{\varphi \,\&\, \psi} = \hat{\varphi} \,\&\, \hat{\psi}$, $\widehat{\neg\varphi} = \neg\hat{\varphi}$,

(5) $\widehat{\exists\,\theta\varphi} = \exists\,\mathfrak{F}_0\hat{\varphi}$,

(6) $\widehat{\exists x\varphi} = \exists\, x_1 \ldots \exists\, x_m\, \hat{\varphi} \,\&\, x_1 \vee \ldots \vee x_m = 1 \,\&\, \bigvee_{i<j} x_i \wedge x_j = 0$
$$\&\, \bigwedge_{i \leqslant n} \bigvee_{a_j \notin A_i} x'_j \in \mathfrak{F}_i.$$

We now want to show that this translation does the same for the filtered Boolean powers of A as the Feferman–Vaught translation did.

If $C \leqslant A[B]$ is a sub-Boolean power, we let $\mathfrak{F}_i = \bigcap\{\mathfrak{u} \in B^* \restriction \forall c \in C\ c(\mathfrak{u}) \in A_i\}$ for $i = 1, \ldots, n$. For each $c \in C$ we let $c_i = [c = \hat{a}_i]$ $(i \leqslant m)$ where $\hat{a}_i \in A[B]$ is the constant map with value a_i. To each closed congruence θ on C corresponds a filter $\mathfrak{F}_\theta$ on B via $(x, y) \in \theta \Leftrightarrow [x = y] \in \mathfrak{F}_\theta$.

2.14. THEOREM. *Let $C \leqslant A[B]$ be a sub-Boolean power of A and $\varphi(c_1, \ldots, c_k, \theta_1, \ldots, \theta_j) \in L$. Then*

$$C \models \varphi(c_1, \ldots, c_k, \theta_1, \ldots, \theta_j) \quad \textit{iff} \quad B \models \hat{\varphi}(c_{11}, \ldots, c_{km}, \mathfrak{F}_{\theta_1}, \ldots, \mathfrak{F}_{\theta_j}),$$

where $\mathfrak{F}_1, \ldots, \mathfrak{F}_n, c_{11}, \ldots, c_{km}, \mathfrak{F}_{\theta_1}, \ldots, \mathfrak{F}_{\theta_j}$ are defined as above.

Proof. We proceed by induction over the complexity of φ.

(1) $\varphi \equiv f(c_1, \ldots, c_k) = c_{k+1}$,

$$\hat{\varphi} \equiv \Big[\bigwedge_{A \models f(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}}} (c_{1i_1} \wedge \ldots \wedge c_{ki_k}) \vee c'_{k+1, i_{k+1}}\Big] = 1.$$

$B \models \hat{\varphi}$ iff $f(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}} \Rightarrow c_{1i_1} \wedge \ldots \wedge c_{ki_k} \leqslant c_{k+1, i_{k+1}}$,

iff $f(a_{i_1}, \ldots, a_{i_k}) = a_{i_{k+1}}$ & $c_1(\mathfrak{u}) = a_{i_1}$ & ... & $c_k(\mathfrak{u}) = a_{i_k} \Rightarrow c_{k+1}(\mathfrak{u}) = a_{i_{k+1}}$,

iff $\forall \mathfrak{u} \in B^*\ f(c_1(\mathfrak{u}), \ldots, c_k(\mathfrak{u})) = c_{k+1}(\mathfrak{u})$,

iff $C \models f(c_1, \ldots, c_k) = c_{k+1}$.

(2) $\varphi \equiv R(c_1, \ldots, c_k)$,

$$\hat{\varphi} \equiv \Big[\bigvee_{A \models R(a_{i_1}, \ldots, a_{i_k})} c_{1i_1} \wedge \ldots \wedge c_{ki_k}\Big] = 1.$$

$B \models \hat{\varphi}$ iff $\forall \mathfrak{u} \in B^*\ \exists (a_{i_1}, \ldots, a_{i_k}) \in R^A\ c_1(\mathfrak{u}) = a_{i_1}$ & ... & $c_k(\mathfrak{u}) = a_{i_k}$,

iff $\forall \mathfrak{u} \in B^*\ (c_1(\mathfrak{u}), \ldots, c_k(\mathfrak{u})) \in R^A$,

iff $C \models R(c_1, \ldots, c_k)$.

(3) $\varphi \equiv (c, d) \in \theta$,

$$\hat{\varphi} \equiv \Big[\bigwedge_{i \leqslant m} c_i \wedge d_i\Big] \in \mathfrak{F}_\theta,$$

$B \models \hat{\varphi}$ iff $[c = d] \in \mathfrak{F}_\theta$,

iff $C \models (c, d) \in \theta$.

(4) $\wedge, \neg$.

(5) $\exists\, \theta$ are obvious.

(6) $\varphi \equiv \exists x\, \psi$,

$$\hat{\varphi} \equiv \exists x_1 \ldots \exists x_m\ \hat{\psi}\ \&\ x_1 \vee \ldots \vee x_m = 1\ \&\ \bigvee_{i<j} x_i \wedge x_j = 0\ \&\ \bigwedge_{i \leqslant n} \bigwedge_{a_j \notin A_i} x'_j \in \mathfrak{F}_i.$$

Each sequence satisfying $x_1 \vee \ldots \vee x_m = 1\ \&\ \bigvee_{i<j} x_i \wedge x_j = 0$ defines a disjoint cover of B^* and thus defines $f \in A[B]$, taking value a_i on $\mathfrak{u} \in x_i$ $(i = 1, \ldots, m)$. The condition $\bigwedge_{i \leqslant n} \bigwedge_{a_j \notin A_i} x'_j \in \mathfrak{F}_i$ means that, for $\mathfrak{u} \subseteq \mathfrak{F}_i$, $f(\mathfrak{u}) \in A_i$, because $f(\mathfrak{u}) = a_j \notin A_i$ would imply $x'_j \in \mathfrak{F}_i \subseteq \mathfrak{u}$ and $\mathfrak{u} \in x_j$, i.e. $x_j \in \mathfrak{u}$ — a contradiction; thus $f \in C$. Conversely, each $f \in C$ defines a sequence $x_i = [f = \hat{a}_i]$ $(i = 1, \ldots, m)$ satisfying those three conditions, because for $i \leqslant n$ and $a_j \notin A_i$ we have $[f \neq \hat{a}_j] \in \mathfrak{F}_i$. This shows that $A \models \varphi$ iff $B \models \hat{\varphi}$, which finishes the proof.

2.15. COROLLARY. *For each structure A the class of all sub-Boolean powers of A has a decidable theory.*

Proof. Let $\mathfrak{K}$ be the class of all sub-Boolean powers of A and let $\mathfrak{K}_\omega$ be the class of all countable members of $\mathfrak{K}$. Note that the members of $\mathfrak{K}_\omega$ are precisely those for which the corresponding Boolean algebra is countable. Clearly $\mathrm{Th}(\mathfrak{K}) = \mathrm{Th}(\mathfrak{K}_\omega)$. If φ is a sentence in the language of $\mathfrak{K}$ and $\hat{\varphi}$ is its Ershov translation, then by Theorem 2.14 $\mathfrak{K}_\omega \models \varphi$ iff $\forall \mathfrak{F}_1 \ldots \forall \mathfrak{F}_n\ \hat{\varphi}$ belongs to the theory (with quantification over filters) of all countable Boolean algebras, which by Rabin's theorem is decidable.

2.16. THEOREM. *If A is a finite structure and $\mathfrak{A}$ is a set of sub-structures of A such that each isomorphism between substructures of A isomorphic to a members of $\mathfrak{A}$ extends to an automorphism of A, then each countable member of $\Gamma^a\mathfrak{A}$ is a sub-Boolean power of A.*

Moreover, for each finite set of finite structures $\mathfrak{A}$ such a structure A exists.

The proof of this theorem is rather complicated and will be given in 2.22.

2.17. COROLLARY. *If $\mathfrak{A}$ is a finite set of finite structures, then $\Gamma^a\mathfrak{A}$ has a decidable theory.*

2.18. COROLLARY. *For each finite set $\mathfrak{A}$ of finite structures the theory of the countable member of $\Gamma^a\mathfrak{A}$ with quantification over closed congruences is decidable.*

Proof. Let A be a structure satisfying the properties of 2.17, $A_1, \ldots, A_m$ all substructures of A and $A_1, \ldots, A_n$ those isomorphic to some member of $\mathfrak{A}$. Then each countable member of $\Gamma^a\mathfrak{A}$ is a sub-Boolean power of A satisfying $\theta \equiv \forall x_1 \ldots \forall x_n \mathop{\&}_{i \leqslant n} x_i \in \mathfrak{F}_i \Rightarrow x_1 \vee \ldots \vee x_n = 1$. Conversely, each sub-Boolean power of A satisfying θ belongs to $\Gamma^a\mathfrak{A}$. Let $\mathfrak{K}$ be the class of countable members of $\Gamma^a\mathfrak{A}$. For each $\varphi \in L$ (see 2.14) we have $\mathfrak{K} \models \varphi$ iff $\forall F_1 \ldots \forall F_m\ \theta \to \hat{\varphi}$ holds for all countable Boolean algebras, which is decidable by Rabin's theorem.

We close this section with some remarks about what can be done for Boolean products which do not satisfy the maximum property. We have already handled the case of sub-Boolean powers, but in the general case we can still go up to existential formulas; however, we cannot say anything about more complex formulas.

2.19. THEOREM. *Let $f\colon A \rightarrowtail \prod_{x\in\Delta} A_x$ be a full atomic Boolean product and let $\alpha_0(\vec{x}), \ldots, \alpha_n(\vec{x})$ be atomic formulas over A with $\vec{x} = (x_1, \ldots, x_k)$; for some $m < n$ let*

$$\varphi_i = \exists\vec{x}\ \ \neg\alpha_i(\vec{x})\wedge\alpha_m(\vec{x})\wedge\ldots\wedge\alpha_n(\vec{x}) \quad \text{for} \quad i = 0, \ldots, m-1,$$

$$\varphi_m = \exists\vec{x}\ \ \alpha_m(\vec{x})\wedge\ldots\wedge\alpha_n(\vec{x})$$

and

$$\varphi = \exists\vec{x}\ \ \neg\alpha_0(\vec{x})\wedge\ldots\wedge\neg\alpha_{m-1}(\vec{x})\wedge\alpha_m(\vec{x})\wedge\ldots\wedge\alpha_n(\vec{x}).$$

(1) *$A\models\varphi_m$ iff $[\varphi_m] = X$.*

(2) *$A\models\varphi_0$ iff $[\varphi_m] = X$ & $[\varphi_0] \neq \emptyset$.*

(3) *If X has no isolated points: $A\models\varphi \Leftrightarrow [\varphi_m] = X$ & $\mathop{\&}_{i<m} [\varphi_i] \neq \emptyset$.*

Proof. We prove (3) and we shall see that for the special cases (1) and (2) the assumption that X has no isolated points is not necessary. Cleary, if $A\models\varphi$, the solution $\vec{a}\in A^k$ shows $[\varphi_m] = X$ and $[\varphi_i] \neq \emptyset$ for $i = 0, \ldots, m-1$, and so we assume for the converse that $[\varphi_m] = X$ and $[\varphi_i] \neq \emptyset$ for $i = 0, \ldots, m-1$. We pick $x_0, \ldots, x_{m-1}\in X$ such that $x_i\in[\varphi_i]$ $(i = 0, \ldots, m-1)$. As X has no isolated points, we have infinitely many choices for each x_i and thus we can assume them to be all different. For each x_i pick a clopen neighbourhood $N_i \subseteq [\varphi_i]$ such that for $i<j$ $N_i\cap N_j = \emptyset$ and $\vec{a}_i\in A^k$ such that for $x\in N_i$

$$A_x\models\neg\alpha_i(\vec{a}_i(x))\wedge\alpha_m(\vec{a}_i(x))\wedge\ldots\wedge\alpha_n(\vec{a}_i(x)) \quad (i = 0, \ldots, m-1).$$

For each point $x\in M := X\setminus\bigcup_{i<m} N_i$ we find $\vec{a}_x\in A^k$ such that

$$A_x\models\alpha_m(\vec{a}_x(x))\wedge\ldots\wedge\alpha_n(\vec{a}_x(x)),$$

and so the sets

$$N_x := [\alpha_m(\vec{a}_x)\wedge\ldots\wedge\alpha_n(\vec{a}_x)]$$

form a clopen cover of M; thus finitely many of them do so, say $N_{x_m}, \ldots, N_{x_r}$, and they have a disjoint clopen subcover $N_m, \ldots, N_r$ with $N_i \subseteq N_{x_i}$ $(i = m, \ldots, r)$. By the patchwork property there exists an $\vec{a}\in A^k$ such that for $j = 0, \ldots, r$ and $x\in N_j$ $\vec{a}(x) = \vec{a}_j(x)$ $(j<m)$ or $\vec{a}(x) = \vec{a}_{x_j}(x)$ $(j\geqslant m)$, respectively. Now by costruction $A\models\neg\alpha_i(\vec{a})$ for $i = 0, \ldots, m-1$, and $A\models\alpha_i(\vec{a})$ for $i = m, \ldots, n$, which proves $A\models\varphi$.

Remark. Observe that in 2.19 (3) the points with singleton stalk may be isolated.

2.20. COROLLARY. *Let $A \subseteq \prod_{x\in X} A_x$ be a full Boolean product.*

(1) *Each universal Horn-formula true in A is also true in $\prod_{x\in X} A_x$.*

(2) *A is a pure substructure of $\prod_{x\in X} A_x$ (i.e. each positive existential sentence over A true in $\prod_{x\in X} A_x$ is also true in A).*

(3) *If X has no isolated points, then A is an existential substructure of $\prod_{x\in X} A_x$ (i.e. each existential sentence over A true in $\prod_{x\in X} A_x$ is also true in A).*

Proof. Recall that a universal Horn-formula is a universally quantified conjunct of basic Horn-formulas and a basic Horn-formula is a disjunct of negated atomic formulas and at most one atomic formula; so a basic Horn-formula in either $\alpha(\vec{x})$ or $\neg\alpha(\vec{x})$ or $\beta_1(\vec{x})\wedge\ldots\wedge\beta_n(\vec{x})\to\alpha(\vec{x})$ respectively.

As $\forall\vec{x}\bigwedge_{i<n}\alpha(\vec{x}) \Leftrightarrow \bigwedge_{i<n}\forall\vec{x}\ \alpha(\vec{x})$, we can assume for our purposes that the formula φ is a universally quantified basic Horn-formula, and so its negation satisfies the conditions of 2.19 (1) or (2). Both A and $\prod_{x\in X} A_x$ are Boolean products over the same set X (with different topologies), and so, by 2.19, $A\models\varphi$ iff $\prod_{x\in X} A_x\models\varphi$. (2) is the special case of (1) where all basic Horn-formulas are of the form $\neg\alpha_i(\vec{x})$. For the proof of (3) we can assume that φ is of the form as in 2.19 because

$$\exists\vec{x}\bigvee_{i\leqslant k}\psi_k(\vec{x}) \Leftrightarrow \bigvee_{i\leqslant k}\exists\vec{x}\ \ \psi_k(\vec{x}).$$

Now, if $\prod_{x\in X} A_x = \varphi$, we clearly have $[\varphi_m] = X$ and $[\varphi_i] \neq \emptyset$. As X with the topology belonging to A has no isolated points, we can conclude that $A\models\varphi$ by 2.19 (3).

2.21. COROLLARY. *Assume that $\varphi \equiv \forall\vec{x}\ \ \beta(\vec{x})\to\gamma(\vec{x})$ where $\beta(\vec{x})$, $\gamma(\vec{x})$ are primitive positive. Then a full Boolean product satisfies φ provided all stalks satisfy φ.*

Proof. Assume that $A \subseteq \prod_{x\in X} A_x$ is a full Boolean product and all stalks satisfy φ. If $\vec{a}\in A^k$ such that $A\models\beta(\vec{a})$, we have $[\beta(\vec{a})] = X$ and hence $[\gamma(\vec{a})] = X$, but this implies $A\models\gamma(\vec{a})$ by 2.19 (1). We thus have proved $A\models\varphi$.

2.22. *Proof of* 2.16. $\mathfrak{A}$ is a finite set of finite structures which we order by embeddability $\leqslant$. We now prove by induction on $|\mathfrak{A}|$ that, for each countable $R \in \Gamma^a(\mathfrak{A}), R \to \prod_{i\in I} A_i$ and $i \in I$ there is a clopen neighbourhood N of i and for $j \in N$ embeddings $\sigma_j\colon A_j \to A$ such that for $r \in R$ and $a \in A$ the set $\{j \in N \mid \sigma_j(r(j)) = a\}$ is always clopen in I.

$|\mathfrak{A}| = 1$: Pick the same embedding for each $i \in I$.

$|\mathfrak{A}| > 1$: Pick $i \in I$. Let $\delta(x_1, \ldots, x_n)$ $(n = |A_i|)$ be the conjunct of all atomic and negatomic formulas holding for the n elements of A_i; then $N = [\delta(r_1, \ldots, r_n)]$ is a clopen neighbourhood of i for suitable $r_1, \ldots, r_n \in R$. The set $M = [\exists x\ \delta(r_1, \ldots, r_n) \wedge \bigwedge_{k=1}^{n} x \neq r_k]$ is an open set avoiding i. Let σ_i be any embedding of A_i into A. For each $j \in N$ the map $\lambda_j\colon r_k(i) \to r_k(j)$ $(k = 1, \ldots, n)$ is an embedding of A_i into A_j which is an isomorphism for $j \in N \setminus M$. For $j \in N \setminus M$ we pick $\sigma_j := \delta_i \circ \lambda_j^{-1}$. As all stalks over M properly contain A_i, we have by the induction hypothesis for each $j \in M$ a clopen neighbourhood N_j and embedding $\sigma_{jk}\colon A_k \to A$ for $k \in N_j$ such that for $r \in R$ and $a \in A$ the set $\{k \in N_j \mid \sigma_{jk}(r(k)) = a\}$ is clopen in I. As R is countable and all clopen sets of I are of the form $[r = s]$ or $[r \neq s]$ for $r, s \in R$, we can pick countably many $j_1, j_2, \ldots \in M$ such that the $N_{j_1}, N_{j_2}, \ldots$ cover M. Now, let $N_1 = N_{j_1}$, $N_2 = N_{j_2} \setminus N_1, \ldots, N_k = N_{j_k} \setminus \bigcup_{h<k} N_h, \ldots$, then $N_1, N_2, \ldots$ form a disjoint cover of M and they are all clopen. Moreover, for $t \in N_k$ set $\mu_t := \sigma_{j_k t}$. Assume that $\varphi_0, \ldots, \varphi_m$ are the different embeddings of A_i into A, and $\alpha_0, \ldots, \alpha_m$ are the automorphisms of A such that $\alpha_i \circ \varphi_i = \sigma_i$. For each $k \in \boldsymbol{N}$, $t \leqslant m$, the set $N_{kt} = \{j \in N_t \mid \mu_j \circ \lambda_j = \varphi_k\}$ is clopen. Now we pick $\sigma_j := \alpha_t \circ \mu_j$ where $j \in N_{kt}$.

$$\{j \in N \mid \sigma_j(r(j)) = a\} = [r = r_k] \cap N \quad \text{whenever} \quad \sigma_i(r_k(i)) = a$$

and

$$\{j \in N \mid \sigma_j(r(j)) = a\} = \bigcup_{k\in\boldsymbol{N}} \bigcup_{t=0}^{m} \{j \in N_{kt} \mid \sigma_j(r(j)) = a\}$$
$$= \bigcup_{k\in\boldsymbol{N}} \bigcup_{t=0}^{m} \{j \in N_{kt} \mid \mu_j(r(j)) = \alpha_t^{-1}(a)\},$$

which is open. The open sets form a finite disjoint cover of I and hence they are clopen. Now we are in a position to prove our theorem.

As in the proof for M, we know that we find embeddings $\sigma_i\colon A_i \to A$ $(i \in I)$ such that for each $r \in R$ and $a \in A$ the set $\{j \in N \mid \sigma_j(r(j)) = a\}$ is clopen in I. Consider the map $\hat{\sigma}\colon R \to A^I$, $\hat{\sigma}(r)(i) := \sigma_i(r(i))$. Clearly, $\hat{\sigma}$ is an embedding and $\hat{\sigma}(R)$ satisfies the patchwork property because R does. We only have to show $\hat{\sigma}(R) \subseteq A[I^*]$. If $r \in R$ and $a \in A$, then $[\hat{\sigma}(r) = \hat{a}] = \{j \in I \mid \sigma_j(r(j))\}$ is clopen, where $\hat{a}$ is the constant map $\hat{a}\colon I \to A$ with value a, but then $\hat{\sigma}(r)$ is continuous.

3. Axiomatic classes of Boolean products

In this chapter we investigate classes of structures $\mathfrak{A}$ for which the class $\Gamma\mathfrak{A}$ is axiomatic. The way we do this is via the generalization of Comer's theorem by investigating for a Boolean products $A \to \prod_{i\in I} A_i$ the special congruences

$$\Phi(x, y) := \{(a, b) \mid [x = y] \subseteq [a = b]\} \qquad (x, y \in A).$$

These congruences satisfy the assumptions of Theorem 1.10.2:

(1) $\Phi(x, y)$ is a congruence.

(2) $\Phi(x, x) = \Delta$.

(3) $(x, y) \in \Phi(x, y)$.

(4) $(x, y) \in \Phi(u, v) \to \Phi(x, y) \subseteq \Phi(u, v)$.

(5) $\Phi(x, y) \circ \Phi(u, v) = \Phi(z, w)$ for some z, w.

(6) $\Phi(x, y) \cap \Phi(u, v) = \Phi(z, w)$ for some z, w.

(7) $\Phi(x, y) \circ (\Phi(u, v) \cap \Phi(z, w)) \supseteq (\Phi(x, y) \circ \Phi(u, v)) \cap (\Phi(x, y) \circ \Phi(z, w))$.

(8) $\Phi(x, y) \subseteq \Phi(u, v) \Rightarrow \exists z \exists w\ \Phi(x, y) \circ \Phi(z, w) = \Phi(u, v)\ \&\ \Phi(x, y) \cap \Phi(z, w) = \Delta$.

(9) For each m-ary fundamental relation R and $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m) \in R$ we have

$$\mathop{\&}_{i\leqslant m} (a_i, b_i) \in \Phi(x, y) \circ \Phi(u, v) \Rightarrow \exists c_1 \ldots \exists c_m\ (c_1, \ldots, c_m) \in R\ \& \mathop{\&}_{i\leqslant m} \big((a_i, c_i) \in \Phi(x, y)\ \&\ (c_i, b_i) \in \Phi(u, v)\big).$$

If we also want to axiomatize atomic Boolean products, we have to replace (9) by

(10) For each m-ary fundamental relation R and $a_1, \ldots, a_m \in A$ there is a smallest $\Phi(x, y)$ such that

$$\exists b_1 \ldots \exists b_m (b_1, \ldots, b_m) \in R \mathop{\&}_{i\leqslant m} (a_i, b_i) \in \Phi(x, y).$$

In most cases we have a formula $\tau(x, y, u, v)$ expressing $(u, v) \in \Phi(x, y)$. Then we can reformulate (1)–(10) in a first-order fashion using τ. In several applications we find a formula τ expressing $(u, v) \in \Phi(x, y)$ at least for some pairs $(x, y) \in A^2$ but not for all. If we have a formula $\alpha(x, y)$ selecting the "good" pairs from A^2, we can again formulate (1)–(10) by relativizing all quantifiers to α, but then we have to add the additional condition:

(11) For $a, b \in A$ there is a smallest $\Phi(x, y)$ such that $(a, b) \in \Phi(x, y)$.

We want to give this translation into first-order sentences in the special case of a theory with a constant 0, where we only consider the congruences of the form $\Phi(0, x)$ where $A \models \alpha(x)$ and we have a ternary formula $\tau(x, u, v)$ expressing $(u, v) \in \Phi(0, x)$ provided $A \models \alpha(x)$. Then the axioms are:

1. $\alpha(x) \to \tau(x, u, u)$,
   $\alpha(a) \wedge \tau(x, u, v) \to \tau(x, v, u)$,
   $\alpha(a) \wedge \tau(x, u, v) \wedge \tau(x, v, w) \to \tau(x, u, w)$,
   for each n-ary fundamental operation f:

$$\alpha(x) \wedge \tau(x, u_1, v_1) \wedge \ldots \wedge \tau(x, u_n, v_n) \to \tau\big(x, f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)\big),$$

2. $\alpha(0) \wedge \big(\tau(0, x, y) \to x = y\big)$,

3. $\alpha(x) \to \tau(x, 0, x)$,

4. $\alpha(x) \wedge \alpha(y) \wedge \tau(x, 0, y) \wedge \tau(y, u, v) \to \tau(x, u, v)$,

5. $\alpha(x) \wedge \alpha(y) \to \exists z \ \alpha(z) \wedge \tau(z, 0, x) \wedge \tau(z, 0, y) \wedge$
   $\wedge \forall u \forall v \ \tau(z, u, v) \to \exists w \ \tau(x, u, w) \wedge \tau(y, w, v)$,

6. $\alpha(x) \wedge \alpha(y) \to \exists z \ \alpha(z) \wedge \tau(x, 0, z) \wedge \tau(y, 0, z) \wedge \forall u \forall v \ \tau(x, u, v) \wedge$
   $\wedge \tau(y, u, v) \to \tau(z, u, v)$,

7. $\alpha(x) \wedge \alpha(y) \wedge \alpha(z) \wedge \tau(x, u, s) \wedge \tau(y, s, v) \wedge \tau(x, u, t) \wedge \tau(z, t, v)$
   $\to \exists w \ \tau(x, u, w) \wedge \tau(y, w, v) \wedge \tau(z, w, v)$,

8. $\alpha(x) \wedge \alpha(y) \wedge \tau(x, 0, y) \to \exists z \ \alpha(z) \wedge \tau(x, 0, z) \wedge \exists u \ \tau(y, 0, u) \wedge$
   $\wedge \tau(z, u, x) \wedge \forall v \forall w \ \tau(y, v, w) \wedge \tau(z, v, w) \to v = w$,

9. for each m-ary fundamental relation R:

$$\alpha(x) \wedge \alpha(y) \wedge \exists c_1 \ldots \exists c_m$$

$$\bigwedge_{i \leqslant m} \tau(x, a_i, c_i) \wedge \tau(y, c_i, b_i) \wedge R(a_1, \ldots, a_m) \wedge R(b_1, \ldots, b_m)$$

$$\to \exists c_1 \ldots \exists c_m \ R(c_1, \ldots, c_m) \wedge \bigwedge_{i \leqslant m} \tau(x, a_i, c_i) \wedge \tau(y, c_i, b_i),$$

10. for each m-ary fundamental relation R:

$$\exists x \exists b_1 \ldots \exists b_m \, \alpha(x) \wedge R(b_1, \ldots, b_m) \wedge \bigwedge_{i \leqslant m} \tau(x, a_i, b_i) \wedge$$

$$\wedge \forall y \forall c_1 \ldots \forall c_m \ \alpha(y) \wedge R(c_1, \ldots, c_m) \wedge \bigwedge_{i \leqslant m} \tau(y, a_i, c_i) \to \tau(y, 0, x),$$

11. $\exists x \ \alpha(x) \wedge \tau(x, a, b) \wedge \forall y \ \alpha(y) \wedge \tau(y, a, b) \to \tau(y, 0, x)$.

As we shall soon see, the form of the axioms is rather unsatisfactory for some of the applications and so we shall sometimes have to refine the method. In the following chapter we refer to these axioms as to axioms 1–11.

If we have a class $\mathfrak{K}$ of structures such that for some formulas α, τ the axioms 1–11 hold, we know by Theorem 1.10.2 that the members of $\mathfrak{K}$ have a Boolean product representation and the stalks are homomorphic images of the members of $\mathfrak{K}$ such that the special congruences $\Phi(x, y)$ induce the two trivial cogruences. If the formulas α, τ are both positive, we infer that each stalk is a homomorphic image of some member of $\mathfrak{K}$ satisfying axiom (S):

$$\text{(S)} \quad \exists x \ \alpha(x, x) \wedge \exists x \ \exists y \ x \neq y \wedge \alpha(x, y) \wedge \forall x \forall y \forall u \forall v \ \alpha(x, y) \to [\tau(x, y, u, v) \Leftrightarrow (x = y \to u = v)].$$

Clearly, if we have a constant 0, 1-ary α, and 3-ary τ, we replace (S) by

$$(\mathrm{S}_0) \quad \alpha(0) \wedge \exists x \ x \neq 0 \wedge \alpha(x) \wedge \forall x \forall u \forall v \ \tau(x, u, v) \Leftrightarrow (x = 0 \to u = v).$$

If a class of structures satisfies (S), it is not clear that each Boolean product of these structures satisfies axioms 1–10, because α and τ need not define the desired congruences $\Phi(x, y)$ unless both α and τ are primitive (= existential conjuncts of atomic and negatomic formulas).

For the rest of this chapter we make the following general assumptions:

(A) Let $\mathfrak{K}$ be an axiomatic class of structures closed under subdirect products and homomorphic images. Let α, τ be primitive-positive formulas. For any formula s let $\mathfrak{K}_s := \{A \in \mathfrak{K} |\ A \models (s)\}$.

3.1. Theorem. *Under the assumptions* (A) *the class* $\Gamma \mathfrak{K}_s \subseteq \mathfrak{K}$ *is axiomatized by axioms* 1–9, 11, *relative to* $\mathfrak{K}$. $\Gamma^a \mathfrak{K}_s$ *is axiomatized by axioms* 1–11 *relative to* $\mathfrak{K}$.

Moreover, $A \in \mathfrak{K}$ *has a full Boolean product representation with stalks in* $\mathfrak{K}_s$ *(i.e. no singleton-stalks) iff it satisfies the additional axiom*

$$(12) \qquad \exists x \exists y \ \alpha(x, y) \ \wedge \ \forall u \forall v \ \tau(x, y, u, v).$$

This theorem is a first step towards characterizing $\Gamma \mathfrak{K}$ or $\Gamma^a \mathfrak{K}$ for an axiomatic class $\mathfrak{K}$. Unfortunately, we only get good general results for inductive classes $\mathfrak{K}$.

3.2. Corollary. *Under the assumptions* (A) *let* $\mathfrak{K} \subseteq \mathfrak{K}_s$ *have a set* Σ of positive $\forall \exists$*-axioms relative to* $\mathfrak{K}_s$. *Then* $\Gamma \mathfrak{A}$ *is axiomatized by* Σ *together with axioms* 1–9, 11.

Proof. By 2.21 all members of $\Gamma \mathfrak{A}$ satisfy Σ and, as the axioms in Σ are positive, they hold for all stalks of $R \in \Gamma \mathfrak{A}$.

3.3. Corollary. *Let* $\mathfrak{A}$ *be a class of structures having a set* Σ *of positive* $\forall \exists$*-axioms (relative to some axiomatic class* $\mathfrak{K}$*). Let* α, τ *be primitive-positive formulas such that* $\mathfrak{A} \models (\mathrm{S})$.

Then $\Gamma \mathfrak{A}$ *is axiomatized (relative to* $\mathfrak{K}$*) by* Σ *and axioms* 1–9, 11.

We can sharpen this result considerably if we assume that $\mathfrak{A}$ has an *encoding formula*, i.e. a primitive-positive formula $\tau(x, y, u, v)$ such that $\mathfrak{A} \models \tau(x, y, u, v) \leftrightarrow (x = y \to u = v)$, which means that we are considering the case above with a being a tautology, e.g. $a(x, y) \equiv x = x$. We say that a formula τ has no *negated relations* if in its prenex normal form it has no subformula of the form $\neg R(x_1, \ldots, x_n)$.

3.4. Corollary. *Let $\mathfrak{A}$ be a class of structures having a set Σ of $\forall\exists$-axioms without negated relations and having an encoding formula τ. Then $\Gamma\mathfrak{A}$ is axiomatized by axioms* 1–9, 11 *and a set $\bar{\Sigma}$ of positive $\forall\exists$-axioms related to Σ, as the proof shows.*

Proof. Let u, v be two new variables not occurring in Σ. In the normal form of $\tau \in \Sigma$, replace each inequality $s \neq t$ by $\tau(s, t, u, v)$ to obtain $\bar{\tau}(u, v)$ and put $\bar{\Sigma} := \{\forall u \forall v\ \bar{\sigma}(u, v) \mid \sigma \in \Sigma\}$. Clearly, if $A \models (\mathrm{S})$ with a being any tautology, then $A \models \Sigma$ iff $A \models \bar{\Sigma}$ but $\bar{\Sigma}$ is a positive set of $\forall\exists$-sentences and we are back to 3.3.

For relational structures it is unsatisfactory not to be able to negate relations, but we can overcome this in the same way as we did in 3.4. For each relation R we need an *encoding formula* for R, i.e. a primitive formula $\tau_R(a_1, \ldots, a_n, x, y)$ which is equivalent to $R(a_1, \ldots, a_n) \to x = y$. In this terminology the encoding formula above is an encoding formula for the equality $=$. Recall that an axiomatic class is *inductive* (closed under directed unions) iff it is $\forall\exists$-axiomatic.

3.5. Corollary. *Let $\mathfrak{A}$ be an inductive axiomatic class of structures having encoding formulas for equality and all fundamental relations. Then $\Gamma\mathfrak{A}$ is axiomatic, satisfying axioms* 1–9.

Recall that in 1.9 we characterized Boolean products as those subdirect products which are closed under the pointwise discriminator. We can use this result to get a much simpler axiomatization of Boolean products in a special situation.

3.6. Discriminator formulas

A formula $\delta(x, y, z, u)$ is said to be a *discriminator formula* for $\mathfrak{A}$ if it is primitive positive and $\mathfrak{A} \models \delta(x, y, z, u) \Leftrightarrow [(x = y \to z = u) \wedge (x \neq y \to x = u)]$. Clearly, the formula $\exists z\ \delta(x, y, u, z) \wedge \delta(x, y, v, z)$ is then an encoding formula for $\mathfrak{A}$ and $\exists r \exists s\ \ \delta(x, y, u, r) \wedge \delta(x, y, v, s) \wedge \delta(r, s, v, u)$ is equivalent to $x = y \vee u = v$ on $\mathfrak{A}$. Conversely, if $\mathfrak{A}$ has an encoding formula together with a primitive positive formula equivalent to $x = y \vee u = v$, then $\mathfrak{A}$ has a discriminator formula.

(1) If $\mathfrak{A}$ has a discriminator formula δ, then $\Gamma\mathfrak{A}$ is axiomatized relative to the class $\boldsymbol{P}_s\mathfrak{A}$ of all subdirect products of members of $\mathfrak{A}$ by the axiom

$$\text{(D)} \qquad \forall x \forall y \forall z \exists u\ \ \delta(x, y, z, u).$$

Clearly, each member of $\Gamma\mathfrak{A}$ then satisfies axioms 1–9 with the formulas

$$a(x, y) \equiv x = x \quad \text{and} \quad \tau(x, y, u, v) \equiv \exists z\ \ \delta(x, y, u, z) \wedge \delta(x, y, v, z).$$

If we axiomatize $\Gamma\mathfrak{A}$ with axiom (D), there are two ways in which new stalks might come into play: firstly, when we construct the full Boolean product by the reduction 1.5, and secondly, when we look into members of $\boldsymbol{SP}\mathfrak{A}$ satisfying (D). In both cases the new structures turn out to be substructures of members of $\mathfrak{A}$ for which δ is still a discriminator formula. In any case we know that the new subdirect factors are substructures of (ultraproducts of) members of $\mathfrak{A}$ which satisfy (D), as (D) is positive. Now a substructure of a member of $\mathfrak{A}$ satisfies (D) iff δ is a discriminator formula for it, and hence we have

(2) If $\mathfrak{A}$ has a discriminator formula δ and contains all members of $\boldsymbol{S}\mathfrak{A}$ for which δ is a discriminator formula, then $\Gamma\mathfrak{A}$ is axiomatized relative to $\boldsymbol{SP}\mathfrak{A}$ by the axiom (D).

(3) If $\mathfrak{A}$ is axiomatic with the assumptions of (2), then each member of $\Gamma\mathfrak{A}$ is a full Boolean product with stalks in $\mathfrak{A}$.

We close this section with some examples of encoding formulas and discriminator formulas.

3.7. Examples

(1) *Semilattices.* Let $S_2 = (\{0, 1\}, \cup, 0)$ be the 2-element join-semilattice with zero. As formulas a, τ for $\mathfrak{A} = \boldsymbol{I}(S_2)$ pick $a(x, y) \equiv x = 0$, $\tau(x, y, u, v) \equiv u \cup y = v \cup y$.

(2) *Distributive lattices.* Let $D_2 = (\{0, 1\}, \cup, \cap)$ be the 2-element distributive lattice. The formula $\tau(x, y, u, v) \equiv u \cup x \cup y = v \cup x \cup y \wedge u \cap \cap x \cap y = v \cap x \cap y$ is an encoding formula for D_2.

(3) *Bounded distributive lattices.* Let $\mathfrak{A}$ be any axiomatic class of bounded distributive lattices with join-irreducible unit. Pick a, τ as follows:

$$a(x, y) \equiv x = 0 \wedge \exists z\ \ y \cap z = 0 \wedge y \cup z = 1 \quad \text{and}$$

$$\tau(x, y, u, v) \equiv u \cup y = v \cup y.$$

(4) *Heyting algebras.* In all subdirectly irreducible Heyting algebras ($=$ bounded distributive relatively pseudocomplemented lattices) 1 is join-irreducible.

For pseudocomplemented lattices or relatively pseudocomplemented semilattices with join-irreducible 1 the formulas $a(x, y) \equiv 1 = x \wedge y \cup \cup (y \to 0) = 1$ and $\tau \equiv u \cap y = v \cap y$ or $\tau \equiv u \to v \geqslant y \wedge v \to u \geqslant y$ do the job.

(5) *Fields*. For each field F the following formula δ is a discriminator formula:

$$\delta(x, y, z, u) \equiv \exists r\ (y-x)^2\cdot r = y-x \wedge u = z+(x-z)\cdot r\cdot(y-x).$$

(6) *Discriminator varieties*. If $\mathscr{V}$ is a discriminator variety (see 1.9) and $\mathfrak{A}$ its class of subdirectly irreducibles, then $t(x, y, z) = u$ is a discriminator formula.

4. Atomic compactness

In this chapter we investigate atomic compact structures in $\Gamma\mathfrak{A}$ and then we concentrate on injectives, weak injectives and subretracts in $\Gamma\mathfrak{A}$ and more general in $\boldsymbol{SP}\mathfrak{A}$. A structure A is said to be *atomic compact* if each set Σ of atomic formulas over A has a solution in A provided its finite subsystems have solutions in A. It is known that A is atomic compact iff A is a *pure retract*, i.e. A is a retract of each pure extension of A. Special cases are injectives, weak injectives and subretracts in a class $\mathfrak{K}$. $I \in \mathfrak{K}$ is *injective* in $\mathfrak{K}$ iff for each embedding $A \overset{f}{\rightarrowtail} B$ each morphism $g\colon A \to I$ *lifts* to B, i.e. ex. $\bar{g}\colon B \to I$ such that $g = \bar{g}\circ f$. If we demand that only ontomorphisms or isomorphisms lift to B, we say that I is a *weak injective* or a *subretract* respectively in $\mathfrak{K}$. A special case of atomic compact structures are *compact* structures (structures which can carry a compatible compact Hausdorff topology). In fact, atomic compactness has been introduced as an algebraic version of topological compactness. If we speak about algebras only, we use the term *equationally compact* instead of atomic compact. The most important basic facts to keep in mind are that products and retracts of atomic compact structures are again atomic compact, and the same is true for the injectives in a class $\mathfrak{K}$. Regarding the weak injectives in $\mathfrak{K}$, we only know that they are closed under *subdirect retracts*, i.e. retracts of a direct product such that the projections are still onto. These well-known facts are easy to verify. Clearly, each finite structure is compact — with the discrete topology — and hence must be atomic compact. As each complete Boolean algebra B is a retract of some 2^I, the Boolean power $A[B]$ is a retract of A^I for each finite structure A because $A[\cdot]$ is a functor for each finite structure A. Thus each Boolean power of a finite structure by a complete Boolean algebra and all products of those are atomic compact. We are going to show that all atomic compact structures in $\Gamma\mathfrak{A}$ look like that under certain assumptions on $\mathfrak{A}$.

As the first result we keep in mind

4.1. THEOREM. *Let A_i $(i \in I)$ be a sequence of finite structures and B_i $(i \in I)$ a sequence of complete Boolean algebras; then $\prod_{i\in I} A_i[B_i]$ is an atomic compact structure.*

In general, not all atomic compact structures look like that, e.g. the real interval $[0, 1]$ is a distributive lattice but it does not have a representation as a Boolean product of smaller structures. Even if the class $\mathfrak{A}$ of structures is nice, the atomic compact members of $\boldsymbol{SP}\mathfrak{A}$ usually do not belong to $\Gamma\mathfrak{A}$, but for those which belong to $\Gamma\mathfrak{A}$ we can show that they look like those in 4.1.

Throughout this chapter we assume that $\mathfrak{A}$ is an $\forall\exists$-axiomatic class of structures and α, τ are primitive-positive formulas such that $\mathfrak{A}\models$(S) and $\Gamma^\alpha\mathfrak{A}$ is axiomatized by some positive $\forall\exists$-axioms together with axioms 1–11.

4.2. LEMMA. *Assume that $A \in \Gamma\mathfrak{A}$ is atomic compact.*

(1) *The congruences $\Phi(a, b) := \{(x, y) \mid A \models \tau(a, b, x, y)\}$ $(A \models \alpha(a, b))$ form a complete Boolean algebra.*

(2) *If φ is primitive-positive, then $[\varphi]$ is clopen.*

(3) *If U_i $(i \in I)$ are disjoint clopen subsets of the base-space and $a_i \in A$ $(i \in I)$, then there exists an $a \in A$ such that $U_i \subseteq [a = a_i]$ for $i \in I$.*

Proof. As A is atomic compact, each set Σ of primitive positive formulas has a solution iff each finite subset of Σ has a solution.

(1) Let $M \subseteq \{(a, b) \mid A \models \alpha(a, b)\}$; as our set Σ of primitive positive formulas we choose

$$\Sigma = \{\alpha(x, y) \wedge \tau(x, y, a, b) \mid A \models \alpha(a, b)\} \cup \{\tau(c, d, x, y) \mid A \models \alpha(c, d) \wedge \wedge\, \tau(c, d, a, b) \text{ for all } (a, b) \in M\}.$$

Σ is finitely solvable since the join of finitely many $\Phi(a, b)$'s is again of this form, and thus the solution (e, f) of Σ describes $\Phi(e, f) = \sup\{\Phi(a, b) \mid (a, b) \in M\}$.

(2) Let φ be primitive positive and x, y two new variables not in φ. $\bar{\varphi}(x, y, \bar{z})$ is the quantifier-free part of φ, where each equation $p = q$ is replaced by $\tau(x, y, p, q)$. The set

$$\Sigma = \{\alpha(x, y) \wedge \tau(x, y, a, b) \wedge \bar{\varphi}(x, y, \bar{z}) \mid [a = b] \subseteq [\varphi]\}$$

is finitely solvable and so its solution c, d, $\bar{e}$ gives a solution for φ on $[\varphi] = [c = d]$, a clopen set.

(3) By (1) we conclude that the whole base-space of A is of the form $[a = b]$, and so we can assume all clopen sets to be of that form, in particular $U_i = [b_i = c_i]$, $A \models \alpha(b_i, c_i)$. The set

$$\Sigma = \{\tau(b_i, c_i, x, a_i) \mid i \in I\}$$

is finitely solvable and its solution is the desired a.

Remark. If α is a tautology, we can allow in 4.2 (2) also φ to be an existential conjunct of atomic formulas and inequalities. In this case we replace each inequality $p \neq q$ by $\tau(p, q, a, b)$, where $\Phi(a, b) = \nabla$.

4.3. THEOREM. *Let α be a tautology. $A \in \Gamma\mathfrak{A}$ is atomic compact iff $A \simeq \prod_{i\in I} A_i[B_i]$ where $A_i \in \mathfrak{A}$ is finite and B_i is a complete Boolean algebra for each $i \in I$.*

Proof. For each finite structure C there is an existential conjunct δ_C of atomic formulas and inequalities such that $B = \delta_C$ iff C is a weak substructure of B. Clearly, for each finite structure C there is a finite set $\{C_1, \ldots, C_n\}$ of (partial) structures with at most one more element such that $B = \delta_C$ and $B \simeq C \Rightarrow B \models \delta_{C_i}$ for some $i \leqslant n$. Let $\mu_C = \delta_C \wedge \bigwedge_{i=1} \neg\delta_{C_i}$. By the above remark $[\mu_C]$ is clopen for each finite $C \in \mathfrak{A}$ and $i \in [\mu_C]$ iff the stalk A_i is isomorphic to C. Thus the sets $[\mu_C]$, $C \in \mathfrak{A}$ finite, form clopen disjoint subsets of the base space of A, and by 4.2 (1) the Boolean algebra B_C of clopen subsets of $[\mu_C]$ is complete. We now show that $T := \bigcup \{[\mu_C] | \ C \in \mathfrak{A}$ finite$\}$ is dense in the base space of A. If $a \neq b$ and $T \subseteq [a = b]$, then for $\varkappa > |A|$ we consider the set $\Sigma = \{\tau(x_i, x_j, a, b) | \ i < j < \varkappa\}$, which is finitely solvable in A since all stalks over $[a \neq b]$ are infinite. A global solution cannot exist since no stalk of size $\varkappa > |A|$ is possible, and thus T is dense in the base space of A. Let $\{A_i | \ i \in I\}$ be, up to an isomorphism, the set of all finite stalks of A and $B_i = B_{A_i}$ the corresponding complete Boolean algebra. Consider the morphism $A \xrightarrow{f} \prod_{i\in I} A_i[B_i]$. As T is dense, f is an embedding which by 4.1 (3) is onto and hence an isomorphism.

In the presence of a non-tautological α we cannot imitate this proof because we cannot express inequality in a primitive formula (only inequality between pairs satisfying α). Still we have a primitive positive formula δ_B satisfied precisely by those structures D which have a weak morphism $B \to D$. Assuming further that each finite member B of $\mathfrak{A}$ has only finitely many covers in $\mathfrak{A}$ with respect to embeddability, we can imitate the same proof as before:

4.4. THEOREM. *Let $\mathfrak{F}$ be the class of all finite members of $\mathfrak{A}$ and assume that:*

(i) *Each morphism in $\mathfrak{F}$ is an embedding.*

(ii) *Each member of $\mathfrak{F}$ has only finitely many covers in $\mathfrak{F}$ w.r. to embeddability.*

Then $A \in \Gamma\mathfrak{F}$ is atomic compact iff $A \simeq \prod_{i\in I} A_i[B_i]$ where $A_i \in \mathfrak{F}$ and B_i is a complete Boolean algebra for each $i \in I$.

If α is a tautology, 4.3 gives a full characterization of atomic compact structures in $\Gamma\mathfrak{A}$, but in general there could exist infinite atomic stalks and we cannot expect a similar characterization. Practically, nothing is known about conditions on $\mathfrak{A}$ that force all atomic compact structures of $\boldsymbol{SP}\mathfrak{A}$ into $\Gamma\mathfrak{A}$. The situation is quite different if we consider subretracts in $\boldsymbol{SP}\mathfrak{A}$. Each member of $\boldsymbol{SP}\mathfrak{A}$ can be embedded into a member of $\Gamma\mathfrak{A}$, and so, if we can state the axioms of $\Gamma\mathfrak{A}$ in such a way that they are preserved under retractions, then each subretract of $\boldsymbol{SP}\mathfrak{A}$ belongs to $\Gamma\mathfrak{A}$, and thus 4.3 or 4.4 can be applied. So we first investigate sentences which are preserved by retractions.

4.5. LEMMA. *Let $f\colon A \rightarrowtail B$, $g\colon B \twoheadrightarrow A$ be such that $g\circ f = 1_A$. Let μ, ν be positive and let δ, λ be existential formulas or sentences.*

(1) $B \models \mu \Rightarrow A \models \mu$.

(2) $A \models \delta \Rightarrow B \models \delta$.

(3) $B \models \forall\bar{x}\ (\delta(\bar{x}) \to \mu(\bar{x})) \Rightarrow A \models \forall\bar{x}\ (\delta(\bar{x}) \to \mu(\vec{x}))$,

(4) $\varphi \equiv \forall\bar{x}\ \big(\delta(\bar{x}) \to \exists\,\bar{y}\ [\mu(\bar{x}, \bar{y}) \wedge \forall\bar{z}\ (\lambda(\bar{x}, \bar{y}) \to \nu(\bar{x}, \bar{y}, \bar{z}))]\big)$,
$B \models \varphi \Rightarrow A \models \varphi$.

Proof. We show (4). Assume $B \models \varphi$ and $A \models \delta(\bar{a})$ for some sequence $\bar{a}$ in A. Then $B \models \delta(f\bar{a})$ since δ is existential, and hence $B \models \mu(f, \bar{a}, \bar{b})$ and $B \models \forall\bar{z}\ (\lambda(f\bar{a}, \bar{z}) \to \nu(f\bar{a}, \bar{b}, \bar{z}))$ for suitable $\bar{b}$ in B. As $\bar{a} = gf\bar{a}$ and μ is positive, we have $A \models \mu(\bar{a}, g\bar{b})$. Now assume $A \models \lambda(\bar{a}, \bar{c})$ for some $\bar{c}$ in A. Then $B \models \lambda(f\bar{a}, f\bar{c})$ and hence $B \models \nu(f\bar{a}, \bar{b}, f\bar{c})$. Again we get $A \models \nu(\bar{a}, g\bar{b}, \bar{c})$ since ν is positive and $gf\bar{c} = \bar{c}$, and this proves $A \models \varphi$.

If we now review the axioms for $\Gamma^\alpha\mathfrak{A}$ with respect to preservation under retract, we immediately see that the positive $\forall\exists$-axioms and axioms 1–4, 6, 7, 9–11 are preserved under retracts and we only have to consider axioms 5 and 8. Consider $\gamma(u, v, w, x, y, z) \equiv \tau(y, z, u, v) \wedge \wedge\, \tau(y, z, w, x) \wedge \exists\, r\ \ \tau(u, v, y, r) \wedge \tau(w, x, r, z)$. Clearly, $\mathfrak{A} \models \alpha(u, v) \wedge \alpha(w, x) \wedge \wedge\, \alpha(y, z) \to [\gamma(u, v, w, x, y, z) \Leftrightarrow (u = v \wedge w = x \Leftrightarrow y = z)]$, and this implies that $A \in \mathfrak{A}$ satisfies $\gamma(a, b, c, d, e, f)$ iff $\Phi(a, b)\circ\Phi(c, d) = \Phi(e, f)$. Now we can formulate axiom 5 using this primitive-positive formula γ:

5a. $\alpha(u, v) \wedge \alpha(w, x) \to \exists\, y \exists\, z\ \ \gamma(u, v, w, x, y, z) \wedge \alpha(y, z)$,

5b. $\alpha(u, v) \wedge \alpha(w, x) \wedge \alpha(y, z) \wedge \gamma(u, v, w, x, y, z) \wedge \tau(y, z, p, q)$
$\to \exists\, r\ \ \tau(u, v, p, r) \wedge \tau(w, x, r, q)$.

These axioms are covered by Lemma 4.5 as well, and so only the relative complementedness (axiom 8) remains to be investigated. In order to handle axiom 8 we need a formula $\varrho(x, y, u, v)$ expressing the fact that $\Phi(x, y) \cap \Phi(u, v) = \Delta$. On the stalks, ϱ must be equivalent to $x = y \vee \vee\, u = v$ under the proviso $\alpha(x, y)$ and $\alpha(u, v)$. If we have a positive-primitive formula ϱ with this property, we say that ϱ expresses *disjointness*; clearly disjointness of congruences is preserved under any homomorphism and thus axiom 8 is true for each homomorphic image of a member of $\Gamma\mathfrak{A}$. In all the examples in 3.7 such a formula ϱ is easy to find.

4.6. THEOREM. *If $\mathfrak{A}$ has a primitive-positive formula ϱ expressing disjointness, then each retract of a member of $\Gamma^\alpha\mathfrak{A}$ belongs to $\Gamma^\alpha\mathfrak{A}$.*

4.7. Corollary. *If $\mathfrak{A}$ has a primitive-positive formula ϱ expressing disjointness, then each subretract in $\boldsymbol{SP}\mathfrak{A}$ belongs to $\Gamma\mathfrak{A}$.*

If α is a tautology, then by 3.6 the existence of ϱ is equivalent to $\mathfrak{A}$ having a discriminator formula δ, and we have

4.8. Corollary. *If $\mathfrak{A}$ is an inductive class of structures having a discriminator formula, then each subretract in $\boldsymbol{SP}\mathfrak{A}$ is of the form $\prod\limits_{i\in I} A_i[B_i]$, where all $A_i \in \mathfrak{A}$ are finite and all B_i are complete Boolean algebras.*

In the case where α is not a tautology we have a similar result if $\mathfrak{A}$ is (up to isomorphism) a finite set of finite structures.

4.9. Corollary. *If $\mathfrak{A}$ is a finite set of structures, α, τ, ϱ being primitive-positive such that*

$$\mathfrak{A}\models(\mathrm{S})\wedge\big(\alpha(x, y)\wedge\alpha(u, v)\to[\varrho(x, y, u, v)\Leftrightarrow x = y\vee u = v]\big),$$

then each subretract in $\boldsymbol{SP}\mathfrak{A}$ is of the form

$$A_1[B_1]\times\ldots\times A_n[B_n]$$

where $A_1, \ldots, A_n \in \mathfrak{A}$ and $B_1, \ldots, B_n$ are complete Boolean algebras.

If we have a product of structures which is injective, neither of the factors has to be injective, but the matter is quite different if we consider weak injective or subretracts. Here we can characterize the weak injectives at least in the case where $\mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$ has *factorizable congruences*, i.e. if $A, B\in\mathfrak{K}$ and $\theta\in\mathrm{Con}(A\times B)$, then

$$\exists a\ (a, b)\theta(a, b') \Leftrightarrow \forall a\ \ (a, b)\ \theta(a, b').$$

In other words, $\theta = \theta_A\times\theta_B$ for some $\theta_A\in\mathrm{Con}A$ and $\theta_B\in\mathrm{Con}B$.

4.10. Lemma. *Let $\mathfrak{K}$ be a class with factorizable congruences A, B, $A\times B\in\mathfrak{K}$.*

If $A\times B$ is a (weak) injective (subretract) in $\mathfrak{K}$, then both A and B are (weak) injective (subretracts) in $\mathfrak{K}$. For (weak) injectives also the converse is true.

Proof. Assume that $A\times B$ is a weak injective (subretract) in $\mathfrak{K}$, $f\colon C \rightarrowtail D$ an embedding in $\mathfrak{K}$ and $g\colon C\to A$ an (onto-)morphism (isomorphism). There exists an $h\colon D\times B\to A\times B$ such that $h\circ(f\times 1_B) = g\times 1_B$, and we consider $k = p_1\circ h\colon D\times B\to A$. $\mathrm{Ker}\,k = \theta_D\times\alpha_B$ and for $c\in C$ we have

$$k(fc, b) = p_1\circ h\circ(f\times 1_B)(c, b) = p_1\circ(g\times 1_B)(c, b) = gc$$

and thus $\theta_B = \nabla_B$, which implies

$$\forall d\in D\,\forall b, b'\in B\ \ k(d, b) = k(d, b'),$$

and so for each $b\in B$ the map $d\to k(d, b)$ is a homomorphism $k_1\colon D\to A$ which satisfies $k_1\circ f = g$.

We get a similar result for (bounded) Boolean powers $A[B]^*$, the structure of all continuous maps $B^*\to A$ with the natural embedding $d_A\colon A\to A[B]^*$ which maps each $a\in A$ onto the constant map with value a.

4.11. Lemma. *If $A, A[B]^*\in\mathfrak{K}$ and $A[B]^*$ is an injective (a weak injective, a subtract) in $\mathfrak{K}$ ($\mathfrak{K}$ closed under bounded Boolean powers), then so is A.*

Proof. Assume that $f\colon C\rightarrowtail D$ is an embedding and $g\colon C\to A$ a morphism (onto-, iso-). Then we have an embedding $f^*\colon C[B]^*\to D[B]^*$ and a morphism $g^*\colon C[B]^*\to A[B]^*$ (onto-, iso-) such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, tail, "f"] \arrow[dr, "d_C"] \arrow[dd, "g"'] & & D \arrow[d, "d_D"] \\
& C[B]^* \arrow[r, tail, "f^*"] \arrow[d, "g^*"] & D[B]^* \arrow[dl, dashed, "h"] \\
A \arrow[r, tail, "d_A"] & A[B]^* &
\end{tikzcd}$$

By the injectivity-condition on $A[B]^*$ we have a commutative diagram-completion $h\colon D[B]^*\to A[B]^*$. For any $x\in B^*$ we have the projection $p_x\colon A[B]^*\to A$, which satisfies $p_x\circ d_A = 1_A$, and thus $p_x\circ h\circ d_D\colon D\to A$ is the desired morphism because

$$(p_x\circ h\circ d_D)\circ f = p_x\circ h\circ f^*\circ d_c = p_x\circ g^*\circ d_c = p_x\circ d_A\circ g = 1_A\circ g = g.$$

If B is a complete Boolean algebra, then $A[B]^*$ is a subdirect retract of A^{B^*}, and thus $A[B]^*$ is an injective or a weak injective just in case A is, but for subretracts the converse of 4.11 is not true in general. In view of the results above we only seem to be able to characterize fully (weak) injectives in $\mathfrak{K}$ if $\mathfrak{K}$ has factorizable congruences. Therefore we first study classes such that $\mathfrak{K}$ has factorizable congruences.

4.12. Factorizable congruences

We say that a class $\mathfrak{K}$ has a *congruence formula* $\sigma(x, y, u, v)$ iff for $A\in\mathfrak{K}$ we have $(x, y)\in\theta_A(a, b)$ (= the smallest congruence on A containing (a, b)) $\Leftrightarrow A\models\sigma(a, b, x, y)$. We only consider the case where σ is primitive-positive.

1) *If $\mathfrak{K}$ has a congruence formula, then $\mathfrak{K}$ has factorizable congruences.*

Proof. Assume $A, B, A\times B \in \mathfrak{K}$, $(a, b)\,\theta\,(a, b')$ for some $a \in A$, $\theta \in \operatorname{Con} A\times B$. Clearly, $\sigma(a, a, x, x)$ and $\sigma(b, b', b, b')$ and hence $(x, b)\,\theta\,(x, b')$ for all $x \in A$.

2) *If σ is a congruence formula for $\mathfrak{A}$, it is also a congruence formula for $\Gamma\mathfrak{A}$.*

Proof. The fact that σ defines congruences is expressed by axioms like axiom 1 in § 3, which by 2.21 are preserved by forming Boolean products. Now, clearly, $(a, b) \in \theta(c, d)$ implies $(a_i, b_i) \in \theta(c_i, d_i)$ in all stalks, and thus $\sigma(c, d, a, b)$ holds.

3) *If $\mathfrak{A}$ is a finite set of finite structures with factorizable congruences, then $\mathfrak{A}$ has a congruence-formula σ.*

Proof. Assume that $\mathfrak{A}$ has factorizable congruences and let $A = \prod_{i\leqslant n} A_i$ be a finite product of members of $\mathfrak{A}$ with $a, b, c, d \in A$ such that for each quadruple $x, y, u, v \in \mathfrak{B} \in \mathfrak{A}$ with $(u, v) \in \theta_B(x, y)$ there exist an $i \leqslant n$ and an isomorphism $f_i\colon B \to A_i$ such that $f_i(u) = a_i, f_i(v) = b_i, f_i(x) = c_i$, $f_i(y) = d_i$ and such that, for each $i \leqslant n$, $(a_i, b_i) \in \theta_{A_i}(c_i, d_i)$; then $\theta_A(c, d) = \prod_{i\leqslant n} \theta_{A_i}(c_i, d_i)$ since A has factorizable congruences and $(a, b) \in \theta_A(c, d)$. By Malcev's Lemma there exists a primitive-positive formula σ such that $\sigma(c, d, a, b)$ which by construction is a congruence formula for $\mathfrak{A}$.

4) *If $\mathfrak{A}$ has a congruence formula, then each quotient of $A \in \Gamma\mathfrak{A}$ again satisfies axiom* 8.

Proof. On $\mathfrak{A}$ (and $\Gamma\mathfrak{A}$) the following axioms hold:

(i) $\alpha(a, b)\wedge\tau(a, b, x, y)\wedge\tau(c, d, y, z)$
$$\to \exists r\ \sigma(c, d, x, r)\wedge\tau(a, b, r, z),$$

(ii) $\alpha(c, d)\wedge\alpha(e, f)\wedge\sigma(a, b, x, y)\wedge\tau(c, d, y, z)\wedge\sigma(a, b, x, u)\wedge$
$$\wedge\,\tau(e, f, u, z)\to\exists v\ \sigma(a, b, x, v)\wedge\tau(c, d, v, z)\wedge\tau(e, f, v, z).$$

The first axiom implies $\theta\circ\Phi(a, b) = \Phi(a, b)\circ\theta$ and the second
$$\big(\theta\circ\Phi(b, c)\big)\wedge\big(\theta\circ\Phi(e, f)\big) = \theta\circ\big(\Phi(b, c)\wedge\Phi(e, f)\big);$$
so each homomorphism maps disjoint $\Phi(a, b)$'s onto disjoint congruences, which makes axiom 8 hold for each homomorphic image of $A \in \Gamma\mathfrak{A}$.

This result enables us to characterize fully (weak) injectives in the case where $\mathfrak{A}$ has factorizable congruences.

4.13. Theorem. *Let $\mathfrak{A}$ be an $\forall\exists$-class of structures with an encoding formula τ and a congruence-formula σ. $I \in \mathfrak{K} := \boldsymbol{SP}\mathfrak{A}$ is a (weak) injective in $\mathfrak{K}$ iff $I \simeq \prod_{i\in I} A_i[B_i]$ where, for each $i \in I$, $A_i \in \mathfrak{A}$ is finite and a (weak) injective in $\mathfrak{K}$ and B_i is a complete Boolean algebra.*

Proof. By 4.12 (4) I satisfies axiom 8 and thus belongs to $\Gamma^a\mathfrak{A}$, whence by 4.3 has the desired form. By 4.10 and 4.11 all A_i are (weak) injectives in $\mathfrak{K}$. To see the converse we just use the fact that weak injectives are closed under subdirect retracts.

4.14. Theorem. *Let $\mathfrak{A}$ be finite set of finite structures satisfying* (S) *and having factorizable congruences. $I \in \mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$ is a (weak) injective in $\mathfrak{K}$ iff $I \simeq \prod_{i\leqslant n} A_i[B_i]$ where, for each $i \leqslant n$, $A_i \in \mathfrak{A}$ is a (weak) injective in $\mathfrak{K}$ and B_i is a complete Boolean algebra.*

Proof. By 4.12, I belongs to $\Gamma\mathfrak{A}$ and the proof proceeds as before.

In these theorems we reduce the question of determining the (weak) injectives in $\mathfrak{K}$ to determining those members of $\mathfrak{A}$ which are weak injectives in $\mathfrak{K}$. A proof of this might still be very difficult, and so we investigate under which conditions on $\mathfrak{K}$ we can even reduce to weak injectivity within $\mathfrak{A}$.

4.15. Lemma. *Assume that $\mathfrak{K}$ is congruence-distributive and all members of $\mathfrak{A}$ are finitely subdirectly irreducible (i.e. Δ is $\cap$-irreducible). Then $A \in \mathfrak{A}$ is a weak injective in $\mathfrak{K}$ iff A is a weak injective in $\mathfrak{A}$. Moreover, if each $A \in \boldsymbol{S}\mathfrak{A}$ is finitely subdirectly irreducible, then $A \in \mathfrak{A}$ is injective in $\mathfrak{K}$ iff A is injective in $\mathfrak{A}$.*

Proof. Assume that A is a weak injective in $\mathfrak{A}$ and $B \overset{f}{\rightarrowtail} C$ is an embedding in $\mathfrak{K}$ and $g\colon B \twoheadrightarrow A$ an ontomorphism. $C \in \boldsymbol{SP}\mathfrak{A}$ has an embedding $h\colon C \rightarrowtail \prod_{i\in I} C_i$ $(C_i \in \mathfrak{A})$.

As A is finitely subdirectly irreducible, the congruence $\operatorname{Ker} g$ of B is meet-prime in $\operatorname{Con} B$, and so by Jónsson's lemma there is an ultrafilter U on I such that the restriction of the congruence defined by U to B is smaller than $\operatorname{Ker} g$. Thus we have the following situation:

$$\begin{tikzcd}
 & B \arrow[r, tail, "f"] \arrow[dl, two heads, "g"'] \arrow[d] & C \arrow[r, tail, "h"] \arrow[d] & \Pi C_i \arrow[d, "p_U"] \\
A & B/_U \arrow[l, dashed, "k"] \arrow[r, tail] & C/_U \arrow[r, tail] & \Pi C_i/_U
\end{tikzcd}$$

$\prod C_i/_U$ is an ultraproduct of members of $\mathfrak{A}$ and thus itself belongs to $\mathfrak{A}$, and so the weak injectivity of A in $\mathfrak{A}$ yields a diagram completion $l\colon \prod C_i/_U \to A$ and altogether we have $l\circ p_U\circ h\colon C \to A$, which is the desired diagram-completion shoving the weak injectivity of A. If we consider injectives we do not assume g, to be onto and, in order for the same proof to work, we have to assume that the image of g is also finitely subdirectly irreducible. Then exactly the same proof works with $A' = g(B)$.

If $\mathfrak{A}$ is a finite set of finite structures, we can eliminate $\forall$-quantifiers in formulas over $\mathfrak{A}$ by introducing enough constants in the language of $\mathfrak{A}$.

This allows us then to have α and τ of arbitrary quantifier-type and we still can get 4.15 to hold. We just have to investigate what influence the introduction of new constants has on the weak injectives.

Let $\mathfrak{A}$ be a finite set of finite structures and $\mathfrak{A}^*$ the same set endowed with a finite set of new constants. Let $\mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$ and $\mathfrak{K}^* = \boldsymbol{SP}\mathfrak{A}^*$. We have a map $U\colon \mathfrak{K}^* \to \mathfrak{K}$, which forgets the new constants again.

4.16. Lemma. *If $A \in \mathfrak{K}$ is a (weak) injective in $\mathfrak{K}$, then there exists a (weak) injective A^* in $\mathfrak{K}$ such that $A = U(A^*)$.*

Proof. Assume that A is a (weak) injective in $\mathfrak{K}$. Then A has an embedding $A \to \prod_{i\in I} A_i$ $(A_i \in \mathfrak{A})$ which has a left inverse $g\colon \prod_{i\in I} A_i \to A$. We now endow the A_i $(i \in I)$ with the new constants to obtain A_i^* and define the new constants on A by $g\colon \prod_{i\in I} A_i^* \to A^*$. We now claim that A^* is a (weak) injective in $\mathfrak{K}^*$. Assume that $B^* \rightarrowtail C^*$ is an embedding in $\mathfrak{K}^*$ and $B^* \to A^*$ is an (onto-)morphism. Now by the (weak) injectivity of A we have a diagram completion $U(C^*) \to U(A^*) = A$, which clearly preserves all the constants because they are already inside B^*; so in fact we have a diagram completion $B^* \to A^*$ which shows the (weak) injectivity of A^*.

If we have an $\exists\forall$-conjunct of atomic formulas, we can introduce new constants $C = \{c_1, \ldots, c_n\}$ naming each element of each $A \in \mathfrak{A}$, and then we replace $\forall x_1 \ldots \forall x_n\, \beta(x_1, \ldots, x_n)$ by $\bigwedge_{c_1,\ldots,c_n\in C} \beta(c_1, \ldots, c_n)$, obtaining a primitive-positive formula over $\mathfrak{A}^*$ ($\mathfrak{A}$ with the new constants from C).

4.17. Corollary. *Let $\mathfrak{A}$ be a finite set of finite structures with factorizing congruences and such that $\mathfrak{A} \models (\mathrm{S})$ for some $\exists\forall$-conjuncts of atomic formulas. Then $I \in \mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$ is a (weak) injective in $\mathfrak{K}$ iff $I \simeq \prod_{i\leqslant n} A_i[B_i]$ where, for each $i \leqslant n$, $A \in \mathfrak{A}$ is a (weak) injective in $\mathfrak{A}$ and B_i is a complete Boolean algebra.*

In the congruence-distributive case we can also use 4.15 for a reduction to the question of injectivity of the stalks $\mathfrak{A}$.

5. Model companions

In search of model-complete theories A. Robinson invented the notion of the *model-companion* T^* of a theory T, i.e. each model of T^* is a substructure of some model of T and vice versa (T and T^* are *model-consistent*) and T^* is model complete. If T is an $\forall\exists$-theory, then each model of T^* also satisfies T. Abusing terminology, we also say that $\mathrm{Mod}(T^*)$ is the *model companion* of $\mathrm{Mod}(T)$. Not each theory has a model companion but if T has a model companion T^* then $\mathrm{Mod}(T^*)$ is the class of all structures which are *existentially closed* in $\boldsymbol{S}\,\mathrm{Mod}(T)$, i.e. each finite set of atomic and negatomic formulas over $A \in \mathrm{Mod}(T^*)$ which has a solution in some extension $B \in \mathrm{Mod}(T)$ of A $(A \subseteq B)$ already has a solution in A. In other words, each embedding $f\colon A \to B \in \mathrm{Mod}(T)$ is existential. Sometimes the notion of algebraically closed structure is used. A structure A is *algebraically closed* in $\mathfrak{A}$ if each embedding $f\colon A \to B \in \mathfrak{A}$ is pure (i.e. each finite set of atomic formulas over A which has a solution in some extension $B \in \mathfrak{A}$ of $\mathfrak{A}$ has a solution in A. If $\mathfrak{A}$ has an encoding formula τ and all *relations defined algebraically* (i.e. $R(a_1, \ldots, a_n) \Leftrightarrow \varphi(a_1, \ldots, a_n)$ for some open formula ϱ without relations), the two notions coincide for members of $\mathfrak{A}$, because each inequality $x \neq y$ can be replaced by $\tau(x, y, a, b)$ where a, b are different elements in A. The only exception is the singleton-structure, which is always algebraically closed but existentially closed only in case it is not embeddable into any larger structure in $\mathfrak{A}$. For each class of structures $\mathfrak{K}$ the class of all members of $\boldsymbol{S}\mathfrak{K}$ existentially closed in $\mathfrak{K}$ is denoted by $\mathfrak{K}^*$. If both $\mathfrak{K}$ and $\mathfrak{K}^*$ are axiomatic, then $\mathfrak{K}^*$ is the model companion of $\mathfrak{K}$ and we have $\mathrm{Th}(\mathfrak{K}^*) = \mathrm{Th}(\mathfrak{K})^*$.

In this section we want to use the Boolean product techniques to investigate the existentially closed members of $\boldsymbol{SP}\mathfrak{A}$ for some classes $\mathfrak{A}$. Throughout this section we assume that $\mathfrak{A}$ is an axiomatic class of structures which has a discriminator formula δ and algebraically defined relations (hence $\Gamma\mathfrak{A} = \Gamma^a\mathfrak{A}$ and we can assume the members of $\mathfrak{A}$ to be algebras only), and we assume $\mathfrak{A}$ to contain all structures of $\boldsymbol{HSP}\mathfrak{A}$ for which δ is a discriminator formula. Let $\mathfrak{A}^*$ be the class of all algebraically (= existentially) closed members of $\mathfrak{A}$. If $\mathfrak{A}^*$ is axiomatic, note that it has a set of $\forall\exists$-axioms. Under the assumptions just made, each embedding $f\colon A \rightarrowtail B$ in $\Gamma\mathfrak{A}$ induces embeddings between the stalks. If A_i is a stalk of A and B_j is a stalk of B with the projections $p_i\colon A \twoheadrightarrow A_i$, $q_j\colon B \twoheadrightarrow B_j$, then f *induces* a morphism $f_{ij}\colon A_i \to B_j$ iff $\mathrm{Ker}\, p_i \subseteq f^{-1}(\mathrm{Ker}\, q_j)$. In this case, f_{ij} is defined by $f_{ij}(p_i a) := \lambda_j(f(a))$. For each congruence $\theta \in \mathrm{Con} B$ we call $f^{-1}(\theta) := \{(x, y) \mid (fx, fy) \in \theta\} \in \mathrm{Con} A$ the *restriction* of θ to A. In 2.13 we studied *closed* congruences (congruences of the form $\{(x, y) \mid N \subseteq [x = y]\}$) which are characterized by $(x, y) \in \theta \;\&\; A \models \tau(x, y, u, v) \Rightarrow (u, v) \in \theta$, and we saw that the maximal closed congruences (together with ∇) determine the stalks. Recall that τ was defined by $\tau(x, y, u, v) \equiv \exists z\, \delta(x, y, u, z) \wedge \wedge\, \delta(x, y, v, z)$.

5.1. Lemma. *Let $f\colon A \rightarrowtail B$ be an embedding in $\Gamma\mathfrak{A}$.*

(1) *A congruence θ is closed iff it preserves the discriminator, i.e.*

$$(x, x'), (y, y'), (z, z') \in \theta \;\&\; \delta(x, y, z, u) \;\&\; \delta(x', y', z', u') \Rightarrow (u, u') \in \theta.$$

(2) *If Φ is a closed congruence on B, then its restriction $f^{-1}(\Phi)$ to A*

is a closed congruence on A, which, in case Φ is maximal, is either maximal or ∇.

(3) *If θ is a closed congruence on A, then*

$$\langle f\theta\rangle := \{(u, v)|\ B \models \tau(fx, fy, u, v),\ (x, y) \in \theta\}$$

is a closed congruence on B whose restriction to A is θ.

Proof. We prove (1) in three steps.

(a) If θ preserves the discriminator, then $(x, y) \in \theta\ \&\ \tau(x, y, u, v) \Rightarrow (u, v) \in \theta$.

$\tau(x, y, u, v) \equiv \exists z\ \delta(x, y, u, z) \wedge \delta(x, y, v, z)$.

$\delta(x, x, u, u)$ and $\delta(x, x, v, v)$ implies $u\theta z\theta v$.

(b) If $\big((x, y) \in \theta\ \&\ \tau(x, y, u, v)\big) \Rightarrow (u, v) \in \theta$, then $\theta = \{(u, v)|\ N \subseteq [u = v]\}$ where $N = \bigcap\{[x = y]|\ (x, y) \in \theta\}$. If $N \subseteq [u = v]$, then $[u \neq v] \subseteq \bigcup\{[x \neq y]|\ (x, y) \in \theta\}$ and by the compactness of $[u \neq v]$ we get

$$[u \neq v] \subseteq [x_1 \neq y_1] \cup \ldots \cup [x_n \neq y_n] \quad ((x_i, y_i) \in \theta, i \leqslant n).$$

We claim $[x_1 = y_1] \cap \ldots \cap [x_n = y_n] = [x_0 = y_0]$ for some $(x_0, y_0) \in \theta$, and then we are done as $[x_0 = y_0] \subseteq [u = v] \Leftrightarrow \tau(x_0, y_0, u, v)$. Clearly, we only have to consider the case of $n = 2$. By the patchwork property we can assume $x_1 = x_2$ and we pick z such that

$$[x_1 = y_1] \subseteq [z = y_2] \quad \text{and} \quad [x_1 \neq y_1] \subseteq [z = x_1].$$

Thus $[z = y_1] = [x_1 = y_1] \cap [x_1 = y_2]$ and $(x_1, y_1) \in \theta$ implies $(z, y_2) \in \theta$. As $(x_1, y_2) \in \theta$, we get $z\theta y_2 \theta x_1 \theta y_1$.

(c) $\theta = \{(u, v)|\ N \subseteq [u = v]\}$ preserves the discriminator. Assume $(x, x'), (y, y'), (z, z') \in \theta$, $\delta(x, y, z, u)$, $\delta(x', y', z', u')$. As $N \subseteq [x = x'] \cap \cap [y = y'] \cap [z = z']$, we have for $i \in N$ either $x_i = y_i\ \&\ x'_i = y'_i$ or $x_i \neq y_i\ \&\ x'_i \neq y'_i$, and hence either $u_i = z_i = z'_i = u'_i$ or $u_i = x_i = x'_i = u'_i$, and hence $N \subseteq [u = u']$.

In order to prove (2) assume that θ is a closed congruence on B, $(x, y) \in f^{-1}(\theta)$ and $A \models \tau(x, y, u, v)$. As τ is existential, we have $B \models \tau(fx, fy, fu, fv)$ and hence $(fu, fv) \in \theta$. Observe that $f^{-1}(\theta)$ is maximal or ∇ iff for $(x, y), (u, v) \in A^2$ such that $\Phi(x, y) \cap \Phi(u, v) = \Delta$ we have either $(x, y) \in f^{-1}(\theta)$ or $(u, v) \in f^{-1}(\theta)$. $\Phi(x, y) \cap \Phi(u, v) = \Delta$ is equivalent to

$$\exists r \exists s\ \delta(x, y, u, r) \wedge \delta(x, y, v, s) \wedge \delta(r, s, u, v);$$

the same holds in B and since θ is maximal, implies $(fx, fy) \in \theta$ or $(fu, fv) \in \theta$.

For (3) observe that $\langle f\theta\rangle$ is defined as a directed union of closed congruences and thus is itself a closed congruence. Assume (u, v)

$\in f^{-1}(\langle f\theta\rangle)$, i.e. for some $(x, y) \in \theta$ $B \models \tau(fx, fy, fu, fv)$. Now let $s, t \in A$ be the *unique* (!) elements such that $A \models \delta(x, y, u, s) \wedge \delta(x, y, v, t)$. We have to prove $s = t$. Now $B \models \tau(fx, fy, fu, fv)$ implies $fs = ft$ and thus $s = t$ as f is one-to-one.

5.2. Theorem. *Let $\mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$, $\mathfrak{K}^+ = \{A \in \mathfrak{K}|\ A$ algebraically closed in $\mathfrak{K}\}$.*

(1) $\mathfrak{A}^+ \subseteq \Gamma\mathfrak{A}$.

(2) $\Gamma\mathfrak{A}^* \subseteq \mathfrak{K}^+$.

(3) $\mathfrak{A}^*$ *axiomatic* $\rightarrow \Gamma\mathfrak{A}^* = \mathfrak{K}^+$.

Proof. (1): Axiom (D): $\exists u\ \delta(x, y, z, u)$ is primitive-positive and holds is some extensions of each $A \in \mathfrak{K}^+$; thus $A \models$(D) and $\mathfrak{K}^+ \subseteq \Gamma\mathfrak{A}$.

(2): Assume that $A \in \Gamma\mathfrak{A}$ is embedded into some $B \in \mathfrak{K}$ satisfying a primitive-positive formula π over A. As each $B \in \mathfrak{K}$ is embedded into some member of $\Gamma\mathfrak{A}$, we can assume that $B \in \Gamma\mathfrak{A}$ and that each stalk of B satisfies π. By 2.19 (1) we have to show that each stalk of A satisfies π. Let A_i be a stalk of A with the projection $p_i\colon A \twoheadrightarrow A_i$ and $\Phi_i = \mathrm{Ker}\, p_i$. Φ_i is a maximal closed congruence on A, and so $\langle f\Phi_i\rangle$ is a closed congruence on B whose restriction to A is Φ_i. For some $a, b \in A$ with $(a, b) \notin \Phi_i$ let θ_j be a maximal closed congruence on B containing $\langle f\Phi_i\rangle$ but not containing (fa, fb) $(f\colon A \rightarrowtail B)$. We have $f^{-1}\theta_j = \Phi_i$ since Φ_i was maximal, and so we have an embedding $f_{ij}\colon A_i \rightarrowtail B_j$, $A_i \in \mathfrak{A}^*$ and $B_j \models \pi$ and thus $A_i \models \pi$.

(3): If $\mathfrak{A}^*$ is axiomatic it has a set of $\forall\exists$-axioms which, by the technique of 3.4, can be transformed into a set Σ of positive $\forall\exists$-axioms. As each member of $\mathfrak{A}$ can be embedded into some member of $\mathfrak{A}^*$, each $A \in \Gamma\mathfrak{A}$ is embeddable into some member of $\Gamma\mathfrak{A}^*$, which then satisfies Σ. Thus each $A \in \mathfrak{K}^+$ satisfies Σ and hence belongs to $\Gamma\mathfrak{A}^*$, which proves $\mathfrak{K}^+ = \Gamma\mathfrak{A}^*$.

Now we want to concentrate on the existentially closed members of $\Gamma\mathfrak{A}$. As in the above theorem, we want to use 2.19 in order to show that certain structures in $\Gamma\mathfrak{A}$ are existentially closed. For aribtrary primitive sentences we shall need 2.19 (3), which requires the assumption that the base-space has no atoms. This can be expressed by the following $\forall\exists$-sentence:

$$\text{(a)} \quad \forall x \forall y\ x \neq y\ \exists u \exists v\ u \neq v \wedge \tau(x, y, u, v) \wedge$$
$$\wedge \exists r \exists s\ \delta(u, v, x, r) \wedge \delta(u, v, y, s) \wedge r \neq s.$$

Clearly, this axiom (a) is satisfied by some extension of each $A \in \mathfrak{K}$, and thus (a) holds for all members of $\mathfrak{K}^*$.

Let $\varphi(x)$ be a primitive formula with one free variable x such that for some $A \in \mathfrak{A}$ with a singleton substructure $\{a\}$ of A, $A \models \varphi(a)$. Then each $B \in \mathfrak{K}$ can be embedded into $B \times A$ and thus satisfies the $\forall\exists$-sentence

$$(\varphi, n) \quad \forall x_1 \ldots \forall x_n \exists u \exists v \bigwedge_{i<n} \tau(u, v, x_i, x_{i+1}) \wedge \overline{\varphi(x_1)}(u, v) \wedge u \neq v,$$

where $\overline{\varphi(x_1)}$ is the formula $\varphi(x_1)$ with each equality $p = q$ replaced by $\tau(u, v, p, q)$. Let $\Gamma_0\mathfrak{A}^*$ denote the class of all members of $\Gamma\mathfrak{A}^*$ satisfying (a) and all (φ, n)'s.

Observe that the (φ, n)'s disappear if no member of $\mathfrak{A}$ has a singleton substructure.

5.3. Theorem. *Let* $\mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$, $\mathfrak{K}^* = \{A \in \mathfrak{K} |\ A$ *exist, closed in* $\mathfrak{K}\}$.

(1) $\mathfrak{K}^* \subseteq \Gamma\mathfrak{A}$.

(2) $\Gamma_0\mathfrak{A}^* \subseteq \mathfrak{K}^*$.

(3) $\mathfrak{A}^*$ *axiomatic* $\Rightarrow \Gamma_0\mathfrak{A}^* = \mathfrak{K}^*$.

Proof. (1) is an immediate consequence of 5.2 (1) and (3) follows from (2) and 5.2 (3) observing that $\Gamma_0\mathfrak{A}^*$ is defined by $\forall\exists$-axioms relative to $\Gamma\mathfrak{A}^*$. It remains to prove (2): Assume $A \in \Gamma_0\mathfrak{A}^*$ and $f\colon A \rightarrowtail B \in \mathfrak{K}$ and $B \models \varphi$ where φ is as in 2.19. As in 5.2, we can assume $B \in \Gamma\mathfrak{A}$ and hence B satisfies $[\varphi_m] = X(B)$ and $[\varphi_i] \neq \emptyset$ for $i < m$. If we can show the same for A, we are done because then by 2.19 (3) $A \models \varphi$. As φ_m is primitive-positive and $B \models \varphi_m$, we have $A \models \varphi_m$ by 5.2 (2). Pick a stalk B_j of B with projection $q_j\colon B \twoheadrightarrow B_j$ such that $B_j \models \varphi_1$ (for $\varphi_2, \ldots, \varphi_{m-1}$ proceed similarly). Let $\Phi_i = f^{-1}(\mathrm{Ker}\, q_j)$. If Φ_i is maximal, we have $f_{ij}\colon A_i \rightarrowtail B_j$ and $A_i \models \varphi_1$, as $A_i \in \mathfrak{A}^*$. Now assume $\Phi_i = \nabla$. Then B_j has a singleton-substructure $\{fa_1, \ldots, fa_k\}$ where $a_1, \ldots, a_k$ are the members of A occurring in φ_1, such that

$$B_j \models fa_1 = fa_2 \wedge \ldots \wedge fa_{k-1} = fa_k \wedge \varphi_1(fa_1, \ldots, fa_k).$$

$A \in \Gamma_0\mathfrak{A}^*$ implies $A \models \forall x_1 \ldots \forall x_k \exists u \exists v \bigwedge_{i<k} \tau(u, v, x_i, x_{i+1}) \wedge \overline{\varphi_1(x_1, \ldots, x_k)}$ $(u, v) \wedge u \neq v$ and thus $[\varphi_1(a_1, \ldots, a_k)] \neq \emptyset$ in particular. This proves $A \models \varphi$, and so A is existentially closed.

5.4. Corollary. *Let* $\mathfrak{A}$ *be an inductive axiomatic class of structures such that*

(i) $\mathfrak{A}$ *has a discriminator-formula* δ,

(ii) $\mathfrak{A}$ *has algebraically defined relations*,

(iii) $\mathfrak{A}$ *contains all members of* $\boldsymbol{HSP}\mathfrak{A}$, *for which* δ *is a discriminator formula*,

(iv) $\mathfrak{A}$ *has a model-companion* $\mathfrak{A}^*$.

Then $\mathfrak{K} = \boldsymbol{SP}\mathfrak{A}$ *has a model companion* $\mathfrak{K}^*$ *which equals* $\Gamma_0\mathfrak{A}^*$.

Observe that $\mathfrak{A}^*$ has a model-complete theory and so $\Gamma\mathfrak{A}^* = \Gamma^e\mathfrak{A}^*$. So if $\mathfrak{A}$ is (up to isomorphism) a finite set of finite structures, we can also assume that we have constants for all elements of members of $\mathfrak{A}$, and thus we can assume δ to be an $\exists\forall$-formula and the relations can be defined by universal formulas. In any case examples 3.7(4)–(6) satisfy the assumption of 5.4 and we get important classes of structures having model-companions.

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